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SYSTEMATIC EXAMINATION OF LITTLEWOOD'S BOUNDS ON $L(1, \chi)$

DANIEL SHANKS

1. **Introduction.** This investigation was largely conducted in close collaboration with D. H. and Emma Lehmer. My joint paper with them [1] overlaps some with the present paper but each paper also treats topics not in the other, and to minimize duplication the papers refer to each other for those aspects of the problem.

We confine ourselves to the real characters $\chi_d = (d/n)$ and examine the functions

$$L(s, \chi_d) = \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) \frac{1}{n^s} = \prod_{q=2}^{\infty} \frac{q^s}{q^s - (d/q)} \tag{1}$$

for $s = 1$. If $L(s, \chi_d)$ satisfies the Riemann hypothesis, and $d \neq m^2$, then Littlewood [2] deduces the bounds

$$[\{1 + o(1)\}\, (12e^{\gamma}/\pi^2) \ln \ln |d|]^{-1} < L(1, \chi_d) < \{1 + o(1)\}\, 2e^{\gamma} \ln \ln |d|. \tag{2}$$

He gives nothing about the $o(1)$ here, neither its sign nor the manner in which it approaches zero as a function of d.

We wish to study the possibility of approaching these bounds or, perhaps, surpassing them, and to obtain a measure for this we temporarily ignore the $o(1)$ and define the *upper* and *lower Littlewood indices* by

$$L(1, \chi_d)/2e^{\gamma} \ln \ln |d| = \mathrm{ULI}, \qquad L(1, \chi_d)\, (12/\pi^2)\, e^{\gamma} \ln \ln |d| = \mathrm{LLI}. \tag{3}$$

We will examine, systematically, the possibility of finding d with

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$$\text{ULI} \geqq 1 \quad \text{or} \quad \text{LLI} \leqq 1. \tag{4}$$

Littlewood himself [**2**], followed by Chowla [**3**], got halfway there by constructing arbitrarily large $|d|$ having

$$\text{ULI} \geqq \tfrac{1}{2}(1-\varepsilon) \quad \text{or} \quad \text{LLI} \leqq 2(1+\varepsilon) \tag{5}$$

for any positive ε. Relative to these constructions (called LC in the following) the question now is whether we can attain the extra factor of 2. If LC obtains a certain large (or small) $L(1, \chi_D)$ for a discriminant D, then we would have to obtain a comparable $L(1, \chi_d)$ with

$$\ln \ln |d| = \tfrac{1}{2} \ln \ln |D| \quad \text{or} \quad |d| = \exp((\ln |D|)^{1/2}). \tag{6}$$

Thus, if their $D = 10^{450}$, our d must be the much smaller $d = 10^{14}$.

The first step of LC in obtaining a large (or small) $L(1, \chi_D)$ is to select D such that

$$(D/q) = +1 \qquad (\text{or } (D/q) = -1) \tag{7}$$

for all primes $q \leqq$ some p. That maximizes (or minimizes) the first $\pi(p)$ factors in the Euler product in (1) for $s = 1$. There are such D by the Chinese Remainder Theorem satisfying

$$D < 4 \prod_{q=2}^{p} q = U_p. \tag{8}$$

The bound on the right, U_p, and some further construction then yields (5). But U_p is surely grossly too large since there are, in fact,

$$S_p = \prod_{q=3}^{p} \left(\frac{q-1}{2}\right) \tag{9}$$

distinct solutions D of (7), all being less than U_p.

If one could identify the smallest of these D by some algebraic or analytic technique, one could seek to improve (5) with these smallest D. Since no such technique is known, we will compute the smallest d numerically and begin our study with four introductory examples of (3) so computed.

2. **Four examples and their computation.** In (10a-d) below, we list four d, each being the *smallest* discriminant having a prescribed quadratic character. The characters are designated as follows: aR_p (aN_p) means a positive $d \neq m^2$ of

the form $8k+a$ which is a quadratic residue (nonresidue) of all odd primes $q \leqq p$. Similarly, $-aR_p(-aN_p)$ is such a negative $d=-(8k+a)$. For each d in (10a-d) we give the class number $h(d)$ of $Q(d^{1/2})$ and, for $d>0$, the regulator $\ln \varepsilon$. Then $L(1, \chi)$ equals

$$2h(d) \ln \varepsilon / d^{1/2} \quad \text{or} \quad \pi h(d)/(-d)^{1/2}$$

for $d>0$ or $d<0$, and the indices are computed by (3).

