8

Jnl of Rec Math 3(1970)

2778
2779
2780
2781

[JRM]

Please enter 3 2778 2779 2780 2781

o. 2 (April 1969) pp.

December 1962) pp.

fornia, December 1,

Palindromic Powers *

Gustavus J. Simmons
Sandia Laboratory and University of New Mexico
Albuquerque, New Mexico

A number is a palindrome if it is unchanged by reversal, i.e., 121, 14541, etc. Palindromic numbers which are also powers, for example

$$121 = (11)^2,$$

have several interesting number theoretic properties, which we shall explore. First, however, we shall examine some elementary properties of the palindromes themselves.

If one calculates the frequency with which palindromes occur in the sequence of integers, it is apparent that the probability of a randomly selected integer being a palindrome goes to zero as the number of digits in the integer increases. Table 1 gives the number of palindromes, N_n, among the integers less than 10^n for $n \leqq 12$ and shows the regular way in which N_n increases.

TABLE 1. Number of Palindromes $<10^n$

no!

n	N_n	n	N_n	n	N_n
1	9	5	1,098	9	109,998
2	18	6	1,998	10	199,998
3	108	7	10,998	11	1,099,998
4	198	8	19,998	12	1,999,998

The table was constructed by noting that if a k-digit number is to be a palindrome the outer pair of digits, which must be alike, can be selected in nine ways. The inner $k-2$ digits must also be a palindrome and, hence, could by any of N_{k-2} $(k-2)$-digit palindromes or any of N_{k-2} $(k-4)$-digit palindromes preceded and followed by a zero, etc. This gives the following recursion relation for the number of palindromes with exactly k digits, n_k,

$$n_k = 9\,(N_{k-2} + N_{k-4} + \ldots + N_1 + 1) \tag{1}$$

where i is either 1 or 2 and $N_1 = N_2 = 9$. The solution to this equation is

$$n_k = 9 \cdot 10^{[(k-1)/2]} \tag{2}$$

where $[X]$ represents the integer part of X.

Equation (2) makes it possible to give an asymptotic estimate of the probability, P_n, that an integer less than 10^n is a palindrome. There are two cases:

$$n = 2m$$

$$P_n = 2 \cdot 10^{-m} \tag{3}$$

*This work was supported by the U.S. Atomic Energy Commission.

$$n = 2m + 1$$

$$P_n = (1.1)\, 10^{-m} \tag{4}$$

There is no apriori reason to suspect that the properties of being a square and of being a palindrome should be in any way related, i.e., one would expect them to be independent attributes of the integers. If this were the case, the probability that a randomly selected integer less than 10^n would be both a square and a palindrome would be

$$P_n \cdot 10^{-n/2}$$

and the expected number of palindromic squares would be

$$P_n \cdot 10^{-n/2} \tag{5}$$

If $n = 2m$, equations (3) or (4) and (5) imply that the expected number of palindromic squares would be asymptotic to 2 and if $n = 2m + 1$ asymptotic to 3.5.

Table 2 is an exhaustive tabulation of the 55 palindromic squares less than 2.5×10^{13} obtained with the CDC 6600 computer which shows that there are many more numbers which are both palindromes and squares than one would expect if these were independent properties. An asterisk marks those cases where both X and X^2 are paldindromic.

TABLE 2. Palindromic Squares

X 2778	X^2 2779	X cont 2778	X^2 cont 2779
* 1	1	24,846	617,323,716
* 2	4	30,693	942,060,249
* 3	9	* 100,001	10,000,200,001
* 11	121	* 101,101	10,221,412,201
* 22	484	* 110,011	12,102,420,121
26	676	* 111,111	12,345,654,321
* 101	10,201	* 200,002	40,000,800,004
* 111	12,321	798,644	637,832,238,736
* 121	14,641	*1,000,001	1,000,002,000,001
* 202	40,804	*1,001,001	1,002,003,002,001
* 212	44,944	*1,002,001	1,004,006,004,001
264	69,696	*1,010,101	1,020,304,030,201
307	94,249	*1,011,101	1,002,325,232,201
836	698,896	*1,012,101	1,024,348,434,201
* 1,001	1,002,001	1,042,151	1,086,078,706,801
* 1,111	1,234,321	*1,100,011	1,210,024,200,121
* 2,002	4,008,004	*1,101,011	1,212,225,222,121
2,285	5,221,225	*1,102,011	1,214,428,244,121
2,636	6,948,496	1,109,111	1,230,127,210,321
*10,001	100,020,001	*1,110,111	1,232,346,432,321
*10,101	102,030,201	*1,111,111	1,234,567,654,321
*10,201	104,060,401	1,270,869	1,615,108,015,161
*11,011	121,242,121	*2,000,002	4,000,008,000,004
*11,111	123,454,321	*2,001,002	4,004,009,004,004
*11,211	125,686,521	2,012,748	4,051,154,511,504
*20,002	400,080,004	2,294,675	5,265,533,355,625
*20,102	404,090,404	3,069,307	9,420,645,460,249
22,865	522,808,225		

