Knots, Links and Tangles

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We start with some terminology from differential topology [1]. Let C be a circle and $n \geq 2$ be an integer. An **immersion** $f : C \to \mathbb{R}^n$ is a smooth function whose derivative never vanishes. An **embedding** $g : C \to \mathbb{R}^n$ is an immersion that is one-to-one. It follows that $g(C)$ is a manifold but $f(C)$ need not be (f is only locally one-to-one, so consider the map that twists C into a figure eight).

A **knot** is a smoothly embedded circle in $\mathbb{R}^3$; hence a knot is a closed spatial curve with no self-intersections. Two knots J and K are **equivalent** if there is a homeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ taking J onto K. This implies that the complements $\mathbb{R}^3 - J$ and $\mathbb{R}^3 - K$ are homeomorphic as well.

A **link** is a compact smooth 1-dimensional submanifold of $\mathbb{R}^3$. The connected components of a link are disjoint knots, often with intricate intertwinings. Two links L and M are **equivalent** if, likewise, there is a homeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ taking L onto M.

We can project a knot or a link into the plane in such a way that its only self-intersections are transversal double points. Ambiguity is removed by specifying at each double point which arc passes over and which arc passes under. Over all possible such projections of K or L, determine one with the minimum number of double points; this defines the **crossing number** of K or L.

There is precisely 1 knot with 0 crossings (the circle), 1 knot with 3 crossings (the trefoil), and 1 knot with 4 crossings. Note that, although the left-hand trefoil T_L is not ambiently isotopic (i.e., deformable) to the right-hand trefoil T_R, a simple reflection about a plane gives T_R as a homeomorphic image of T_L. Under our definition of equivalence, chiral pairs as such are counted only once.

There are precisely 2 knots with 5 crossings, and 5 knots with 6 crossings. In particular, there is no homeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ taking the granny knot $T_L \# T_L$ onto the square knot $T_L \# T_R$, where $\#$ denotes the connected sum of manifolds [2, 3]. (See Figure 1.) Also, there are precisely 8 knots with 7 crossings, and 25 knots with 8 crossings.

A link L is **splittable** if we can embed a plane in $\mathbb{R}^3$, disjoint from L, that separates one or more components of L from other components of L. There are precisely 1, 0, 1, 1, 3, 4, 15 nonsplittable links with 0, 1, 2, 3, 4, 5, 6 crossings, respectively.

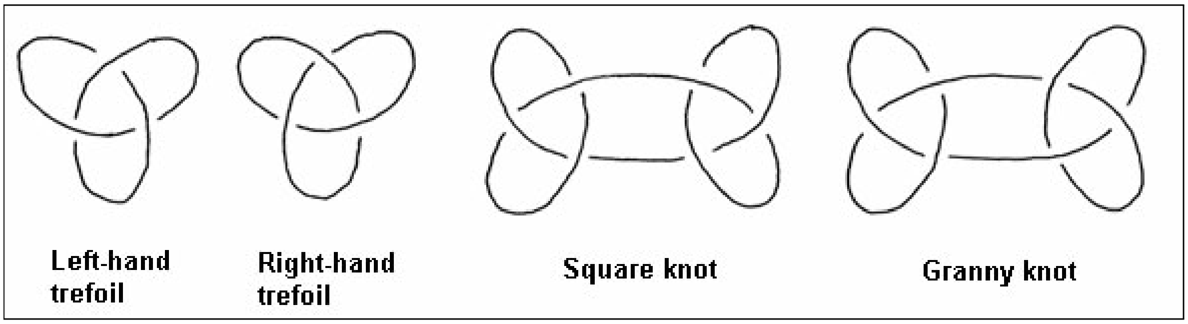

Figure 1: Four famous knots (T_L and T_R are prime and equivalent; $T_L \# T_R$ and $T_L \# T_L$ are composite and distinct).