$$\text{(10a)} \quad \begin{array}{ccc} d=1R_{139}=2871842842801 \ \text{(prime)}, & & h(d)=1, \\ \ln \varepsilon=7023729.36, & L(1, \chi)=8.28929, & \text{ULI}=0.6933. \end{array}$$

$$\text{(10b)} \quad \begin{array}{ccc} d=5N_{139}=49107823133 \ \text{(prime)}, & & h(d)=1, \\ \ln \varepsilon=18804.68, & L(1, \chi)=0.16972, & \text{LLI}=1.1773. \end{array}$$

$$\text{(10c)} \quad \begin{array}{cc} d=-7R_{157}=-47375970146951 \ \text{(composite)}, & h(d)=19213042, \\ L(1, \chi)=8.76934, & \text{ULI}=0.7136. \end{array}$$

$$\text{(10d)} \quad \begin{array}{cc} d=-3N_{181}=-30059924764123 \ \text{(prime)}, & h(d)=296475, \\ L(1, \chi)=0.16988, & \text{LLI}=1.2637. \end{array}$$

These four (first solution) d are clearly much stronger than the LC constructions D that yield (5). The example (10b) is especially strong; it nearly attains (4). The first $-3N_{181}$ is not quite that strong, but if it had a class number, say 230000 instead of its listed $h(d)$, it could well be a violation of the RH, subject to investigation of its factor $\{1+o(1)\}$.

A brief word about computation. These four d, and most of those that follow, were obtained with Lehmer's delay line sieve DLS-157 [**4**]. This is a specialized computer that determines solutions N of the system of congruences:

$$N \equiv a_q \pmod q \qquad (q=2, 3, 5, \ldots, 157).$$

If it had not been available, the computation of, say, the first $-3N_{181}$ above on a commercial computer would be incredibly time-consuming and expensive; in a word, impractical. Again, the classical algorithms for computing $h(d)$ and ε are far too slow for the huge regulator in (10a) and $h(d)$ in (10c), and it was necessary to devise new algorithms for computing $h(d)$ [**5**] and $\ln \varepsilon$ [**6**] that are far more efficient. Suffice it to say that without Lehmer's DLS-157 and without these two new algorithms much of the data that follows would have been almost impossible to obtain.

3. **Even discriminants.** Presently, we will study the variations in the ULI and LLI for all such first solutions of $1R_p$, $5N_p$, etc., as p is systematically increased: $p=3, 5, 7, 11, \ldots$. But these four characters all have odd d and it is desirable to gather more data by examining even d also.

For any $N \neq -k^2$ we write

Dirichlet series

(11) $$L_N(1)=\sum_{m=1}^{\infty}\left(\frac{-4N}{m}\right)\frac{1}{m}$$

for the even $d=-4N$. All even terms $m=2r$ in (11) vanish. Correspondingly, the leading (and strongest) factor in the Euler product in (1) is now lost since $(d \mid q)=0$ for $q=2$. Using Littlewood's analysis for $d=-4N$, everything goes as before except at the very end when these leading factors of 2 or $\frac{2}{3}$ drop off. One therefore has, instead of Littlewood's (2), the stronger result:

(12) $$[\{1+o(1)\}\,(8e^{\gamma}/\pi^2)\ln\ln|4N|]^{-1}<L_N(1)<\{1+o(1)\}\,e^{\gamma}\ln\ln|4N|.$$

For even d we therefore modify (3) and define the indices by

(13) $$L_N(1)/e^{\gamma}\ln\ln|4N|=\text{ULI},\qquad L_N(1)\,(8/\pi^2)\,e^{\gamma}\ln\ln|4N|=\text{LLI}.$$

The bounds (12) are valid for every $N \neq -k^2$, not merely for fundamental discriminants. Consider

$$-3R_{167}=-29772062022491=-N.$$

One has

$$(-N \mid q)=+1 \quad \text{for } q=3 \text{ to } 167 \qquad \text{and} \qquad (-N \mid q)=-1 \quad \text{for } q=2.$$

With a discriminant $-4N$, for this N, we can "neutralize" the "wrong" character with respect to $q=2$, and (12) then holds for its $L_N(1)$.

In (14a–d) we list four examples analogous to (10a–d). Each has a wrong character for $q=2$ that is neutralized with a factor of 4. Their indices are now computed by (13) and are seen to be comparable to those in (10a–d). In effect, we simply ignore $q=2$ by this device and study only the sequence of $(d \mid q)$ for $q=3, 5, \ldots$.

