us the graph of the back surface from which we can draw the front surface and so have a lens without knowing the equations of the curves. (Assuming the focal length to be unity, the fulfilment of the sine condition will require $y=\sin\theta$.) Similarly we could work with the surface

$$\zeta=a+b\rho^2+c\rho^4,$$

where $$b=\frac{1}{2r} \quad \text{and} \quad c=\frac{p}{8r^3}, \quad \text{instead of } \frac{1}{8r^3},$$

as in the case of the sphere. The choice of p would help in dealing with the aberrations. This would mean including terms like $\rho_1{}^4$, etc. in P_{12}, and the equations obtained from Fermat's theorem would be solved by approximate methods. Finally, it would be interesting to compare the results for all methods and develop the formulæ for more general optical systems.

XXXVII. *Table of Coefficients for Repeated Integration with Differences.*

By Herbert E. Salzer *†.

[Received August 2, 1945.]

Formulas for doubly or multiply repeated integration, employing either advancing or backward differences of the integrand, are obtained by integrating the Gregory–Newton interpolation formula with advancing differences or the Newton formula with backward differences. Although it is true that a k-fold primitive of $f(x)$ is expressible as $1/(k-1)!$ times a single primitive of $(x-t)^{k-1}f(t)$, that fact is of no help when only $f(x)$ and its differences are tabulated. Thus it is convenient to have a table facilitating repeated integration in terms of the integrand and its differences.

A k-fold quadrature introduces an arbitrary polynomial of the $(k-1)$th degree whose coefficients are determined by the values of the primitive at k near-by points, or instead, the primitive and its first $k-1$ derivatives at a point. A useful case (occurring in the solution of differential equations) is where the integration proceeds stepwise. Then a particular k-fold primitive (*i. e.* apart from the arbitrary polynomial) is obtained by making x_0 the lower limit of the repeated integral and x_1 the last upper limit, where $x_1-x_0=h=$the tabular interval. Then, using Δ notation for advancing differences and ∇ for backward differences, one finds for

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Advancing Differences :

$$\underbrace{\int_{x_0}^{x_1}\cdots\int_{x_0}^{x}\int_{x_0}^{x}}_{k\text{-fold}} f(x)(dx)^k = h^k \underbrace{\int_0^1\cdots\int_0^p\int_0^p}_{k\text{-fold}} f(x_0+ph)(dp)^k$$

$$= h^k\left\{\frac{f(x_0)}{k\,!} + \sum_{n=1}^{m} G_n^{(k)} \Delta^n f(x_0)\right\} + R_m. \quad . \quad . \qquad (1)$$

Backward Differences :

$$\underbrace{\int_{x_0}^{x_1}\cdots\int_{x_0}^{x}\int_{x_0}^{x}}_{k\text{-fold}} f(x)(dx)^k = h^k \underbrace{\int_0^1\cdots\int_0^p\int_0^p}_{k\text{-fold}} f(x_0+ph)(dp)^k$$

$$= h^k\left\{\frac{f(x_0)}{k\,!} + \sum_{n=1}^{m} H_n^{(k)} \nabla^n f(x_0)\right\} + R'_m. \quad . \quad . \qquad (2)$$

The coefficients $G_n^{(k)}$ and $H_n^{(k)}$ are defined by

$$G_n^{(k)} = \frac{1}{n\,!}\underbrace{\int_0^1\cdots\int_0^p\int_0^p}_{k\text{-fold}} p(p-1)\ldots(p-n+1)(dp)^k, \quad . \quad . \quad . \qquad (3)$$

and

$$H_n^{(k)} = \frac{1}{n\,!}\underbrace{\int_0^1\cdots\int_0^p\int_0^p}_{k\text{-fold}} p(p+1)\ldots(p+n-1)(dp)^k. \quad . \quad . \quad . \qquad (4)$$

For the most important case of a single quadrature, the exact values of $G_n^{(k)}$ and $H_n^{(k)}$ have already been tabulated up to $n=20$ *. Also, in the paper by W. E. Milne, "On the Numerical Integration of Certain Differential Equations of the Second Order," Am. Math. Monthly, xl. pp. 322–327 (1933), there are tabulated the exact values of $G_i^{(2)}$ and $H_i^{(2)}$ for $i=1, 2, \ldots, 7$ (" i " for " n " only here to avoid confusion with n in Milne's x_{n-1} and x_{n+1}), where $G_i^{(2)} \equiv (-1)^i A_i(x_{n-1})$ and $H_i^{(2)} \equiv A_i(x_{n+1})$. The tables in this paper give the exact values of $G_n^{(2)}$ and $H_n^{(2)}$ up to $n=20$ (because of the greater importance of double quadrature and second-order differentiale quations). Then $G_n^{(k)}$ and $H_n^{(k)}$ are given in decimal form for $k=2$, 3, 4, 5 and 6, n going up to $22-k$ ($k=2$ is repeated for convenience), with an accuracy well within $1\frac{1}{2}$ units in the last decimal place for $k=2$ and well within 2 units in the last place for $k=3$, 4, 5 and 6.