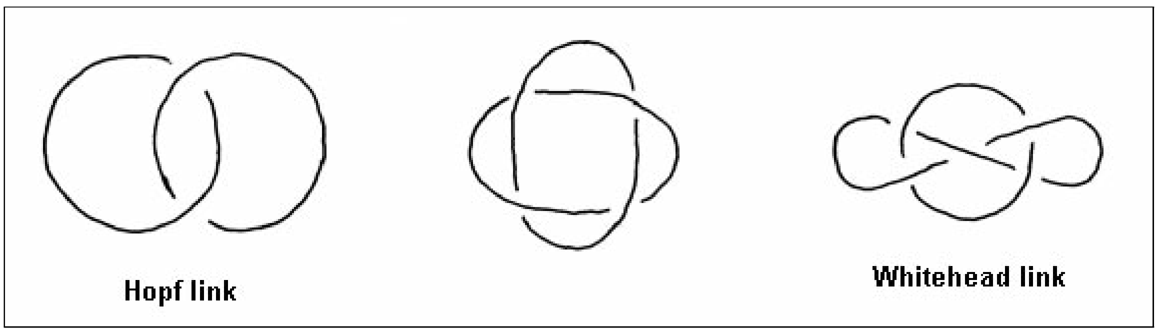

Figure 2: All two-component prime links with crossing number ≤ 5.

A knot K or nonsplittable link L is **prime** if it is not a circle and if, for any plane P that intersects K or L transversely in exactly two points, P slices off merely an unknotted arc away from the rest. (See Figure 2.) Otherwise it is **composite**. For example, $T_L \# T_L$ and $T_L \# T_R$ are composite knots, each being nontrivial connected sums of knots. Every knot decomposes as a unique connected sum of prime knots [4].

People have known for a long time that there exist non-equivalent links with homeomorphic complements [5, 6]. This cannot happen for knots, as recently proved by Gordon & Luecke [7, 8].

Let B denote the compact unit ball in $\mathbb{R}^3$ and ∂B denote its boundary. A **tangle** U is a smooth 1-dimensional submanifold of B meeting ∂B transversely at the four points

$$NE = \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right), \quad NW = \left(\tfrac{-1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right), \quad SW = \left(\tfrac{-1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0\right), \quad SE = \left(\tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0\right)$$

and meeting ∂B nowhere else. Thus U is a union of two smoothly embedded line

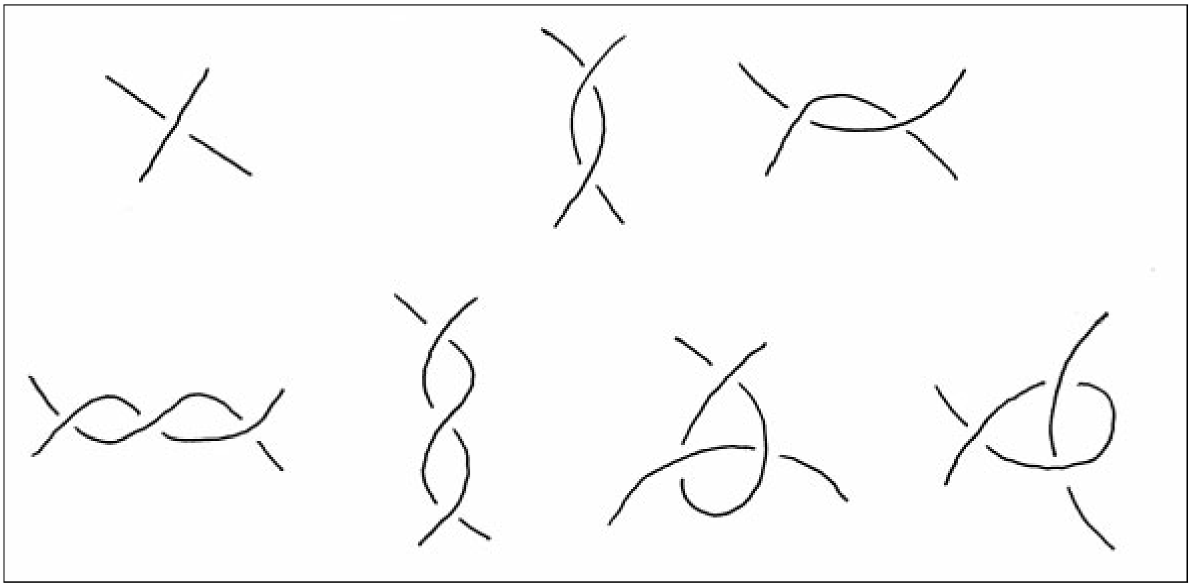

Figure 3: All prime alternating tangles with crossing number ≤ 3.

segments in B with distinct endpoints on ∂B, together with an arbitrary number of smoothly embedded circles in the interior of B, all disjoint but often intertwined. Two tangles U and V are **(strongly) equivalent** if there is a homeomorphism $B \to B$ that takes U onto V, is orientation-preserving on B, and leaves ∂B fixed pointwise. The crossing number of a tangle is defined via projections as before. Tangles form the building blocks of knots and links [9, 10, 11]; the first precise asymptotic enumeration results discovered in this subject concerned tangles (as we shall soon see).