(14a) $$d=4(-3R_{167})=-4\cdot 29772062022491,$$
$$L_N(1)=4.54327,\qquad \text{ULI}=0.7333.$$

(14b) $$d=-4(-7N_{167})=-4\cdot 17382121592383, \quad L_N(1)=0.27109, \qquad \text{LLI}=1.3548.$$

(14c) $$d=4(5R_{163})=4\cdot 4745628949021, \quad L_N(1)=4.30219, \qquad \text{ULI}=0.7063.$$

(14d) $$d=4(1N_{167})=4\cdot 11571384229697, \quad L_N(1)=0.26008, \qquad \text{LLI}=1.2950.$$

4. **Systematic examination of the LLI.** In Table 1 we list the indices LLI for the smallest d having the character $-3N_p$, $5N_p$, $4(-7N_p)$ and $4(1N_p)$ for $p=3, 5, 7, \ldots$. (The LLI of the examples above are found in Table 1 in the appropriate rows and columns.) The discriminants d themselves, their $h(d)$ and $L(1, \chi_d)$, are not given in Table 1 but can be found in the tables in [1] and [7]. This is what we observe in Table 1:

(a) All LLI listed are far stronger for these smallest d than for the LC construction in (5).

(b) If we set aside the smaller d, those for $p<50$, we see a certain uniformity here; the LLI are essentially equal, on the average, for all four characters, and appear to remain stable, on the average (or change only very slowly), as p increases.

(c) For these $50<p\leqq 181$, the average LLI is about $1\frac{1}{3}$ and the fluctuations take us up to 1.528 for the weak $4(1N_{83})$ and down to 1.177 for the very strong example (10b).

(d) The $d=-3N_p$ for $p=17$ thru 37 is the famous -163 and its startling LLI$=0.8675$ would imply that $\sum(-163 \mid n)\, n^{-s}$ violates the Riemann hypothesis were it not for its factor $\{1+o(1)\}$. For the present, we will assume that this factor saves the day (since 163 is quite small) but we must return to this $\{1+o(1)\}$ problem later. Similarly, the LLI shown for the even smaller $d=-28=4(-7N_5)$ and $d=68=4(1N_{11})$ are (temporarily) discounted.

(e) With this dubious $d=-163$ excepted, we see no indications here for violations of the RH. We are making a real effort here to obtain cases of LLI<1 but they do not appear (for large d); the strongest examples such as $5N_{139}$ press towards the bound, but do not cross it.

5. **Systematic examination of the ULI.** In Table 2 we list the ULI for the characters $1R_p(\neq m^2)$, $-7R_p$, $4(5R_p)$, and $4(-3R_p)$. The ULI behave quite differently from the LLI.

(a) For $p<13$, the ULI can even be weaker than (5) but they increase rapidly with p and become distinctly stronger.

(b) Quite unlike point (b) of § 4, the growth of the ULI is very obvious as are the differences among the four characters, especially the outer two.

TABLE 1. LLI for first discriminant of the character.

p	$-3N_p$	$5N_p$	$4(-7N_p)$	$4(1N_p)$
3	1.6855	0.4436	1.0317	1.0560
5	1.3744	1.6125	1.0317	1.0560
7	1.3744	1.3880	1.8407	1.0560
11	1.1937	1.3880	1.6717	1.0560
13	1.1937	1.2467	1.4888	1.7017
17	0.8675	1.1377	1.4565	1.5780
19	0.8675	1.2470	1.1671	1.5108
23	0.8675	1.2470	1.6268	1.4011
29	0.8675	1.2908	1.4350	1.4011
31	0.8675	1.3876	1.3874	1.1893
37	0.8675	1.3876	1.4031	1.1893
41	1.3002	1.3876	1.4031	1.4815
43	1.3002	1.3249	1.3838	1.5256
47	1.2315	1.3593	1.3838	1.3750
53	1.2617	1.3593	1.2898	1.4138
59	1.2617	1.3593	1.2898	1.4194
61	1.3058	1.1855	1.2898	1.3409
67	1.3944	1.3144	1.2607	1.3409
71	1.3269	1.4284	1.2607	1.3042
73	1.3423	1.4220	1.2607	1.3042
79	1.3423	1.4220	1.3514	1.2411
83	1.2869	1.3633	1.2979	1.5283
89	1.2832	1.3633	1.2979	1.4297
97	1.2832	1.3633	1.2979	1.4297
101	1.2832	1.2210	1.4066	1.3877
103	1.2832	1.2210	1.3432	1.3877
107	1.2974	1.2809	1.3454	1.3877
109	1.3182	1.2809	1.3303	1.3877
113	1.3182	1.2809	1.3303	1.3877
127	1.2422	1.2243	1.3303	1.4173
131	1.3604	1.2176	1.3248	1.3541
137	1.3604	1.1773	1.3248	1.3541
139	1.3114	1.1773	1.3130	1.3279
149	1.3422	1.2393	1.3555	1.3010
151	1.3422	1.2393	1.3555	1.3010
157	1.3422	1.2393	1.3555	1.3343
163	1.3422	1.2393	1.3555	1.3343
167	1.3223	1.3433	1.3548	1.2950
173	1.2789	1.2846		
179	1.2637			
181	1.2637			
Average LLI for $p > 50$				
	1.3096	1.2899	1.3218	1.3629

