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ON THE LEFT FACTORIAL FUNCTION !N[1]

1. Definition of $!n$ (left factorial of n). For every (cardinal or ordinal) number n let be

$$!n = 0! + 1! + \cdots + m! + \cdots, \ (m < n), \text{ i.e.}$$

$$!n = \Sigma\, m!, \ (m < n), \text{ with } 0! = 1.$$

The number $!n$ is called „*left factorial of n*" (to be distinguished from the n factorial, $n!$)[2]. E.g.

n	1	2	3	4	5	6	7	8	9	10	11
$n!$	1	2	6	24	120	720	5040	40 320	322 560	322 5600	5 581600
$!n$	1	2	4	10	34	154	874	5 914	46 234	368 704	3 594394
(cf. §2.1) M_n	1	2	2	2	2	2	2	2	2	2	2

1.1. One has the following recursive formula:

$$!(n+1) = !n + n! \ (n \in N).$$

We shall give some properties of the sequence $!n$, in particular in connexion with its prime divisors (if not stated otherwise $n, m, \ldots,$ will be assumed to be natural numbers).

The main question is to know whether for every natural number $n > 2$ one has $M_n = 2$ (cf. 2.1) or equivalently whether $!n \not\equiv 0 \pmod{n}$.

[1] Some results of this paper were presented at the 5th Congress of mathematicians, physicists and astronomers of Yugoslavia (Ohrid, 14–19.9.1970).

[2] The number $n!$ is defined as the cardinality k$S!$ of the set $S!$ of all permutations of S, S being any set of cardinality n.

1.2. Lema. *If* $m \mid !n$ *and* $m \leqslant n$, *then* $m \mid !m$.[1]

As a matter of fact, it suffices to consider the case $m<n$; the relation $m<n$ implies

$$!n = !m + m! + (m+1)! + \cdots + (n-1)!$$

and therefore $!n$ is divisible by m.

1.3. Lemma *If the relations*

(1.3.1) $$!n \equiv 0 \pmod n, \quad 2<n\in N$$

are consistent, then $n \equiv 0 \pmod m$, $m\in N$ *implies* $!m \equiv 0 \pmod m$. *In particular, the least number satisfying (1.3.1) should be an odd prime number.*

As a matter of fact, the relation $m \mid n$ implies $m \leqslant n$; let us consider the case $m<n$; then

$$!n = !m + B, \text{ where } B = m! + (m+1)! + \cdots + (n-1)!.$$

Since each term of B is divisible by m, and since $m \mid !n$, the number $!m = !n - B$ should be divisible by m.

The last section of the lemma is obvious.

1.3.1. Corollary. *There are infinitely many solutions of*

(1.3.1.1) $$!n \not\equiv 0 \pmod n, \quad 2<n\in N.$$

Since these relations are satisfied for $m=3,4,5,6,7$, the same is so for $n=mk$ for every $k\in N$.

Our hypothesis is that (1.3.1.1) is holding identically.

1.3.2. Problem (Function r_n). For any natural number n let r_n be the least non negative integer which is congruent $!n \pmod n$. The problem is to investigate the function $n \rightarrow v_n$ and in particular to determine the values of r_n and the frequence of every term of the sequence $r_1, r_2, r_3, \ldots$ Our hypothesis says that $r_n \neq 0$ for every $n>2$, i. e. that for every natural number $n>2$ one has

$$!n \not\equiv 0 \pmod n \text{ (cf. } H_2 \text{ in § 2.4).}$$

1.4. Lemma. *If* $m \mid !n$ *and* $m<n\leqslant r$ *then* $m \mid ! r_n$.

The lemma is obvious because m divides $s!$ for every $s \geqslant n$.

1.5. Lemma. *For any natural integer* m *satisfying* $m \nmid !m$ *and any natural number* n *we have*

(2) $$m \mid !n \Rightarrow m>n.$$

Proof. In opposite case there would be some m such that (1) holds and some n such that (2) does not hold, i.e. such that

(3) $$m \mid !n, \quad m<n.$$

[1] The sign $\mid$ means *divides*; $\nmid$ is the negation of $\mid$.

Now, from (3) one infers that m divides $!m$ too, because

$$!n = !m + m! + \cdots + (n-1)!; \tag{4}$$

every member in the second part $(4)_2$ of (4) except the first one is divisible by m; now, by hypothesis $m \mid !n$, and the relation (4) would imply $m \mid !m$, contradicting (1).