A tangle is **trivial** if it is only the union of the two line segments NW-NE and SW-SE, or the union of the two line segments SW-NW and SE-NE. A tangle U is **prime** if it is not trivial; if, for any sphere S in B that is disjoint from U, no portion of U is enclosed by S; and if, for any sphere S in B that intersects U transversely in exactly two points, S encloses merely an unknotted arc of U. (See Figures 3 and 4.)

Finally, a knot, link or tangle is **alternating** if, for some projection, as we proceed along any connected component in the projection plane from beginning to end, the sequence of underpasses and overpasses is strictly alternating. The first non-alternating knots appear with crossing number ≥ 8. General references on knot theory include [12, 13, 14, 15, 16, 17].

0.1. Prime Alternating Tangles. Let a_n denote the number of prime alternating tangles with n crossings (up to strong equivalence) and let $A(x) = \sum_{n=1}^{\infty} a_n x^n$ be

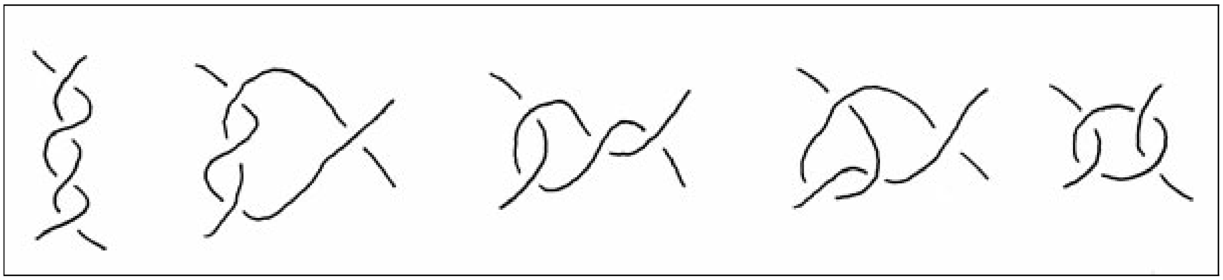

Figure 4: Five of the 4-crossing prime alternating tangles; the other five are obtained by rotating through 90° (and switching crossings to maintain the convention that the NW strand is an underpass).

the corresponding generating function. Then [18]

$$A(x) = x + 2x^2 + 4x^3 + 10x^4 + 29x^5 + 98x^6 + 372x^7 + 1538x^8 + 6755x^9 + 30996x^{10} + \cdots$$

satisfies the equation

$$A(x)(1+x) - A(x)^2 - (A(x)+1)r(A(x)) - x - 2\frac{x^2}{1-x} = 0$$

where the algebraic function $r(x)$ is defined by

$$r(x) = \frac{(1-4x)^{\frac{3}{2}} + (2x^2 - 10x - 1)}{2(x+2)^3} - \frac{2}{1+x} - x + 2$$

Further, $A(x)$ satisfies the irreducible quintic equation

$$\begin{aligned} 0 \quad = \quad & (x^4 - 2x^3 + x^2)A(x)^5 + (8x^4 - 14x^3 + 8x^2 - 2x)A(x)^4 + \\ & (25x^4 - 16x^3 - 14x^2 + 8x + 1)A(x)^3 + (38x^4 + 15x^3 - 30x^2 - x + 2)A(x)^2 + \\ & (28x^4 + 36x^3 - 5x^2 - 12x + 1)A(x) + (8x^4 + 17x^3 + 8x^2 - x) \end{aligned}$$

Sundberg & Thistlethwaite [19] proved the above remarkable formulas, as well as the following asymptotics:

$$a_n \sim \frac{3\alpha}{4\sqrt{\pi}} n^{-\frac{5}{2}} \lambda^{n-\frac{3}{2}} \sim \frac{3}{4}\sqrt{\frac{\beta}{\pi}} n^{-\frac{5}{2}} \lambda^n$$

where

$$\alpha = \frac{5^{\frac{7}{2}}}{3^5\sqrt{2}} \sqrt{\frac{(21001 + 371\sqrt{21001})^3}{(17 + 3\sqrt{21001})^5}} = 3.8333138762...$$

$$\beta = \alpha^2\lambda^{-3} = 0.0632356411...$$

and

$$\lambda = \frac{101 + \sqrt{21001}}{40} = 6.1479304437...$$

A completely different approach to the solution of this problem appears in [20].

