§ 2. *Let $\phi(n)$ be a real, positive, monotone increasing function of the real, positive variable $n$ such that*

$$\phi(n) > 1, \quad \phi(n) \to \infty \quad as \quad n \to \infty,$$

$$0 < n\frac{\phi'(n)}{\phi(n)} < c;$$

*and denote by $F(z)$ the integral function*

$$F(z) = \Sigma \frac{z^n}{\{n\phi(n)\}^n}.$$

*Then, as $z$ increases by real positive values*

$$F(z) > e^{kz/\phi(z)},$$

*where $k$ is a positive constant.*

For if $n$ is regarded as a continuous variable, and $z$ as a constant, the function $\left\{\frac{z}{n\phi(n)}\right\}^n$ has its maximum value when

$$\log\left\{\frac{z}{n\phi(n)}\right\} = 1 + n\frac{\phi'(n)}{\phi(n)},$$

so that for the maximum value

$$e < \frac{z}{n\phi(n)} < k_1 e.$$

Thus, if the equations

(4) $$y = k_1 e m \phi(m), \quad m = \omega(y)$$

are equivalent to one another, it is seen that the maximum of $\left\{\frac{z}{n\phi(n)}\right\}^n$ exceeds $e^{\omega(z)}$. But, from (4),

$$\omega(y) = m = \frac{y}{k_1 e \phi(m)} > \frac{y}{k_1 e \phi(y)},$$

since $m < y$. We therefore have at once the result stated in the theorem.

§ 3. Let now $\phi(n)$ be any function of $n$ satisfying the conditions postulated above, and let $k$ be as before. Write

$$g(z) = \Sigma \left\{\frac{k}{\phi(n)}\right\}^n z^n,$$







so that $g(z)$ is an integral function of $z$; write also

$$f_m(x) = (1+x)^{-m-1} = \sum_{n=0}^{\infty}(-1)^n \binom{n+m}{m} x^n, \quad (m = 0, 1, 2 \ldots),$$

$$f(x) = \sum_{m=0}^{\infty}\left\{\frac{k}{\phi(m)}\right\}^m f_m(x).$$

It is easily shown* that $f(x)$ is analytic for $|x| < 1$, and that, as $x$ tends along any Stolz-path to a point $x_0$ on the unit-circle, Abel's limit

(5) $$\lim_{x \to x_0} f(x) = \frac{1}{1+x_0} g\left(\frac{1}{1+x_0}\right), \quad (|x_0| = 1;\ x_0 \neq -1)$$

exists. On the other hand we have, for $|x| < 1$,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where $$(-1)^n a_n = \sum_{m=0}^{\infty}\left\{\frac{k}{\phi(m)}\right\}^m \binom{n+m}{m};$$

and therefore

(2 *bis*) $$|a_n| > \sum_{m=0}^{\infty}\left\{\frac{k}{m\phi(m)}\right\}^m n^m > e^{n/\phi(n)},$$

in virtue of § 2.

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# SOME PROBLEMS IN POTENTIAL THEORY.

By *Dr. H. Bateman.*

§ 1. In a previous note† it was shown that the potential of a surface of revolution, whose meridian curve is a limaçon, can be expressed in the form

$$V = (\cosh\sigma - \cos\chi)\sum_{n=0}^{\infty}(2n+1)\frac{P_n(\cosh\sigma)}{P_n(\cosh\sigma_0)} Q_n(\cosh\sigma_0) P_n(\cos\chi),$$

* See Landau, *loc. cit.*

† *Messenger of Mathematics*, vol. li. (February, 1922), p. 151.

the potential being unity over the surface $\sigma=\sigma_0$, where

$$\left(\frac{R+X}{2}\right)^{\frac{1}{2}}=\frac{a\sinh\sigma}{\cosh\sigma-\cos\chi},\quad \left(\frac{R-X}{2}\right)^{\frac{1}{2}}=\frac{a\sin\chi}{\cosh\sigma-\cos\chi}.$$

