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The numbers L(m,n) and their Relations with Prepared Bernoulli and Eulerian nos

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A formula for the summation of the series s^r has been given by Hansraj Gupta [2] in terms of G-functions. The present author [1] obtained the sum in terms of functions allied to the G-functions. In this paper the sum is obtained in terms of another class of functions termed L-functions. The numerical values of these are symmetrical integrals. These numbers are interesting because of their close relations with Prepared Bernoullian and Eulerian numbers. From the properties of these L-functions it has been possible to express a Prepared Bernoullian number S_{2m+1} in terms of algebraic functions of m although the formula is of no practical utility in the evaluation of S_{2m+1}. Some algebraic identities have also been obtained from the solutions of difference equations connected with L-and related functions.

1. L (m, n) and the series $\sum_{s=1}^{n} s^r$.

It may be shown that for all positive integers

$$s^r = \sum_{k=1}^{r} L(r, k)\binom{s+k-1}{r} \tag{1·1}$$

$$\therefore \sum_{s=1}^{n} s^r = \sum_{k=1}^{r} L(r, k)\binom{n+k}{r+1} \tag{1·2}$$

$$\left.\begin{array}{l}\text{where } L(m, n) = (m-n+1)\,L(m-1,\ n-1)+n\,L(m-1,\ n)\\ L(m,1) = 1,\ m \geqslant 1;\ L(1, n) = 0;\ n \geqslant 2\end{array}\right\}. \tag{1·3}$$

Table (1) gives a few values of L (m, n).

2. From (1·3) the first three of the following four can be easily obtained.

$$L(m, n) = L(m, m-n+1) \tag{2·1}$$

$$\sum_{r=1}^{m} L(m, r) = m\,! \tag{2·2}$$

$$\sum_{r=1}^{2m} (-1)^{r+1} L(2m, r) = 0 \tag{2·3}$$

$$\sum_{r=1}^{2m+1} (-1)^{r+1} L(2m+1, r) = (-1)^m S_{2m+1} \tag{2·4}$$

where S_{2m+1} are Prepared Bernoullian numbers.

3. Proof of (2·4). Define another set of numbers by the equation

$$L'(m, n) = \sum_{r=1}^{m-n+1} \binom{m-1}{r-1} L(m-r,\ n-1). \tag{3·1}$$

Table (2) gives a few values of $L'(m, n)$.

It may be easily shown that

$$L'(m, n) = L'(m,\ m-n+1). \tag{3·2}$$

Further let

$$h(m) = \sum_{r=1}^{m} (-1)^{r+1} L(m, r) \tag{3·3}$$

$$H(m) = \sum_{r=1}^{m} (-1)^{r+1} L'(m, r). \tag{3·4}$$

Using (3·1) and (3·3) in (3·4), we get

$$H(m) = 1 - \sum_{s=0}^{m-2} \binom{m-1}{s} h(m-s-1). \tag{3·5}$$

By virtue of (2·1) and (3·2), $h(2r)$ and $H(2r)$ vanish. Hence

$$-h(2n-1) = \sum_{s=1}^{n-1} \binom{2n-1}{2s} h(2n-2s-1). \tag{3·6}$$

Now (3·6) is clearly the recurrence relation between Prepared Bernoullian Numbers S_{2m+1} taken alternately positive and negative.

Hence

$$h(2m+1) = (-1)^{m} S_{2m+1}. \tag{3·7}$$

(3·7) gives (2·4).

4. By putting $m = 2n+1$ in (3·5), it may be shown that

$$H(2m+1) = (-1)^{m} S_{2m} \tag{4·1}$$

where S_{2m} are Eulerian numbers.

5. Defining another set of numbers by the equation

$$L''(m, n) = \sum_{r=1}^{m-n+1} \binom{m}{r} L(m-r,\ n-1) \tag{5·1}$$

it may be shown that

$$\sum_{r=1}^{m} (-1)^{r} L''(m, r) = \left(\sin \frac{\pi m}{2} + \cos \frac{\pi m}{2} \right) S_m \tag{5·2}$$

where S_{2m+r} are Prepared Bernoullian and S_{2m} are Eulerian numbers.

Table (3) gives a few values of $L''(m, n)$.

6. A solution of the equation (1·3) can be easily obtained by putting $s = 1, 2, 3$ etc. in (1·1). We get

$$L(m, n) = \sum_{r=0}^{n-1} (-1)^{r} \binom{m+1}{r} (n-r)^{m}. \tag{6·1}$$

7. Using (6·1) as basis let a class of numbers be defined by

$$L(m, n, p) = \sum_{r=0}^{n-1} (-1)^r \binom{m+p}{r} (n-r)^m . \tag{7·1}$$

The following properties may be easily proved.

$$L(m, n, 1) = L(m, n) \tag{7·2}$$

$$L(m, n, p) = (m-n+p)\, L(m-1,\, n-1,\, p) + n\, L(m-1,\, n,\, p) \tag{7·3}$$

$$L(m, n, p) = L(m, n, p-1) - L(m, n-1, p-1) \tag{7·4}$$

$$L(m, n, p) = (-1)^p\, L(m,\, m-n+p,\, p). \tag{7·5}$$

Table (4) gives a few values of $L(m, n, p)$.

8. It is possible to express S_{2m+1} as a finite series in powers of m. However, for the purpose of evaluation of S_{2m+1}, the series is of no use as the expression is not very simple.