Let $\hat{a}_n$ denote the number of n-crossing prime alternating tangles with exactly two components. That is, no circles are allowed. A two-component tangle is also known as a **knot with four external legs**. The sequence [18, 21, 22]

$$\{\hat{a}_n\}_{n=1}^{\infty} = \{1,\ 2,\ 4,\ 8,\ 24,\ 72,\ 264,\ 1074,\ 4490,\ 20296,\ 92768,\ldots\}$$

is believed to possess a leading term of the form $\hat{\lambda}^n$ with $\hat{\lambda} < \lambda$, but more intensive analysis is needed to compute $\hat{\lambda}$.

0.2. Prime Alternating Links. Let b_n denote the number of prime alternating links with n crossings (up to equivalence), then the sequence [23, 24]

$$\{b_n\}_{n=1}^{\infty} = \{0,\ 1,\ 1,\ 2,\ 3,\ 8,\ 14,\ 39,\ 96,\ 297,\ 915,\ 3308,\ 12417,\ldots\}$$

satisfies the following asymptotics [25]:

$$b_n \sim \frac{3}{16\gamma}\sqrt{\frac{\beta}{\pi}}n^{-\frac{7}{2}}\lambda^n$$

where

$$\gamma = \frac{1}{2}\left(\frac{371}{\sqrt{21001}} - 1\right) = 0.7800411357...$$

and λ, β are as before. This is a somewhat more precise result than that proved in [19].

Let c_n denote the number of prime links with n crossings (including both alternating and non-alternating links), then we have [23, 26, 27]

$$\{c_n\}_{n=1}^{\infty} = \{0,\ 1,\ 1,\ 2,\ 3,\ 9,\ 16,\ 50,\ 132,\ 452,\ 1559,\ldots\}$$

The value c_{12} is not known. Stoimenow [28], building on Ernst & Sumners [29] and Welsh [30], proved that

$$4 \le \liminf_{n\to\infty} c_n^{1/n} \le \limsup_{n\to\infty} c_n^{1/n} \le \frac{\sqrt{13681} + 91}{20} = 10.3982903484...$$

but further improvements in the upper bound are likely. The two-component analogs [23]

$$\{\hat{b}_n\}_{n=1}^{\infty} = \{0,\ 1,\ 0,\ 1,\ 1,\ 3,\ 6,\ 14,\ 42,\ 121,\ 384,\ 1408,\ 5100,\ 21854,\ldots\}$$

$$\{\hat{c}_n\}_{n=1}^{\infty} = \{0,\ 1,\ 0,\ 1,\ 1,\ 3,\ 8,\ 16,\ 61,\ 185,\ 638\ldots\}$$

also await study.

0.3. Prime Alternating Knots. Let d_n denote the number of prime alternating knots with n crossings (up to equivalence), then the sequence [31]

$$\{d_n\}_{n=1}^{\infty} = \{0,\ 0,\ 1,\ 1,\ 2,\ 3,\ 7,\ 18,\ 41,\ 123,\ 367,\ 1288,\ 4878,\ 19536,\ldots\}$$

is more difficult and only *conjectured* to satisfy the following asymptotics [32]:

$$d_n \sim \eta \cdot n^{\xi} \cdot \kappa^n$$

where

$$\xi = -\frac{\sqrt{13}+1}{6} - 3 = -3.7675918792...$$

Thistlethwaite [33] proved that

$$\limsup_{n\to\infty} d_n^{1/n} < \lambda$$

and further claimed that $\lim_{n\to\infty} d_n^{1/n}$ exists. If the conjectured asymptotic form for d_n is true, it would follow that $\kappa < \lambda$. Again, more intensive analysis is needed to compute κ. Might it be true that $\kappa = \hat{\lambda}$ [22]?