To find the capacity of the surface we must determine the form of $V$ at infinity, *i.e.* in the neighbourhood of $\sigma=0$, $\chi=0$. Writing

$$P_n(\cosh\sigma)=1+\frac{n(n+1)}{2}(\cosh\sigma-1)+\frac{(n-1)n(n+1)(n+2)}{1^2.2^2}\left(\frac{\cosh\sigma-1}{2}\right)^2\ldots,$$

$$P_n(\cos\chi)=1+\frac{n(n+1)}{2}(\cos\chi-1)+\frac{(n-1)n(n+1)(n+2)}{1^2.2^2}\left(\frac{\cos\chi-1}{2}\right)^2+\ldots,$$

$$R=a^2\frac{\cosh\sigma+\cos\chi}{\cosh\sigma-\cos\chi}\sim a^2\frac{2}{\cosh\sigma-\cos\chi},$$

$$X=a^2\frac{\cosh^2\sigma+\cos^2\chi-2}{(\cosh\sigma-\cos\chi)^2}\sim 2a^2\frac{\cosh\sigma+\cos\chi-2}{(\cosh\sigma-\cos\chi)^2},$$

we find that

$$V=\frac{2a^2}{R}\sum_{n=0}^{\infty}(2n+1)\frac{Q_n(\cosh\sigma_0)}{P_n(\cosh\sigma_0)}+\frac{2a^4X}{R^3}\sum_{n=0}^{\infty}n(n+1)(2n+1)\frac{Q_n(\cosh\sigma_0)}{P_n(\cosh\sigma_0)}$$
$$+\ldots\ldots.$$

The first term gives an expression for the capacity $C$, viz.,

$$C=2a^2\sum_{n=0}^{\infty}(2n+1)\frac{Q_n(\cosh\sigma_0)}{P_n(\cosh\sigma_0)},$$

while the second term enables us to determine a point where the charge $C$ should be placed in order that its potential may agree with $V$ at infinity up to terms of the second order in $\frac{1}{R}$. This point may be called the centre of charge.

To find the polar equation of the limaçon we write

$$r=\frac{2a^2}{\cosh\sigma_0-\cos\chi_0},\quad \cos\theta=\frac{\cosh\sigma_0\cos\chi_0-1}{\cosh\sigma_0-\cos\chi_0},\quad \sin\theta=\frac{\sinh\sigma_0\sin\chi_0}{\cosh\sigma_0-\cos\chi_0},$$

then $\quad X=a^2+r\cos\theta,\quad Y=\sqrt{(R^2-X^2)}=r\sin\theta,$

and $$r=\frac{2a^2}{\sinh^2\sigma_0}(\cosh\sigma_0+\cos\theta).$$

The area of the surface generated by the revolution of the limaçon about its axis of symmetry is $4\pi k^2$, where

$$k=2a^2\operatorname{cosech}^2\sigma_0(\cosh^2\sigma_0+\tfrac{2}{3})^{\frac{1}{2}}.$$



With the aid of tables for $Q_n(\cosh\sigma_0)$ and $P_n(\cosh\sigma_0)$ we find that

| $\cosh\sigma_0$ | $C/2a^2$ | $k/2a^2$ |
|---|---|---|
| 2 | .718695 | .722009 |
| 1.2 | 3.25824 | 3.29872 |

In the case of a sphere ($\cosh\sigma_0=\infty$) we have, of course, $C=k$.

Of all surfaces of given area the sphere has apparently the greatest capacity. When $\cosh\sigma_0=2$ the limaçon has a point of undulation on the axis of symmetry, the points of contact of the double tangent being consecutive. The value of $C$ in this case differs from $k$ by about 1 part in 200. When $\cosh\sigma_0=1.2$ the double tangent touches the limaçon in two distinct real points, and the curve bends inwards near the vertex. The capacity is slightly reduced by this hollow, $C$ differing from $k$ by about 1 part in 80.