The coefficients for double quadrature are expressible rather simply in terms of $B_\nu^{(n)}(x)$, Bernoulli polynomials of order n and degree ν, defined by Milne-Thomson † from the equation

$$\frac{t^n e^{xt}}{(e^t-1)^n} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu\,!} B_\nu^{(n)}(x). \quad . \quad . \quad . \qquad \text{(M. 127 (2) upper)}$$

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* Jour. of Math. & Phys. vol. xxii. No. 2, pp. 49–50 (June 1943), where $G_n^{(1)} \equiv B_n^{(n)}(1)/n\,!$ and $H_n^{(1)} \equiv (-1)^n B_n^{(n)}/n\,!$

† For the sake of brevity, M. will denote L. M. Milne-Thomson, "Calculus of Finite Differences" (Macmillan, 1933).

($B_\nu^{(n)}(0)$ is denoted by $B_\nu^{(n)}$ and called a " Bernoulli number of order n.")

Thus from

$$B_n^{(n+1)}(t+1)=t(t-1)\ldots(t-n+1), \quad . \quad . \quad (\text{M. } 130\ (2))$$

substitute into (3) for

$$G_n^{(2)}=\frac{1}{n\,!}\int_0^1\int_1^{x+1} B_n^{(n+1)}(t)dt\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Making use of the relation

$$\int_a^x B_\nu^{(n)}(t)dt=\frac{1}{\nu+1}\Big[B_{\nu+1}^{(n)}(x)-B_{\nu+1}^{(n)}(a)\Big], \quad . \quad . \quad . \quad (\text{M. } 127\ (3)\ \text{lower})$$

one finds

$$G_n^{(2)}=\frac{1}{(n+1)\,!}\int_0^1\Big[B_{n+1}^{(n+1)}(x+1)-B_{n+1}^{(n+1)}(1)\Big]dx,$$

and using that relation again,

$$G_n^{(2)}=\frac{1}{(n+2)\,!}\Big[B_{n+2}^{(n+1)}(2)-B_{n+2}^{(n+1)}(1)\Big]-\frac{1}{(n+1)\,!}B_{n+1}^{(n+1)}(1).$$

But from

$$B_\nu^{(n)}(x+1)=B_\nu^{(n)}(x)+\nu B_{\nu-1}^{(n+1)}(x), \quad . \quad . \quad (\text{M. } 128\ (7))$$

$$G_n^{(2)}=\frac{1}{(n+1)\,!}\Big[B_{n+1}^{(n)}(1)-B_{n+1}^{(n+1)}(1)\Big]. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now from the relation

$$B_\nu^{(n+1)}(x)=\Big(1-\frac{\nu}{n}\Big)B_\nu^{(n)}(x)+\nu\Big(\frac{x}{n}-1\Big)B_{\nu-1}^{(n)}(x), \quad . \quad (\text{M. } 129\ (2))$$

one gets

$$B_{n+1}^{(n)}(1)=-n\Big[B_{n+1}^{(n+1)}(1)+\frac{(n+1)(n-1)}{n}B_n^{(n)}(1)\Big];$$

from which

$$G_n^{(2)}=\frac{1}{n\,!}\Big[-B_{n+1}^{(n+1)}(1)-(n-1)B_n^{(n)}(1)\Big]. \quad . \quad . \quad . \quad . \quad (7)$$

Equation (7) was employed as an independent check upon the calculations of $G_n^{(2)}$, using the previously tabulated values of $B_n^{(n)}(1)/n\,!$ for n up to 20** and an independently calculated value of $B_{21}^{(21)}(1)/21!$ to check $G_{20}^{(2)}$. From (7) and (M. 128 (7)) it follows immediately that

$$G_n^{(2)}=\frac{1}{n\,!}\Big[2\,B_n^{(n)}(1)-B_{n+1}^{(n+1)}(2)\Big]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

A more direct derivation of (7) is had from (3), through integration by parts and use of the relation

$$B_n^{(n)}(1)=\int_0^1 x(x-1)\ldots(x-n+1)dx. \quad . \quad . \quad \text{(M. 130 (4))}$$