TABLE 2. ULI for first discriminant of the character.

p	$1R_p$	$-7R_p$	$4(5R_p)$	$4(-3R_p)$
3	0.3359	0.4828	0.4062	0.5994
5	0.3989	0.5053	0.4637	0.5994
7	0.4469	0.5438	0.4975	0.6085
11	0.4770	0.5535	0.5324	0.6301
13	0.5064	0.5710	0.5625	0.6301
17	0.5189	0.5896	0.5726	0.6545
19	0.5485	0.5899	0.5868	0.6325
23	0.5551	0.6100	0.5986	0.6642
29	0.5651	0.6155	0.6090	0.6600
31	0.5787	0.6222	0.6250	0.6742
37	0.5892	0.6401	0.6206	0.6742
41	0.5891	0.6401	0.6267	0.6718
43	0.6039	0.6652	0.6267	0.6655
47	0.6138	0.6652	0.6386	0.6937
53	0.6182	0.6652	0.6546	0.6801
59	0.6249	0.6628	0.6562	0.6801
61	0.6249	0.6628	0.6569	0.6924
67	0.6308	0.6691	0.6616	0.7134
71	0.6386	0.6740	0.6633	0.7134
73	0.6386	0.6781	0.6541	0.7134
79	0.6493	0.6781	0.6668	0.7134
83	0.6629	0.6716	0.6576	0.7134
89	0.6629	0.6906	0.6576	0.6930
97	0.6629	0.6906	0.6730	0.7033
101	0.6696	0.6906	0.6792	0.7069
103	0.6695	0.6906	0.6792	0.7069
107	0.6822	0.6906	0.6885	0.7001
109	0.6822	0.6906	0.6767	0.7001
113	0.6709	0.7036	0.6910	0.7178
127	0.6709	0.7036	0.6921	0.7178
131	0.6933	0.6984	0.7039	0.7178
137	0.6933	0.7079	0.6908	0.7178
139	0.6933	0.7079	0.6908	0.7005
149	0.6988	0.7075	0.7063	0.7285
151	0.6988	0.7067	0.7063	0.7077
157		0.7136	0.7063	0.7213
163		0.7064	0.7063	0.7333
167				0.7333
173				0.7241