§ 2. Since the author does not remember having seen any tables of spheroidal harmonics, the values of $P_n$, $Q_n$ and their first derivatives are given* for a few values of $\cosh\sigma$.

$s=\cosh\sigma=1.1$

| $n$ | $P_n(s)$ | $Q_n(s)$ |
|---|---|---|
| 0 | 1 | 1.52226 12188 |
| 1 | 1.1 | .67448 73407 |
| 2 | 1.315 | .35177 35028 |
| 3 | 1.6775 | .19525 98613 |
| 4 | 2.24293 75 | .11204 51059 |
| 5 | 3.09901 625 | .06564 14207 |
| 6 | 4.38056 81875 | .03900 59434 |
| 7 | 6.29257 53687 | .02341 94953 |
| 8 | 9.14543 95340 | .01417 25085 |
| 9 | 13.40879 07039 | .00862 99941 |
| 10 | 19.79347 69907 | .00528 14300 |
| 11 | 29.37649 19495 | .00324 55538 |
| 12 | 43.79141 66188 | .00200 13984 |
| 13 | 65.51892 72018 | .00123 78316 |
| 14 | 98.33026 58463 | .00076 75299 |
| 15 | 147.96469 99781 | .00047 69708 |
| 16 | 223.16514 25975 | .00029 69847 |
| 17 | 337.26232 21552 | .00018 52360 |
| 18 | 510.59955 43788 | .00011 57137 |
| 19 | 774.24631 91802 | .00007 23842 |
| 20 | 1175.68877 79816 | .00004 53361 |

* In calculating these values use has been made of the values of $\log_e 2$, $\log_e 3$, $\log_e 5$, $\log_e 7$, and $\log_e 10$, given by J. C. Adams, *Proc. Roy. Soc. London*, vol. xxvii. (1878), p. 88.

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$s = \cosh\sigma = 1.1$

| $n$ | $P_n'(s)$ | $Q_n'(s)$ |
|---|---|---|
| 0 | 0 | − 4.76190 47619 |
| 1 | 1 | − 3.71583 40193 |
| 2 | 3.3 | − 2.73844 27398 |
| 3 | 7.575 | − 1.95696 65053 |
| 4 | 15.0425 | − 1.37162 37107 |
| 5 | 27.76143 75 | − 0.94856 05522 |
| 6 | 49.13167 875 | − 0.64956 80830 |
| 7 | 84.70882 39375 | − 0.44148 32880 |
| 8 | 143.52030 92812 | − 0.29827 56535 |
| 9 | 240.18129 60152 | − 0.20055 06435 |
| 10 | 398.28733 26562 | − 0.13430 57656 |
| 11 | 655.84431 28191 | − 0.08964 06135 |
| 12 | 1073.94664 74947 | − 0.05965 80282 |
| 13 | 1750.62972 82891 | − 0.03960 56535 |
| 14 | 2842.95768 19433 | − 0.02623 65750 |
| 15 | 4602.20743 78318 | − 0.01734 72864 |
| 16 | 7429.86338 12644 | − 0.01145 04802 |
| 17 | 11966.65714 35493 | − 0.00754 67913 |
| 18 | 19234.04465 66964 | − 0.00496 72202 |
| 19 | 30858.84065 55649 | − 0.00326 53844 |
| 20 | 49429.65110 47242 | − 0.00214 42364 |

Since these values were calculated with the aid of the difference relations

$$P'_{n+1} - P'_{n-1} = (2n+1)\,P_n,$$
$$Q'_{n+1} - Q'_{n-1} = (2n+1)\,Q_n,$$

the last two or three figures in the above numbers are doubtful when $n$ is large. The difference relations

$$P'_n - sP'_{n-1} = nP_{n-1}, \quad Q'_n - sQ'_{n-1} = nQ_{n-1}$$

are, however, satisfied to 9 decimal places when $n = 20$, so the last figure may be the only one which is wrong.