In similar fashion, from (4) and (M. 130 (2)),

$$H_n^{(2)}=\frac{(-1)^n}{n\,!}\int_0^1\int_0^x B_n^{(n+1)}(-t+1)dt\,dx, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

so that

$$H_n^{(2)}=\frac{(-1)^{n+1}}{n\,!}\int_0^1\int_0^{-x} B_n^{(n+1)}(t+1)dt\,dx=\frac{(-1)^n}{n\,!}\int_0^{-1}\int_0^x B_n^{(n+1)}(t+1)dt\,dx$$

$$=\frac{(-1)^n}{n\,!}\int_0^{-1}\frac{1}{n+1}\left[B_{n+1}^{(n+1)}(x+1)-B_{n+1}^{(n+1)}(1)\right]dx,$$

from (M. 127 (3) lower).

Now, by change of variable $x'=x+1$, and from

$$\int_0^1 B_\nu^{(n)}(t)dt=B_\nu^{(n-1)}. \quad . \quad . \quad . \quad . \quad . \quad \text{(M. 128 (10))}$$

$$H_n^{(2)}=\frac{(-1)^n}{(n+1)\,!}\left[-B_{n+1}^{(n)}+B_{n+1}^{(n+1)}(1)\right]. \quad . \quad . \quad . \quad . \quad (10)$$

But from

$$B_{n+1}^{(n)}=-n\left[B_{n+1}^{(n+1)}+(n+1)B_n^{(n)}\right]$$

(M. 129 (3) for $\nu=n+1$),

$$H_n^{(2)}=(-1)^n\left[\frac{n}{(n+1)\,!}B_{n+1}^{(n+1)}+\frac{B_n^{(n)}}{(n-1)\,!}+\frac{1}{(n+1)\,!}B_{n+1}^{(n+1)}(1)\right], \quad . \quad (11)$$

and since

$$\frac{B_{n+1}^{(n+1)}(1)}{(n+1)\,!}=\frac{B_{n+1}^{(n+1)}}{(n+1)\,!}+\frac{B_n^{(n)}}{n\,!}$$

(obvious from M. 128 (8)),

$$H_n^{(2)}=\frac{(-1)^n B_{n+1}^{(n+1)}(1)}{n\,!}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This last equation was employed as a check on $H_n^{(2)}$, even though it would have been very much easier to compute $H_n^{(2)}$ that way rather than by direct quadrature. Thus an additional check has been performed on the quantities $B_n^{(n)}(1)/n\,!$ published previously. Equation (12) can be obtained directly from (4) and (M. 130 (4)) through integration by parts.

From (7) and (12) it follows at once that

$$G_n^{(2)}=(-1)^{n-1}\left[H_n^{(2)}+\frac{1-n}{n}H_{n-1}^{(2)}\right]. \quad . \quad . \quad . \quad . \quad (13)$$

Coefficients for k-fold repeated quadrature ($k\geqslant 2$) can be generated by simple recursion formulas for $G_n^{(k)}$ in terms of $G_n^{(k-1)}$ and $G_{n+1}^{(k-1)}$, and for $H_n^{(k)}$ in terms of $H_n^{(k-1)}$ and $H_{n+1}^{(k-1)}$. Thus, when all successive primitives of $f(t)$ vanish at $t=0$, one has

$$\underbrace{\int_0^1\cdots\int_0^t\int_0^t}_{k\text{-fold}} f(t)(dt)^k=\frac{1}{(k-1)\,!}\int_0^1(1-t)^{k-1}f(t)dt,$$

so that

$$G_n^{(k)}=\frac{1}{n\,!\,(k-1)\,!}\int_0^1(1-t)^{k-1}t(t-1)\ldots(t-n+1)dt$$

$$=\frac{1}{n\,!\,(k-1)\,!}\int_0^1[1-n-(t-n)](1-t)^{k-2}t(t-1)\ldots(t-n+1)dt$$

$$=\frac{1-n}{n\,!\,(k-1)\,!}\int_0^1(1-t)^{k-2}t(t-1)\ldots(t-n+1)dt-$$

$$\frac{(n+1)}{(n+1)\,!\,(k-1)\,!}\int_0^1(1-t)^{k-2}t(t-1)\ldots(t-n+1)(t-n)dt.$$

Hence,

$$G_n^{(k)}=\frac{1}{1-k}\left[(n-1)G_n^{(k-1)}+(n+1)G_{n+1}^{(k-1)}\right],\ \text{for } k\geqslant 2. \quad . \quad . \quad (14)$$