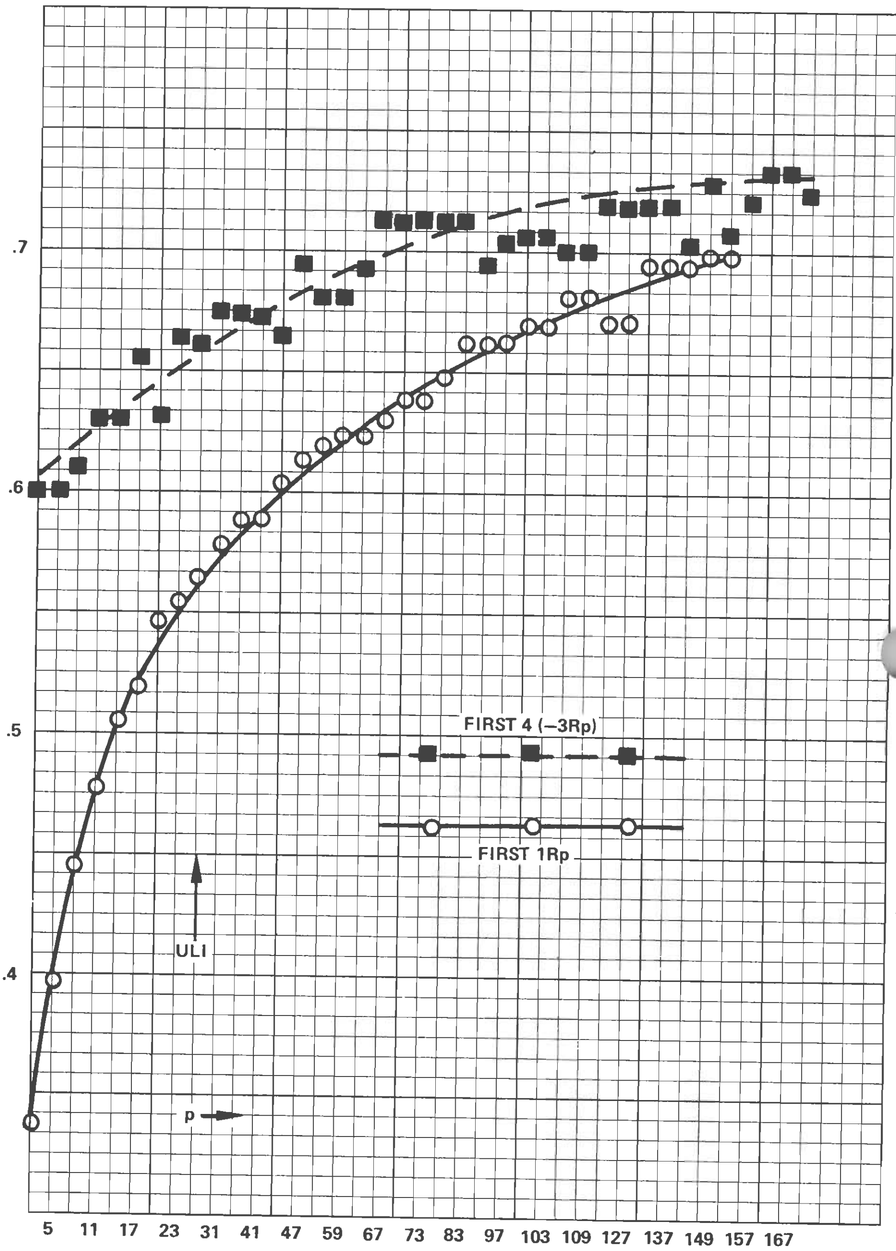

FIGURE 1

In Figure 1 we show this difference graphically. The ULI for $1R_p$ (the so-called "pseudosquares") start very low, increase rapidly and smoothly with p, and only become ragged as p exceeds 100 and ULI approaches 0.7. Those for $4(-3R_p)$

start much higher, increase slowly and exhibit much greater fluctuations. The two intermediate characters, not shown in Figure 1, behave intermediately; they start at an intermediate level, increase at an intermediate rate, and have an intermediate amount of raggedness.

A qualitative explanation of this behavior is based upon the relation of these characters to the perfect squares – the principal characters. All squares not divisible by any prime $\leqq p$ are $1R_p$. For $1R_p$, the S_p solutions (9) will therefore include not only the pseudosquares, $1R_p(\neq m^2)$, but also many perfect squares. Thus, the first pseudosquare will appear very late, especially for smaller p. Thus $1R_3=73>U_3=24$, $1R_5=241>U_5=120$, $1R_7=1009>U_7=840$; and while $1R_{11}=2641<U_{11}$, it is larger than the first 11 solutions: $1^2, 13^2, 17^2, \ldots, 47^2$. For $1R_p$, $\ln \ln d$ is therefore correspondingly large and ULI is correspondingly small. As p increases, this competition with the perfect squares slowly decreases.

The sets of S_p solutions for $-7R_p$ and for $5R_p$ are obtained from that for $1R_p$ by, respectively, the sets $\{1R_p-U_p\}$ and $\{1R_p\pm\frac{1}{2}U_p\}$ and so are not distributed uniformly in U_p but are both biased towards the second half of U_p as a reflection of the many small squares in $1R_p$. Their first solutions are therefore also delayed ([7, p. 435], [**1**]) but this effect diminishes with increasing p more rapidly than the corresponding effect for $1R_p$. Finally, $-3R_p$ differs from a square in *two* ways, being *both* negative and wrong for $q=2$. Its delay is therefore relatively small and is relatively quickly dissipated with increasing p. These differences are also reflected in the fact that while $1R_{151}$ and $-3R_{173}$ are nearly the same size, the second is a valid solution for four extra values of q: 157, 163, 167, 173.

For large p, and therefore large d, these strong effects of the perfect squares will dissipate as the squares become less dense. Thus, we can anticipate that the differences noted, caused by differing relations to the principal characters, will largely disappear. For p, say ≈ 300–400, one would expect a common average ULI of about $\frac{3}{4}$ and sizable fluctuations around this average. In a word, we can expect that the ULI will then be a mirror-image of the LLI and that the different behaviors noted in §4(b) and §5(b) will vanish.

6. **Conclusions from this first experiment.** Setting aside the two complications, the $\{1+o(1)\}$ factor and the strong effect of the squares just discussed, the indices for the first solution d behave fairly uniformly; they are consistently stronger than those of LC (5) but show no sign of ever violating the indicated bounds. For very large p and d – far beyond our data – it is likely that the observed average LLI$\approx\frac{4}{3}$ and anticipated ULI$\approx\frac{3}{4}$ will very slowly deteriorate and sink back towards the LC values. The LC bound on D is actually greater than the U_p of (8); it is [**2**, p. 369]

$$|D|<p^4 U_p. \tag{15}$$

On the average, our first solution should be the much smaller:

$$|d| \approx U_p/S_p \approx 2^{\pi(p)} 2e^{\gamma} \ln p. \tag{16}$$

But the ratio

$$\ln \ln |d| / \ln \ln |D| \tag{17}$$

for (15) and (16) nonetheless very slowly increases to 1. It is likely that the fluctuations in the indices around these deteriorating averages will simultaneously slowly increase and that d with strong indices will therefore continue to appear.