Let e_n denote the number of prime knots with n crossings (including both alternating and non-alternating knots), then we have [31]

$$\{e_n\}_{n=1}^{\infty} = \{0,\ 0,\ 1,\ 1,\ 2,\ 3,\ 7,\ 21,\ 49,\ 165,\ 552,\ 2176,\ 9988,\ 46972,\ldots\}$$

The value e_{17} is not known. Welsh [30] proved that

$$2.68 \le \liminf_{n\to\infty} e_n^{1/n}$$

and clearly Stoimenow's upper bound 10.40 applies to the limit superior. Sharper bounds for both $\{c_n\}$ and $\{e_n\}$ would be good to see.

0.4. Planar Curves. Here are enumeration problems that seem to be even more complicated than those in knot theory [34, 35, 36, 37, 38]. A **closed planar curve** is a smoothly immersed circle in $\mathbb{R}^2$ whose only self-intersections are transversal double points. Define an equivalence relation between closed planar curves in the same manner as between knots, with the additional condition that the homeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$ is orientation-preserving. (See Figure 5.)

An **open planar curve** is a smoothly immersed line in $\mathbb{R}^2$, given by $h : \mathbb{R} \to \mathbb{R}^2$, whose only self-intersections are transversal double points and which satisfies $h(x) = (x, 0)$ for all sufficiently large $|x|$. Such a curve is also known as a **knot with two external legs**. Define an equivalence relation between open planar curves in the same manner as between closed planar curves. Note that, unlike closed curves, open

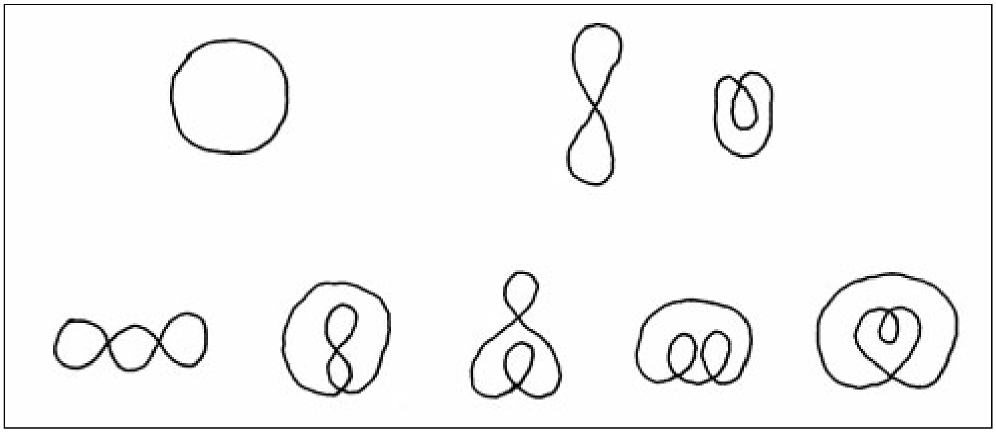

Figure 5: All closed planar curves with crossing number ≤ 2.

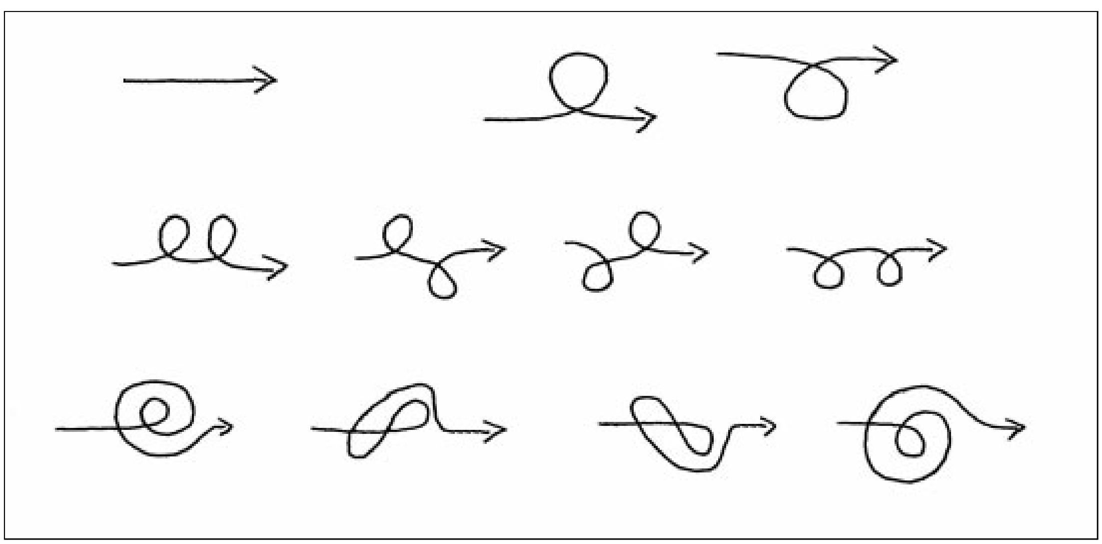

Figure 6: All open planar curves with crossing number ≤ 2.