(4)

quare and of being a
n to be independent
a randomly selected
be

(5)

nber of palindromic

less than 2.5×10^{13}
many more numbers
se were independent
ldindromic.

X^2
17,323,716
42,060,249
00,200,001
21,41[illegible]201
02,4[illegible]21
45,654,321
00,800,004
32,238,736
02,000,001
03,002,001
06,004,001
04,030,201
25,232,201
48,434,201
78,706,801
24,200,121
25,222,121
28,244,121
27,210,321
46,432,321
67,654,321
08,015,161
08,000,004
09,004,004
54,511,504
33,355,625
45,460,249

The following argument shows that there are, in fact, infinitely many palindromic squares. Let X be an n-digit palindrome with 9 or fewer 1's and the remainder of its digits 0. Then X^2 is necessarily a palindrome since a carry cannot be generated. If there are n digits, where $n = 2m$, the first and last digits must be a 1, so that there are

$$M_{2m} = \binom{m-1}{0} + \binom{m-1}{1} + \binom{m-1}{2} + \binom{m-1}{3} \tag{6}$$

or

$$M_{2m} = \frac{m(m^2 - 3m + 8)}{6} \tag{7}$$

such palindromes. If $n = 2m + 1$, the center digit can be either a 0 or a 1, so that

$$M_{2m+1} = 2M_{2m}$$

The first six entries in the following Table 3 can be verified directly from Table 2. $\sum_{i=1}^{n} M_i$ is the number of palindromes of $2n$ or fewer digits of the type described above. Table 2 shows that the actual number of palindromic squares is considerably larger than $\sum_{i=1}^{n} M_i$

TABLE 3. Number of Simple 0-1 Palindromic Squares

n	M_n	$\sum_{i=1}^{n} M_i$	N_n
2	1	2	6
3	2	4	13
4	2	6	19
5	4	10	30
6	4	14	36
7	8	22	≥ 55
8	8	30	
9	16	46	
10	15	61	
11	30	91	
12	26	117	
13	52	169	
14	42	211	
15	84	295	
16	64	359	
17	128	487	
18	93	580	
19	186	766	
20	130	896	

The preceding development has demonstrated a curious correlation, one might even say a preference, for squares to be palindromes.

Question 1

Why should the apparently unrelated properties of being a square and being a palindrome be related at all?

It is possible to extend the argument used above to compute the expected number of palindromic squares to find the expected number of palindromic kth powers less than 10^n in the form

$$c_n 10^{-[(k-2)/k]n} \tag{8}$$

where c_n is a constant between 2 and 4 depending on whether n is even or odd. In particular, equation (8) says that there should not be any palindromic kth powers (if the properties are independent) for $k>2$. In the case of cubes and biquadrates the same anomalous behavior which was found with squares holds. Then, inexplicably, apparently for all higher powers, the numbers behave exactly as we would expect from the probability arguments; namely, there do not appear to be any instances (other than the trivial case $X = 1$) of palindromic powers for $k>4$.

Using the same general methods as above, we can compute the following set of palindromic cubes.

See next page

TABLE 4. Tabulation of Simple 0-1 Palindromic Cubes.