7. **Lochamps and hichamps.** The first solutions of (7) do not necessarily have the strongest indices. They do have minimal values of $\ln \ln |d|$ but their $L(1, \chi)$ need not be the most extreme since the character $(d \mid q)$ has only been forced thru $q=p$ and floats freely for subsequent q. Since we seek to approach or pass the bounds (2) and (12), we will therefore seek (to a limited extent) to locate the strongest possible examples.

Suppose $N>0$, $d=-4N$ in (11). If

$$L_N(1) < L_n(1) \qquad (\text{all } 0<n<N), \tag{18}$$

we say $L_N(1)$ is a *lochamp*. If

$$L_N(1) > L_n(1) \qquad (\text{all } 0<n<N), \tag{19}$$

we say $L_N(1)$ is a *hichamp*. Similarly, there will be a sequence of lochamps and hichamps for positive discriminants $d=4M$, $M>0$. We include odd discriminants $-N$ in the tables by the use of their multiples $d=-4N$, and $L_N(1)$ instead of $L(1, \chi)$, in order to obtain a uniform sequence. It is clear that no indices can be stronger than those for these champions, and if any indices approach or pass the bounds we would find them here.

Table 3 shows the sequence of negative discriminant lochamps thru $N \leqq 50000$. Each $L_N(1)$ there thru $L_{47338}(1)$ satisfies (18). But for $N>50000$ it was not possible to examine *every* N and below the heavy line in Table 3 the $L_N(1)$ shown are merely *tentative*, that is, they are smaller than any $L_n(1)$, $0<n<N$, *that has come to my attention.* For the positive discriminant lochamps in Table 4 the heavy line represents $M=2000$. The entries in these tables come from several sources including calculations of the Lehmers, of myself, and from an unpublished table of $L_N(1)$, $-2000<N<50000$, due to Mohan Lal.

Prior to the $N=163$ in Table 3 we see the well-known, very strong $N=58$, and following 163 no smaller $L_N(1)$ appears until $N=4687$. Only at $N=30493$ does an appreciably smaller $L_N(1)$ develop. The case $N=991027$, with $h(-N)=63$, was

TABLE 3. Lochamps, $-4N$ = Discriminant.

N	$L_N(1)$	LLI
7	0.59371	1.0317
37	0.51647	1.1996
58	0.41251	1.0094
163	0.36910	0.8675
4687	0.36711	1.2117
30178	0.36169	1.2844
30493	0.34182	1.2142
47338	0.33210	1.1974
222643	0.32957	1.1946
546067	0.32523	1.2119
991027	0.29822	1.1302
393292183	0.29449	1.2979
481022602	0.28577	1.2634
1970364883	0.28398	1.2560
2426489587	0.27982	1.2415
3416131987	0.27227	1.2142
8864190043	0.26983	1.2198
71837718283	0.26731	1.2422
85702502803	0.26172	1.2188
569078186623	0.25346	1.2252

TABLE 4. Lochamps, $4M$ = Discriminant.

M	$L_{-M}(1)$	LLI
2	0.62323	0.6587
17	0.50804	1.0560
167	0.45014	1.2168
227	0.40578	1.1239
362	0.38245	1.0959
398	0.33494	0.9660
679733	0.33492	1.2550
2004917	0.30698	1.1855
41941577	0.29228	1.2411
77891897	0.28949	1.2426
261153673	0.28533	1.2210
9447241877	0.27058	1.2243
19553206613	0.26644	1.2176
49107823133	0.25457	1.1773
4813372912697	0.25094	1.2392

TABLE 5. Hichamps, $-4N=$ Discriminant.

N	$L_N(1)$	ULI
2	1.1107	0.8518
5	1.4050	0.7190
11	1.4208	0.5994
14	1.6792	0.6770
26	1.8484	0.6758
41	1.9625	0.6763
89	1.9980	0.6336
101	2.1882	0.6856
194	2.2555	0.6682
314	2.3048	0.6585
341	2.3818	0.6765
689	2.3937	0.6494
1091	2.4254	0.6405
1154	2.5894	0.6817
1889	2.6022	0.6673
2141	2.6479	0.6747
3449	2.6747	0.6661
3506	2.7590	0.6865
5561	2.7805	0.6777
6254	2.7808	0.6744
8126	2.7881	0.6688
8774	2.8173	0.6736
10709	2.8840	0.6841
13166	2.9022	0.6829
15461	2.9561	0.6913
24569	3.0465	0.7005
148139	3.0486	0.6616
275651	3.1504	0.6718
951851	3.2217	0.6655
1692851	3.4046	0.6937
17948531	3.5705	0.6924
42143219	3.7377	0.7134
366393011	3.7642	0.6930
468717779	3.7906	0.6953
1418543411	3.8976	0.7033
4256961491	3.9778	0.7069
14701960979	4.0035	0.7001
36085593491	4.1504	0.7178
461587122779	4.2004	0.7059
864852408419	4.2970	0.7174
3989084684339	4.3938	0.7225
7024878542699	4.4542	0.7285
27484931611331	4.4655	0.7213
29772062022491	4.5433	0.7333

TABLE 6. Hichamps, $4M$ = Discriminant.