2. Function $n \to M_n$

2.1. Definition. *M_n denotes the greatest common divisor of*

$$!n, n! \text{ i.e. } M_n = M(!n, n!).$$

2.2. Lemma. *Every prime divisor of M_n divides M_{n+1}: One has $M_n \mid M_{n+1}$ for every $n \in N$.*

As a matter of fact the relation (1.1.1) implies that $M_n \mid !(n+1)$; since also $M_n \mid (n+1)!$ we infer that M_n divides both $!(n+1)$ and $(n+1)!$ and hence also the greatest commor divisor M_{n+1} of these two numbers.

2.2.1. Corollary. *The sequence M_n is increasing: $M_1 \leqslant M_2 \leqslant \cdots$*

2.3. Our hypothesis states $M_n = 2$ for $1 < n \in N$; this is connected with the determination of prime factors of $!n$. We have the following.

2.4. Main theorem. *The following 3 statements are mutually equivalent:*

H_1: **M – Hypothesis.** *If $1 < n \in N$, then $M_n = 2$ where $M_n = M(!n, n!)$.*

H_2: **Factorial incongruence.** *If $2 < n \in N$, then $!n \not\equiv 0 \pmod n$; in particular if p is odd und prime, then $!p \not\equiv 0 \pmod p$.*

H_3: **Factor hypothesis.** *If $1 < n \in N$, then every divisor of $!n$ is 2 or $\geqslant n$.*

Proof. $H_1 \Rightarrow H_3$. In opposite case, there would be an integer $n > 2$ and a prime p dividing $!n$ and such that $p < n$. Therefore, the more $p \mid !n$ and $p \mid n!$, i.e. $p \mid M_n$ and $M_n \geqslant p > 2$ contradicting the M – hypothesis.

$H_3 \Rightarrow H_2$. In opposite case, there would be some $n > 2$ such that $n \mid !n$, therefore also $n \mid !r$ for every $r \geqslant n$, in particular for $r = n+1$; in other words, n would be a divisor of $n+1$, although $n < n+1$, contradicting the factor hypothesis H_3.

$H_2 \Rightarrow H_1$. Let us assume $M_n > 2$ for some n; let m be the smallest such n, i.e. $M_m > 2$ and $M_{m'} = 2$ for every $2 \leqslant m' < m$. Let p be an odd prime divisor of M_m. Consequently, p divides both $!m$ and $m!$. Therefore, $p \leqslant m$. If $p = m$, the relation $p \mid !m$ says that $p \mid !p$, in contradiction with H_2. If $p < m$, we have $!m = !p + p! + (p+1)! + \cdots + (m-1)!$ from where we get

$$!p = !m - p! - (p+1)! - \cdots - (m-1)!$$

Since every term on the second side is divisible by p, so is the number $!p$ too, contradicting the factorial incongruence. The chain of implications $H_1 \Rightarrow H_3 \Rightarrow H_2 \Rightarrow H_1$ proves completely the theorem.

2.5. A reduction of the factorial hypothesis.

Since for every prime p

$$(p-1)! \equiv -1 \pmod{p}$$
$$(p-2)! \equiv 1 \pmod{p}$$
$$(p-3)! \equiv \frac{p-1}{2} \pmod{p}^{1)}$$

we have the following equivalence:

2.5.1. Theorem. *For every prime number* $p>3$

$$!p \not\equiv 0 \pmod{p} \Leftrightarrow \sum_{k=0}^{p-4} k! \not\equiv \frac{1-p}{2} \pmod{p}.$$

3. On divisors of $!n$.

According to the Main theorem 2.2 our hypothesis says that every prime odd divisor of $!n$ should be $\geqslant n$. Therefore it is interesting to find some cases for which the corresponding induction argument holds.

3.1. Lemma. *If* $p \nmid !n'$ *for every* $n' \leqslant n$ and *if* $p \mid !(n+1)$, *then* $p \geqslant n+1$.

In opposite case, there would be $p \leqslant n$. The relation $!(n+1) = !n + n!$ proves then $p \mid !n$ also, contradicting the assumption $p \nmid !n$.

3.2. Lemma. *Let n be a compound number and p an odd prime number . If for every $m<n$ one has*

$$p \mid !m \Rightarrow p \geqslant m, \text{ then also } p \mid !(n+1) \Rightarrow p \geqslant n+1.$$

Assume that $p \mid !(n+1)$; one has to prove then $p \geqslant n+1$. One has two possibilities:

First case: $p \nmid !n'$ for every $n' \leqslant n$; then $p \geqslant n+1$ in virtue of the preceding lemma.