curves are oriented from the initial point $(-\infty, 0)$ to the final point $(\infty, 0)$. (See Figure 6.)

Let p_n and q_n denote the number of n-crossing closed curves and open curves, respectively. The sequences [39, 40]

$$\{p_n\}_{n=0}^{\infty} = \{1,\ 2,\ 5,\ 20,\ 82,\ 435,\ 2645,\ 18489,\ 141326,\ 1153052,\ 9819315, \ldots\}$$

$$\{q_n\}_{n=0}^{\infty} = \{1,\ 2,\ 8,\ 42,\ 260,\ 1796,\ 13396,\ 105706,\ 870772,\ 7420836,\ 65004584, \ldots\}$$

are *conjectured* to satisfy the following asymptotics [32]:

$$p_n \sim \frac{1}{4} q_n \sim \omega \cdot n^{\theta} \cdot \mu^n$$

where $\theta = \xi + 1 = -2.7675918792....$ Numerically, we have $\mu = 11.4...$ [22]. There is a great amount of work to be done in this area.

0.5. Addendum. At the risk of potential confusion, let us generalize the word **tangle** to include smooth 1-dimensional submanifolds U of B meeting ∂B transversely at *any* four distinct points and meeting ∂B nowhere else. Two such tangles U and V are **weakly equivalent** if there is a homeomorphism $B \to B$ that takes U onto V, but need not be orientation-preserving on B nor need it leave endpoints fixed. Kanenobu, Saito & Satoh [41] gave the number of non-weakly equivalent prime tangles with 4, 5, 6, 7 crossings to be 0, 1, 4, 18 respectively. The four legs (small circles on the spherical surface depicted in Figure 7) of *classical* tangles are fixed on the equator, whereas the legs of weakly equivalent tangles can slide anywhere on the unit sphere, hence there are many more possible untangling strategies.

A different generalization of tangle was provided by Bogdanov, Meshkov, Omelchenko & Petrov [42], in which **2-tangles** correspond to classical tangles and k**-tangles**, $k > 2$, similarly possess $2k$ legs equally spaced on the equator. The number of non-equivalent prime alternating 2-tangles with 2, 3, 4, 5 crossings is given in [42] to be 1, 2, 5, 13 respectively, which at first glance appears to contradict the numbers 2, 4, 10, 29 from [19], until it is understood that 1, 2, 5, 13 do not distinguish projections that differ by only a sequence of flypes. The asymptotics of counts of prime alternating k-tangles, as the number n of crossings $\to \infty$, would be a challenging exercise.

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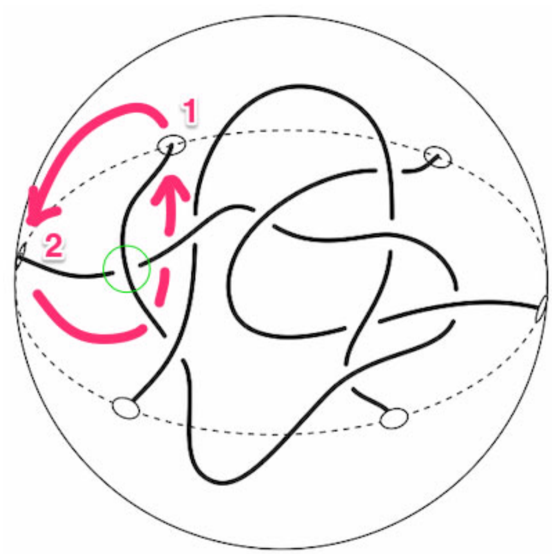

Figure 7: Positions of legs 1 and 2 can be reversed on the sphere (following the arrows), thus removing the crossing indicated by the green circle. Image courtesy of Vadim Meshkov.

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