M	$L_{-M}(1)$	ULI
2	0.6232	0.4780
3	0.7604	0.4690
6	0.9359	0.4544
7	1.0464	0.4881
10	1.1501	0.4947
19	1.3372	0.5122
31	1.4404	0.5142
34	1.4572	0.5140
46	1.5913	0.5410
79	1.7130	0.5495
106	1.7461	0.5446
151	1.7874	0.5404
211	1.8619	0.5479
214	1.9114	0.5619
274	1.9193	0.5538
331	1.9928	0.5673
394	2.0643	0.5805
631	2.1074	0.5748
751	2.1143	0.5706
919	2.1231	0.5662
991	2.1856	0.5803
1054	2.2450	0.5940
1486	2.2636	0.5878
1654	2.2796	0.5886
3019	2.3204	0.5815
3931	2.3487	0.5814
4174	2.3684	0.5847
5119	2.4111	0.5898
7606	2.4731	0.5948
10399	2.5089	0.5958
10651	2.6246	0.6227
18379	2.7086	0.6294
32971	2.7660	0.6295
48799	2.8047	0.6299
61051	2.8364	0.6324
78094	2.8457	0.6296
78439	2.9049	0.6426
111094	2.9134	0.6376
162094	2.9139	0.6307
162451	2.9293	0.6340
187366	2.9461	0.6350
189814	2.9470	0.6350
230239	2.9651	0.6355
241894	2.9807	0.6379

257371	3.0156	0.6443
294694	3.0736	0.6543
584791	3.1077	0.6497
969406	3.1128	0.6427
1138999	3.1509	0.6480
1234531	3.1841	0.6536
3462229	3.2644	0.6546
6810301	3.3194	0.6562
10073779	3.3597	0.6589
10393111	3.4098	0.6683
39136549	3.4616	0.6616
43030381	3.4762	0.6633
100041439	3.5179	0.6615
249623581	3.6001	0.6668
1169755141	3.6343	0.6576
1272463669	3.7146	0.6713
2055693949	3.7496	0.6730
5959962661	3.8389	0.6792
7209891781	3.9018	0.6885
30116328181	3.9041	0.6767
78073081381	4.1608	0.7131
4745628949021	4.3022	0.7063
11256755665549	4.3598	0.7099

discovered by the Lehmers and is exceptionally strong. In Table 4 we find another case of LLI <1 at $d=4\cdot 398$. (Everyone knows of $Q((-163)^{1/2})$ but almost no one knew that $Q(398^{1/2})$ was nearly as strong.)

In Tables 3 and 4 the tentative lochamps having $N>50000$ and $M>2000$ both have an average value of LLI of about 1.22. In a word, we are trying harder than in Table 1 and so are getting indices closer to their presumed bound.

The corresponding hichamps in Tables 5 and 6 that are not already in Table 2 are also somewhat stronger but are clearly also markedly affected by the presence of the squares, as discussed above. Some of the tentative hichamps in Table 6 were extracted from Beach's and Williams' table **[8]** of $(M)^{1/2}$ having exceptionally long continued fractions.

The results of this second experiment confirm those of the first; by trying harder we press a little closer to the bounds but do not pass them except for $d=-163$ and $d=4\cdot 398$. We now return to the postponed problem of $\{1+o(1)\}$ and give it a partial treatment.

8. **Partial analysis of $\{1+o(1)\}$ and conclusions.** Clearly, the next order of business would be to determine if the $o(1)$ on the left sides of (2) and (12) are *positive* and sufficiently large for $d=-163$ and $d=4\cdot 398$ so that the bounds shown are valid. Otherwise, their L functions violate the Riemann hypothesis. Unfortunately,

many complicated terms enter into these $o(1)$ and no such unequivocal determination is now available. Nonetheless, it is desirable to show that the two leading and simplest approximations that were made are of the correct sign and magnitude so that they alone could account for these apparent violations.