From (7·4) we get

$$\sum_{r=1}^{m+p-1} (-1)^r L(m, r, p) = 2 \sum_{r=1}^{m+p-2} (-1)^{r-1} L(m, r, p-1) \tag{8·1}$$

$$= 2^{p-1} \sum_{r=1}^{m+p-2} (-1)^{r-1} L(m, r, 1). \tag{8·2}$$

Using (7·1) and (2·4) in (8·2) we get

$$(-1)^m 2^{p-1} S_{2m+1} = \sum_{t=1}^{2m+1} (-1)^{t+1}\, t^{2m+1} \sum_{r=0}^{2m+p-t} \binom{2m+p+1}{r}. \tag{8·3}$$

In (8·3) putting $p=1$ we get

$$(-1)^m S_{2m+1} = \sum_{t=1}^{2m+1} (-1)^{t+1}\, t^{2m+1} \left\{ \sum_{r=0}^{2m+1-t} \binom{2m+2}{r} \right\}. \tag{8·4}$$

9. Multiplying (8·4) by 2^{p-1} and equating to (8·3), we get

$$2^{p-1} \sum_{t=1}^{2m+1} (-1)^{t+1}\, t^{2m+1} \sum_{r=0}^{2m+1-t} \binom{2m+2}{r}$$

$$= \sum (-1)^{t+1}\, t^{2m+1} \sum_{r=0}^{2m+p-t} \binom{2m+p+1}{r}. \tag{9·1}$$

From (7·4) and (7·5) it may be shown that $L(2m,\, m+n,\, 2n) = 0$. (9·2)

Applying (7·1) to (9·2) we get

$$\sum_{s=0}^{m+n-1} (-1)^s \binom{2m+2n}{s} (m+n-s)^{2m} = 0 \tag{9·3}$$

where $m \geqslant 1$; $n > 1$.

From (7·1) we get

$$L(1, 1, 1) = 1. \tag{9·4}$$

Applying (7·4) repeatedly to itself, we get

$$L(1, n, p) = \sum_{s=0}^{k} (-1)^s \binom{k}{s} L(1, n-s, p-k). \tag{9·5}$$

Then by using (9·4) in (9·5), we get $L(1, n, p) = (-1)^{n-1}\binom{p-1}{n-1}$. (9·6)

From (7·1) and (9·6), we get

$$\sum_{s=0}^{n-1} (-1)^s (n-s) \binom{p+1}{s} = (-1)^{n-1} \binom{p-1}{n-1} \tag{9·7}$$

where $p \geqslant n \geqslant 1$.

References

1. Daljit Singh (1945). "On the series Σs^r." *Math. Student.* Vol. XIII, 2, 59-60.
2. Hansraj Gupta (1940) *Symmetric Functions in the Theory of Integral Numbers.* Lucknow University.

Table (1)

$L(m, n) = (m-n+1)\, L(m-1, n-1) + n\, L(m-1, n)$; $L(1, 1) = 1$; $L(1, 2) = 0$; m, n +ve

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m \ n	1	2	3	4	5	6	7	$\sum_{n=1}^{m} L(m, n)$	$\sum_{n=1}^{m} (-1)^{n-1} L(m, n)$
1	1							1 = 1 !	+ 1 = + S_1
2	1	1						2 = 2 !	0
3	1	4	1					6 = 3 !	− 2 = − S_3
4	1	11	11	1				24 = 4 !	0
5	1	26	66	26	1			120 = 5 !	+ 16 = + S_5
6	1	57	302	302	57	1		720 = 6 !	0
7	1	120	1191	2416	1191	120	1	5040 = 7 !	− 272 = − S_7

Table (2)

$$L'(m, n) = \sum_{r=1}^{m-n+1} \binom{m-1}{r-1} L(m-r, n-1)$$

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m \ n	1	2	3	4	5	6	7	$\sum_{n=1}^{m} L'(m, n)$	$\sum_{n=1}^{m} (-1)^{n-1} L'(m, n)$
1	1							1	+ 1 = + S_0
2	1	1						2	0
3	1	3	1					5	− 1 = − S_2
4	1	7	7	1				16	0
5	1	15	33	15	1			65	+ 5 = + S_4
6	1	31	131	131	31	1		326	0
7	1	63	473	883	473	63	1	1957	− 61 = − S_6

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Table (3)

$$L''(m,n)=\sum_{r=1}^{m-n+1}\binom{m}{r}L(m-r,\,n-1)$$

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$m \backslash n$	1	2	3	4	5	6	7	$\sum_{n=1}^{m} L''(m,n)$	$\sum(-1)^{n-1}L''(m,n)$
1	1							$\boxed{1}$	$+1 = +S_1$
2	1	2						3	$-1 = -S_2$
3	1	6	3					10	$-2 = -S_3$
4	1	14	22	4				41	$+5 = +S_4$
5	1	30	105	65	5			206	$+16 = +S_5$
6	1	62	416	581	171	6		1237	$-61 = -S_6$
7	1	126	1491	3920	2695	420	7	8660	$-272 = -S_7$

Table (4)

$$L(m,n,p)=(m-n+p)\,L(m-1,\,n-1,\,p)+n\,L(m-1,\,n,\,p)$$

$p=1$

$m \backslash n$	1	2	3	4	5
1	1				
2	1	1			
3	1	4	1		
4	1	11	11	1	
5	1	26	66	26	1

$p=2$

$m \backslash n$	1	2	3	4	5	6
1	1	−1				
2	1	0	−1			
3	1	3	−3	−1		
4	1	10	0	−10	−1	
5	1	25	40	−40	−25	−1

$p=3$

$m \backslash n$	1	2	3	4	5	6	7
1	1	−2	1				
2	1	−1	−1	1			
3	1	2	−6	2	1		
4	1	9	−10	−10	9	1	
5	1	24	15	−80	15	24	1

$p=4$

$m \backslash n$	1	2	3	4	5	6	7	8
1	1	−3	3	−1				
2	1	−2	0	2	−1			
3	1	1	−8	8	−1	−1		
4	1	8	−19	0	19	−8	−1	
5	1	23	−9	−95	95	9	−23	−1

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