$s = \cosh\sigma = 1.2$

| $n$ | $P_n(s)$ | $P_n'(s)$ | $Q_n(s)$ | $Q_n'(s)$ |
|---|---|---|---|---|
| 0 | 1 | 0 | 1.19894 76364 | − 2.27272 72727 |
| 1 | 1.2 | 1 | .43873 71637 | − 1.52832 50908 |
| 2 | 1.66 | 3.6 | .19025 30764 | − 0.95651 57816 |
| 3 | 2.52 | 9.3 | .08801 47104 | − 0.57705 97088 |
| 4 | 4.047 | 21.24 | .04214 10845 | − 0.34041 28088 |
| 5 | 6.72552 | 45.723 | .02061 29742 | − 0.19778 99483 |
| 6 | 11.423644 | 95.22072 | .01023 09729 | − 0.11367 00926 |
| 7 | 19.6936752 | 194.230372 | .00513 21902 | − 0.06478 73006 |
| 8 | 34.3150807 | 390.625848 | .00259 53267 | − 0.03668 72396 |
| 9 | 60.27536052 | 777.5867439 | .00132 07936 | − 0.02066 67467 |
| 10 | 106.5442493556 | 1535.85769788 | .00067 56155 | − 0.01159 21612 |



$s = \cosh\sigma = 2$

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 0 | .54930 61443 | − 0.33333 33333 |
| 1 | 2 | 1 | .09861 22886 | − 0 11736 05223 |
| 2 | 5.5 | 6 | .02118 37938 | − 0.03749 64673 |
| 3 | 17 | 28.5 | .00487 11203 | − 0 01144 15531 |
| 4 | 55.375 | 125 | .00116 10758 | − 0.00339 86249 |
| 5 | 185.75 | 526.875 | .00028 29767 | − 0.00099 18706 |
| 6 | 634.9375 | 2168.25 | .00007 00180 | − 0.00028 58810 |
| 7 | 2199.125 | 8781.0625 | .00001 75157 | − 0.00008 16355 |
| 8 | 7691.1484375 | 35155.125 | .00000 44181 | − 0.00002 31451 |
| 9 | 27100.671875 | 139530.5859375 | .00000 11212 | − 0.00000 65271 |
| 10 | 96060.51953125 | 550067.890625 | .00000 02843 | − 0.00000 18419 |

$s = \cosh\sigma = 3$

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 0 | .34657 35903 | − 0.125 |
| 1 | 3 | 1 | .03972 07708 | − 0.02842 64097 |
| 2 | 13 | 9 | .00545 66736 | − 0.00583 76874 |
| 3 | 63 | 66 | .00080 28543 | − 0.00114 30415 |
| 4 | 321 | 450 | .00012 24799 | − 0.00021 77073 |
| 5 | 1683 | 2955 | .00001 91079 | − 0.00004 07227 |
| 6 | 8989 | 18963 | .00000 30267 | − 0.00000 75209 |
| 7 | 48639 | 119812 | .00000 04847 | − 0.00000 13759 |
| 8 | 265729 | 748548 | .00000 00783 | − 0.00000 02499 |
| 9 | 1462563 | 4637205 | .00000 00127 | − 0.00000 00451 |
| 10 | 8097453 | 28537245 | .00000 00021 | − 0.00000 00081 |

§ 3. To obtain a potential function $V$ which satisfies the condition

$$\frac{\partial V}{\partial N} = U\frac{\partial X}{\partial N}$$

over the surface $\sigma = \sigma_0$, we assume for points outside the body

$$V = a^2 U(\cosh\sigma - \cos\chi)\sum_{m=0}^{\infty}(2m+1)\,A_m P_m(\cosh\sigma)\,P_m(\cos\chi)$$

$$= a^2 U\sum_{m=0}^{\infty}(m+1)\,(A_{m+1} - A_m)$$
$$\times\{P_m(\cosh\sigma)\,P_{m+1}(\cos\chi) - P_{m+1}(\cosh\sigma)\,P_m(\cos\chi)\}.$$