Second case: there is some $m \leqslant n$ such that $p \mid !m$; then by induction hypothesis $p \geqslant m$; we have to prove that $p \geqslant n+1$. In opposite case, it would be $p \leqslant n$, i.e. $m \leqslant p \leqslant n$; therefore $!(n+1) = !m + A + n!$ with

$$A = m! + (m+1)! + \cdots + (n-1)!$$

Since the numbers $!(n+1)$, $!m$, $n!$ are divisible by p, so is the number A too; therefore $p \mid !n$, and by induction hypothesis $p \geqslant n$. This relation with $p \leqslant n$ implies $p = n$, absurdity, p being prime and n being compound. This contradiction proves the lemma.

We are not able to prove analoguous statement of the lemma 3.2. for n being a prime number (the statement in just our Factor hypothesis H_2 in § 2.4).

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1) This is consequence of the result that for any prime p and any $0 \leqslant k < p$ one has $(n-k-1)!\, k! \equiv (-1)^{k+1} \pmod{p}$ (KIRIN [1]).

3.3. Problem. *Is it true that the relation*

$$m^2 \mid !n$$

has no solution for natural numbers $m, n > 1$?

It is so at least for $m = 2, 3, 4, 5, 6, 7, 8$, and any $n > 1$.

3.4. Function $s_k(n)$.

Consider the sequence

$$0!,\ 1!,\ 2!, \ldots$$

and the sum of any cosecutive n terms:

$$k! + (k+1)! + \cdots + (k+n-1)! \stackrel{\text{def}}{=} s_k(n).$$

Thus $s_0(n) = !n$. One could examine the new function $s_k(n)$. In particular, what are the prime factors of $s_k(n)$?

4 The function $!z$ **for complex numbers.**

Considering the relation

(4.1) $$\Gamma(a) = \int_0^\infty e^{-x} x^{a-1}\, dx = (a-1)! \text{ for } a \in N$$

for $a = 1, 2, \ldots n$ and adding all these relations we get

$$!n = \int_0^\infty e^{-x} \left(\sum_{a=1}^{n} x^{a-1} \right) dx, \text{ i.e.}$$

(4.2) $$!n = \int_0^\infty e^{-x} \frac{x^n - 1}{x - 1}\, dx.$$

We use the same relation to *define* the function

(4.3) $$!z \stackrel{\text{def}}{=} \int_0^\infty e^{-x} \frac{x^z - 1}{x - 1}\, dx$$

for every *complex* number z satisfying $\operatorname{Re} z > 0$, and define in particular $!1 = 1$. Since

(4.4) $$\frac{x^{z+1} - 1}{x - 1} = x^z + \frac{x^z - 1}{x - 1}$$

on multiplying (4.4) by e^{-x} and on integrating one has

$$\int_0^\infty e^{-x} \frac{x^{z+1} - 1}{x - 1}\, dx = \int_0^\infty e^{-x} x^z\, dx + \int_0^\infty e^{-x} \frac{x^z - 1}{x - 1}\, dx, \text{ i.e.}$$

(4.5) $$!(z+1) = \Gamma(z+1) + !z.$$

In other words

$$!z = !(z+1) - \Gamma(z+1) = !(z+1) - z! . \tag{4.6}$$

This relation enables us to define stepwise the function $!z$ also for z satisfying Re $z \leqslant 0$.

5. Left factorial $!n$ for transfinite numbers.

5.1. This function was introduced in [4] for any cardinal (ordinal) number n. At the same time we proved also that the general continuum hypothesis (GCH) implies

$$!n = n \tag{5.1}$$

for any cardinal transfinite number n (KUREPA [4] Th. 6.2. (0000)).

5.2. On the other side, one knows (cf SIERPIŃSKI [1]) that the G.C.H. implies the choice axiom Z. In other words

$$G.C.H. \Rightarrow (5.1) \wedge Z.$$

5.3. The converse holds also:

The choice axiom Z and the identity $!n = n$ for every infinite cardinality n imply the G.C.H.

Proof. If the general continuum hypothesis were not holding, there would be a cardinal transfinite number n such that for some cardinal transfinite number r

$$n < r < 2^n. \tag{5.2}$$

Now, $n < r$ implies $n! \leqslant !r$ what in connexion with $n! = 2^n$ and (5.1) would imply $2^n \leqslant r$ contradicting the relation (5.2).

The foregoing results in 5.1 and 5.2 yield the following.

5.4. Theorem. *The general continuum hypothesis is equivalent to the identity $!n = n$ on transfinite cardinalities and the choice axiom.*

5.4.1. Problem. We do not know whether it is legitimate to drop in the last theorem the words *„and the choice axiom“. The problem is equivalent to the statement*

$$n! = 2^n$$

for every transfinite cardinal number n.

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