X	X^3
	1
11	1,331
101	1,030,301
111	1,367,631
1,001	1,003,003,001
10,001	1,000,300,030,001
10,101	1,030,607,060,301
11,011	1,334,996,994,331

Obviously, this table can be extended by the single digits 2 and 7 whose cubes are the palindromes 8 and 343. All of these palindromic cubes have palindromic roots. A natural question is: Does there exist a palindromic cube whose root is not a palindrome? The answer is yes.

$$2201 = (10^3 + 1) = (10^3 - 1) + (10^2 + 1) + (10^2 - 1) + 1$$

is such a number. In fact, as an exhaustive computer search has shown, it is the only such number whose cube is less than 2.8×10^{14}. Table 5 is a tabulation of all of the palindromic cubes less than 1.5×10^{12}.

square and being a

expected number of
kth powers less than

(8)

n is even or odd. In
nic kth powers (if the
iquadrates the same
xplicably, apparently
ld expect from the
nces (other than the

he following set of

es.

' whose cubes are the
omic roots. A natural
t a palindrome? The

) + 1
n, it is the only such
lation of all of the

2780 2781

TABLE 5. Palindromic Cubes

X	X^3
1	1
2	8
7	343
11	1,331
101	1,030,301
111	1,367,631
1,001	1,003,003,001
2,201	10,662,526,601
10,001	1,000,300,030,001
10,101	1,030,607,060,301
11,011	1,334,996,994,331

In view of the computer's failure to find a second example of a palindromic cube whose root is not palindromic, we pose the following

Question 2

Is 2201 the only non-palindromic number whose cube is a palindrome?
Table 2 can be used to find the palindromic biquadrates.

TABLE 6. Palindromic Biquadrates

X	X^4
1	1
11	14,641
101	104,060,401
1,001	1,004,006,004,001
10,001	10,004,000,600,040,001

There are infinitely many palindromic biquadrates of the generic form of those shown in Table 6. An exhaustive computer search up to the limit of the machine's double precision arithmetic, 2.8×10^{14}, has failed to reveal a single exception to this apparent rule.

Question 3

Are there any integers not of the form 10. . .01 whose fourth power is a palindrome? If the answer to Question 3 is yes, the diminishing number of non-palindromic roots as we go from palindromic squares to cubes suggests the following question.

Question 4

Does there exist any non-palindromic number whose fourth power is a palindrome?

An exhaustive computer search of the integers less than 2.5×10^{13} has failed to discover a single instance of a number greater than 1 whose fifth, sixth, seventh, eighth, ninth, or tenth power is a palindrome.

Question 5

For any arbitrary $k>4$, does there exist an $X>1$ such that X^k is a palindrome?

The following argument does not answer the previous question: however, it rules out numbers of the type $\overbrace{10\ldots01}^{i}$ for all $k\geq 5$, whereas we have already seen that these numbers are solutions for $k = 2, 3,$ and 4.

There are two cases to be considered. For a given k, let n_k be the maximum number of digits which a binomial coefficient $\binom{k}{j}$ can have. If $i<n_k$, then in forming the kth power of $\overbrace{10\ldots01}^{i}$ at least one pair of coefficients will overlap generating a carry. For example:

$$(11)^5 = (11)(14641) = 161051.$$

Since carry's propagate to the left, this says that the digits to the right of the position in which the carry first appears will differ from the corresponding positions on the left, i.e., 16. . . and . . .51 above. If, on the other hand, $i\geq n_k$, then

$$(\overbrace{10\ldots01}^{i})^k = \sum_{j=0}^{k} \binom{k}{j} 10^{j(i+1)} \tag{9}$$

For $k = 1, 2, 3,$ and 4, the binomial coefficients are single digits and the number given by equation (9) is palindromic as we have already seen. For $k>4, n_k>1$ and the resulting numbers can never be palindromic because of asymmetry; for example

$$(1\overbrace{00}^{i}1)^5 = 1{,}005{,}010{,}010{,}005{,}001.$$

Therefore, numbers of the form $\overbrace{10\ldots01}^{i}$ are never palindromic powers for $k > 4$.

Based on the foregoing observations, we conjecture that there exists a lower bound K, such that for all $k\geq K$ there can never be an integer $X>1$ such that X^k is palindromic. We also conjecture that $K = 5$. A counter-example can, of course, disprove the latter and a procedure for constructing a palindromic power of arbitrary order would disprove the first conjecture. In view of the magnitude of the numbers which have been tested by the computer, direct computation would not appear to be a profitable attack.