In exactly the same manner,

$$H_n^{(k)}=\frac{1}{n\,!\,(k-1)\,!}\int_0^1(1-t)^{k-1}t(t+1)\ldots(t+n-1)dt$$

is seen to satisfy

$$H_n^{(k)}=\frac{n+1}{k-1}\left[H_n^{(k-1)}-H_{n+1}^{(k-1)}\right],\ \text{for k}\geqslant 2. \quad . \quad . \quad . \quad (15)$$

Formulas (14) and (15) are convenient for obtaining $G_n^{(k)}$ and $H_n^{(k)}$, starting from $G_n^{(2)}$ and $H_n^{(2)}$. But if one does not begin with exact values of $G_n^{(2)}$ and $H_n^{(2)}$, the error in $G_n^{(k)}$ and $H_n^{(k)}$ multiplies enormously with k, an upper bound being

$$\frac{2^{k-2}}{n}\binom{n+k-2}{k-1}$$

times the initial error for $k=2$. It was thought sufficient, for most needs that might arise in practice, to begin with about 10 significant figures in $G_n^{(2)}$ and $H_n^{(2)}$ and to apply these recursion formulas up to $k=6$. The number of places given below for the decimal values of $G_n^{(k)}$ and $H_n^{(k)}$ was determined by allowing for the worst possible propagation of error in the recursion scheme.

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TABLE OF COEFFICIENTS.

$$G_n^{(2)} \equiv \frac{1}{n!}\int_0^1\int_0^t t(t-1)\ldots(t-n+1)(dt)^2 \quad \text{and}$$

$$H_n^{(2)} \equiv \frac{1}{n!}\int_0^1\int_0^t t(t+1)\ldots(t+n-1)(dt)^2.$$

n	$G_n^{(2)}$.	$H_n^{(2)}$.
1	$\frac{1}{6}$	$\frac{1}{6}$
2	$-\frac{1}{24}$	$\frac{1}{8}$
3	$\frac{1}{45}$	$\frac{19}{180}$
4	$-\frac{7}{480}$	$\frac{3}{32}$
5	$\frac{107}{10080}$	$\frac{863}{10080}$
6	$-\frac{199}{24192}$	$\frac{275}{3456}$
7	$\frac{6031}{907200}$	$\frac{33953}{453600}$
8	$-\frac{5741}{10\ 36800}$	$\frac{8183}{115200}$
9	$\frac{11\ 29981}{2395\ 00800}$	$\frac{32\ 50433}{479\ 00160}$
10	$-\frac{4\ 35569}{1064\ 44800}$	$\frac{4671}{71680}$
11	$\frac{356\ 61419}{99066\ 24000}$	$\frac{1\ 36957\ 79093}{21\ 79457\ 28000}$
12	$-\frac{15234\ 89833}{47\ 55179\ 52000}$	$\frac{22242\ 34463}{3\ 65783\ 04000}$
13	$\frac{4\ 51830\ 33541}{1569\ 20924\ 16000}$	$\frac{13\ 22828\ 40127}{224\ 17274\ 88000}$
14	$-\frac{1\ 25976\ 80311}{482\ 83361\ 28000}$	$\frac{26396\ 51053}{4\ 59841\ 53600}$
15	$\frac{1905\ 50949\ 97949}{8\ 00296\ 71321\ 60000}$	$\frac{11195\ 67034\ 48001}{2\ 00074\ 17830\ 40000}$
16	$-\frac{933\ 12106\ 33373}{4\ 26824\ 91371\ 52000}$	$\frac{501\ 88465}{9184\ 21504}$
17	$\frac{10414\ 89360\ 40729}{51\ 60701\ 22946\ 56000}$	$\frac{2\ 33402\ 89463\ 44463}{43\ 66747\ 19416\ 32000}$
18	$-\frac{2\ 25017\ 07487\ 19203}{1202\ 13981\ 58049\ 28000}$	$\frac{30112\ 40351\ 85049}{5\ 75186\ 51473\ 92000}$
19	$\frac{734\ 85432\ 83944\ 19537}{4\ 21500\ 27291\ 66028\ 80000}$	$\frac{12365\ 72232\ 34699\ 80029}{2\ 40857\ 29880\ 94873\ 60000}$
20	$-\frac{826\ 51150\ 34638\ 60961}{5\ 07067\ 99749\ 36576\ 00000}$	$\frac{8519\ 31871\ 68012\ 73673}{169022\ 66583\ 12192\ 00000}$