Now

$$X = a^2\frac{\sinh^2\sigma - \sin^2\chi}{(\cosh\sigma - \cos\chi)^2}$$

$$= a^2(\cosh\sigma - \cos\chi)\sum_{m=0}^{\infty}(2m+1)\,[m(m+1)+1]\,Q_m(\cosh\sigma)\,P_m(\cos\chi)$$

$$= a^2 + 2a^2\sum_{m=0}^{\infty}(m+1)^2\{Q_m(\cosh\sigma)\,P_{m+1}(\cos\chi) - Q_{m+1}(\cosh\sigma)\,P_m(\cos\chi)\};$$

hence the boundary condition at $\sigma=\sigma_0$ will be satisfied for all values of $\chi$ if

$$U\sum_{m=0}^{\infty}(m+1)(A_{m+1}-A_m)\times\{P'_m(\cosh\sigma_0)P_{m+1}(\cos\chi)-P'_{m+1}(\cosh\sigma_0)P_m(\cos\chi)\}$$
$$=2U\Sigma(m+1)^2\{Q'_m(\cosh\sigma_0)P_{m+1}(\cos\chi)-Q'_{m+1}(\cosh\sigma_0)P_m(\cos\chi)\}.$$

This leads to the system of equations

$$m(A_m-A_{m-1})P'_{m-1}(\cosh\sigma_0)-(m+1)(A_{m+1}-A_m)P'_{m+1}(\cosh\sigma_0)$$
$$=2m^2Q'_{m-1}(\cosh\sigma_0)-2(m+1)^2Q'_{m+1}(\cosh\sigma_0).$$

The left-hand side of the typical equation becomes a perfect difference when multiplied by $P_m'(\cosh\sigma_0)$, while the right-hand side may be transformed with the aid of the identity

$$Q'_m(\cosh\sigma_0)P'_{m-1}(\cosh\sigma_0)-Q'_{m-1}(\cosh\sigma_0)P'_m(\cosh\sigma_0)=m\,\mathrm{cosech}^2\sigma_0.$$

Consequently the typical equation may be written in the form

$$m(A_m-A_{m-1})P'_m(\cosh\sigma_0)P'_{m-1}(\cosh\sigma_0)$$
$$-(m+1)(A_{m+1}-A_m)P'_m(\cosh\sigma_0)P'_{m+1}(\cosh\sigma_0)$$
$$=2m^2Q'_m(\cosh\sigma_0)P'_{m-1}(\cosh\sigma_0)$$
$$-2(m+1)^2Q'_{m+1}(\cosh\sigma_0)P'_m(\cosh\sigma_0)-2m^3\,\mathrm{cosech}^2\sigma_0.$$

Summing from $m=1$ to $m=n$, we get

$$(n+1)(A_{n+1}-A_n)P'_n(\cosh\sigma_0)P'_{n+1}(\cosh\sigma_0)$$
$$=2(m+1)^2Q'_{m+1}(\cosh\sigma_0)P'_n(\cosh\sigma_0)+\frac{n^2(n+1)^2}{2\sinh^2\sigma_0},$$

therefore

$$A_{m+1}-A_m=2(m+1)\frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)}$$
$$+\frac{m^2}{2}\left[\frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)}-\frac{Q'_m(\cosh\sigma_0)}{P'_m(\cosh\sigma_0)}\right]$$
$$=\tfrac{1}{2}(m+2)^2\frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)}-\tfrac{1}{2}m^2\frac{Q'_m(\cosh\sigma_0)}{P'_m(\cosh\sigma_0)}.$$