Littlewood's (2), prior to the two approximations alluded to, could be written as

$$[\{1+o(1)\}\, B(x)]^{-1} < L(1, \chi_d) < \{1+o(1)\}\, A(x), \tag{20}$$

where

$$B(x) = \exp \sum_{p^m \leqq x} (-1)^{m+1}/mp^m, \qquad A(x) = \exp \sum_{p^m \leqq x} 1/mp^m, \tag{21}$$

and

$$x = (\ln |d|)^{2(1+4\varepsilon)}, \qquad \varepsilon > 0. \tag{22}$$

An integrand in the analysis [**2**, p. 365] includes the factor

$$|L'(\tfrac{1}{2}+\varepsilon+i\eta)/L(\tfrac{1}{2}+\varepsilon+i\eta)|, \tag{23}$$

and the $o(1)$ in (20) depend upon our choice of ε.

Let us define $a(x)$ and $b(x)$ by writing

$$B(x) = \left(1 + \frac{b(x)}{x^{1/2} \ln x}\right) \frac{6e^\gamma}{\pi^2} \ln x, \qquad A(x) = \left(1 + \frac{a(x)}{x^{1/2} \ln x}\right) e^\gamma \ln x. \tag{24}$$

As $x \to \infty$, $a(x)/x^{1/2} \ln x$ and $b(x)/x^{1/2} \ln x \to 0$ and the first approximation is their replacement by 0. The second approximation sets the ε of (22) equal to 0 and so the left side of (20) becomes

$$[\{1+o(1)\}\, 12e^\gamma \pi^{-2} \ln \ln |d|]^{-1}. \tag{25}$$

Now, in all of our examples above we had $|d| < 4 \cdot 10^{14}$, and setting $\varepsilon = 0$ in (22) we obtain $x < 1200$. This is sufficiently small that one can easily compute $b(x)$ and $a(x)$ exactly. We find that throughout this range $b(x)$ is *positive* and fairly stable, remaining mostly between 1 and 2. (We also find that $a(x)$ changes sign frequently and is usually much smaller, but do not need that now.) Therefore,

$$B(x) > 6\pi^{-2} e^\gamma \ln x.$$

This is in the correct direction to absolve $d = -163$ and $4 \cdot 398$, and the difference involved is sufficient to account for the latter's apparent misdemeanor: LLI$=0.966$.

But for $d=-163$, if we set $\varepsilon=0$, we get

$$x=(\ln 163)^2=25.9463, \qquad B(x)=3.7601, \qquad 6\pi^{-2}e^{\gamma}\ln x=3.4853,$$

and even the smaller $B(x)^{-1}$ exceeds $L(1, \chi)=\pi/163^{1/2}$. However, one cannot allow ε to approach 0 too closely for the small $|d|=163$ without losing control over the other approximations leading to the $o(1)$ in (20). It happens that even a quite small ε in (22) will suffice to obtain an x with $B(x)^{-1}<\pi/163^{1/2}$. This is because an increasing x will soon encounter the odd powers of primes $p^m=27, 29, 31$ and thereby yield a $B(x)=4.0695$, whereas, at the earlier square $p^2=25$, $B(x)$ had actually decreased from 3.8360.

That is as far as we will go here. While that leaves it open whether -163 does or does not violate the lower bound, there is enough here, in the correct direction, that we now have no real reason to believe that it does.

We have sought, in two different ways, to exceed the bounds (2) and (12), but with an improbable exception at $d=-163$ we find that we cannot. Our approach has not been at all hit-and-miss but, instead, very systematic. The resulting ULI and LLI are quite uniform and clearly relate to these bounds. All of our strongest cases, such as $d=-991027$ and $d=$first $5N_{139}$, press against the bounds. Our tentative lochamps had LLI$=1.22$. The simplest interpretation of all this persistent behavior is that the extended Riemann hypothesis is true. Of course, that is no proof – not even for a single d.

Any heuristic conclusion is somewhat subjective and I should add that I, personally, regard this as fairly strong evidence. Heuristic reasoning, unlike deductive reasoning, is influenced by collateral evidence. There was considerable evidence for the ERH, of several sorts, prior to this work and that can only strengthen our assessment of the present data.

Suppose we did find a clear violation. We would then know that there were non-Riemannian zeros for that d and we could even give a lower bound for their real parts. If, in place of (23), we were forced out to

$$|L'(\theta+\varepsilon+i\eta)/L(\theta+\varepsilon+i\eta)|$$

because of zeros at $\theta+it$, then (22) would be replaced by

$$x=(\ln|d|)^{(1+4\varepsilon)/(1-\theta)}, \tag{26}$$

and the famous factor of 2 in the bounds would be replaced by the larger factor $1/(1-\theta)$.

Littlewood does not give (26) but the writes [**2**, p. 371], "Hypothesis X, without modification, is essential in proving Theorem 1" [–that is, in proving (2)]. I presume that the need for an enlarged (26) is what he had in mind.

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NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER
BETHESDA, MARYLAND