Hence finally we obtain the following expression for $V$

$$V=\tfrac{1}{2}a^2U\Sigma(m+1)\left[(m+2)^2\frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)}-m^2\frac{Q'_m(\cosh\sigma_0)}{P'_m(\cosh\sigma_0)}\right]$$
$$\times[P_m(\cosh\sigma)P_{m+1}(\cos\chi)-P_{m+1}(\cosh\sigma)P_m(\cos\chi)].$$



We may deduce from this expression the form which $\Phi$ takes at infinity by writing for small values of $\sigma$ and $\chi$ the expansions for $P_m(\cosh\sigma)$ and $P_m(\cos\chi)$ used before. The coefficient of $\cosh\sigma-\cos\chi$ is then

$$a^2U\sum_{m=0}^{\infty}(m+1)^2\left[(m+2)^2\frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)}-m^2\frac{Q'_m(\cosh\sigma_0)}{P'_m(\cosh\sigma_0)}\right],$$

and this is zero. The most important term in the expansion is thus

$$\tfrac{1}{2}a^2U(\cosh\sigma-\cos\chi)(\cosh\sigma+\cos\chi-2)\sum_{m=0}^{\infty}m(m+1)^2(m+2)$$
$$\times\left[(m+2)^2\frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)}-m^2\frac{Q'_m(\cosh\sigma_0)}{P'_m(\cosh\sigma_0)}\right].$$

Now

$$\frac{X}{R^3}=\frac{1}{a^4}\frac{\cosh^2\sigma+\cos^2\chi-2}{(\cosh\sigma-\cos\chi)^2}\cdot\frac{(\cosh\sigma+\cos\chi)^3}{(\cosh\sigma+\cos\chi)^3}$$
$$=\frac{1}{a^4}\frac{(\cosh\sigma+\cos\chi-2)(\cosh\sigma+\cos\chi)-2(\cosh\sigma-1)(\cos\chi-1)}{(\cosh\sigma+\cos\chi)^3}\times(\cosh\sigma-\cos\chi)$$
$$=\frac{1}{4a^4}(\cosh\sigma-\cos\chi)(\cosh\sigma+\cos\chi-2)$$
$$+\text{ terms of the 3rd and higher orders;}$$

hence the most important part of the expansion is equal to

$$\tfrac{1}{2}a^6U\frac{X}{R^3}\sum_{m=0}^{\infty}m(m+1)^2(m+2)$$
$$\times\left[(m+2)^2\frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)}-m^2\frac{Q'_m(\cosh\sigma_0)}{P'_m(\cosh\sigma_0)}\right]$$
$$=-\tfrac{1}{2}a^6U\frac{X}{R^3}\sum_{m=0}^{\infty}(2m+3)(m+1)^2(m+2)^2\frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)}.$$

This gives the moment of the doublet whose potential is a first approximation to the value of $V$ at infinity. The apparent mass of the fluid may be found by means of a theorem due to Munk,* and is

$$\rho B\left[\frac{2\pi a^6}{B}\sum_{m=0}^{\infty}(2m+3)(m+1)^2(m+2)^2\frac{Q'_{m+1}(\cosh\sigma_0)}{P'_{m+1}(\cosh\sigma_0)}-1\right],$$

* "Notes on aerodynamic forces", Technical Note No. 104, National Advisory Committee for Aeronautics", Washington, July, 1922.

where $B$ is the volume of the fluid displaced by the solid, a $\rho$ the density of the fluid. Since

$$B=\frac{4\pi}{3}\cdot 8a^6\frac{\cosh^3\sigma_0}{\sinh^6\sigma_0}\left[1+\frac{1}{\cosh^2\sigma_0}\right],$$

we find that the apparent mass is $k\rho B$, where $k=.5$ for the sphere. When

$\cosh\sigma_0=1.2$ we find $k=.5688$,
$\cosh\sigma_0=2$ „ $k=.548$,
$\cosh\sigma_0=3$ „ $k=.527$.

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# A GENERAL FORM OF THE REMAINDER IN TAYLOR'S THEOREM.

By *G. S. Mahajani*, St. John's College, Cambridge.

1. An examination of the various extant accounts of Taylor's theorem reveals that, for the most part, they obtain the particular form of the remainder with which they happen to be concerned by utilising what we may call the *simple* form of the mean value theorem, which states that if $f(x)$ is continuous in the interval $(a, b)$, end points included, and differentiable in the same interval, end points not necessarily included, then

$$f(b)-f(a)=(b-a)f'(\xi),$$

where $\xi$ is some number between $a$ and $b$ and not coinciding with either.

Now it is well known that the mean value theorem can be expressed in a form more general than the above. If $\phi(x)$ satisfies the same conditions as $f(x)$ and, in addition, is such that $\phi'(x)$ does not vanish anywhere in $(a, b)$, then

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)}=\frac{f'(\xi)}{\phi'(\xi)},$$

where $\xi$, not necessarily the same as before, lies between $a$ and $b$ and does not coincide with either of them.

We propose to show that, by utilising this more general form of the mean value theorem, we can obtain an extremely general form of the remainder in Taylor's theorem.

2. We suppose that $f(x)$ satisfies the strict conditions of order $n+1$ at $a$, being such that it and its first $n+1$ derivatives exist in some neighbourhood of $a$; and that $\phi(x)$ satisfies the conditions of order $p+1$ at $a$. Further, we suppose that $\phi^{p+1}(x)$ does not vanish.



3. Let

$$f(a+h)=f(a)+hf'(a)+\ldots+\frac{h^n}{n!}f^n(a)+R_n,$$

so that $R_n$ is the usual remainder. Evidently

$$R_n=f(a+h)-f(a)-hf'(a)-\ldots-\frac{h^n}{n!}f^n(a)\ldots\text{(1)}.$$

4. Write now

$$\psi(x)=f(a+h)-f(x)-(a+h-x)f'(x)-\ldots-\frac{(a+h-x)^n}{n!}f^n(x)\ \ldots\ldots\text{(2)},$$

$$\chi(x)=\phi(a+h)-\phi(x)-(a+h-x)\phi'(x)-\ldots-\frac{(a+h-x)^p}{p!}\phi^p(x)\ \ldots\ldots\text{(3)}.$$

Then, as is easily seen,

$$\psi'(x)=-\frac{(a+h-x)^n}{n!}f^{n+1}(x),$$

$$\chi'(x)=-\frac{(a+h-x)^p}{p!}\phi^{p+1}(x).$$

5. By the mean value theorem in its general form,

$$\frac{\psi(a+h)-\psi(a)}{\chi(a+h)-\chi(a)}=\frac{\psi'(\xi)}{\chi'(\xi)},$$

where $\xi$ lies between $a$ and $a+h$ and coincides with neither.

In the usual way we have

$$\xi=a+\theta h,$$

where

$$0<\theta<1.$$

Further, as is easily seen,

$$\psi(a+h)=\chi(a+h)=0.$$

Thus

$$\frac{\psi(a)}{\chi(a)}=\frac{\psi'(a+\theta h)}{\chi'(a+\theta h)}=\frac{p!}{n!}(h-\theta h)^{n-p}\frac{f^{n+1}(a+\theta h)}{\phi^{p+1}(a+\theta h)}.$$

6. But (1) and (2) give at once $\psi(a)=R_n$. Thus

$$R_n=\frac{p!}{n!}(h-\theta h)^{n-p}\frac{f^{n+1}(a+\theta h)}{\phi^{p+1}(a+\theta h)}\chi(a)$$

$$=\frac{p!}{n!}(h-\theta h)^{n-p}\frac{f^{n+1}(a+\theta h)}{\phi^{p+1}(a+\theta h)}\left\{\phi(a+h)-\phi(a)-\ldots-\frac{h^p}{p!}\phi^p(a)\right\}.$$