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No. of Graphical Partitions

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ON THE NUMBER OF GRAPHICAL PARTITIONS*

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A graphical partition is one whose parts constitute the degree sequence of at least one simple linear graph (no loops or multiple lines). For graphs with p non-isolated points and q lines, the number of possible graphical partitions is denoted by $g_{p,q}$. In the present paper, enumeration formulae are derived for the simplest cases $q-p\leq 6$, and a general method for obtaining further results is sketched. In addition, a table of $g_{p,q}$ is included which covers the cases $q\leq 27$ for all possible values of p.

1. Let π be a partition of 2q into p positive parts

(1.1) $$\pi \vdash 2q: \quad 2q = \sum_{i=1}^{p} d_i, \; d_1 \geq d_2 \geq \ldots \geq d_p > 0 .$$

In this paper we study the problem: how many partitions of the form (1.1) are graphical? As the term is used here, a graphical partition is one whose parts are the degrees of the points of at least one simple linear graph (i.e. without self-loops or multiple lines). As is clear from equation (1.1), we consider only graphs without isolated points (i.e. no zero parts in π); the extension to graphs with isolated points is trivial (see Section 8.2).

Hakimi [1,2] has devised an algorithm for deciding whether or not a partition π, as given by equation (1.1), is graphical. Let S be the (ordered) sequence of parts of π:

S: $d_1, d_2, \ldots, d_p$. Form a new sequence S' by dropping the leading term d_1 and subtracting 1 from each of the next d_1 terms of S:

$$S' = d_2-1,\ d_3-1,\ \ldots,\ d_{d_1+1}-1,\ d_{d_1+2},\ \ldots .$$

The sequence S' can be constructed in this manner if and only if $d_1 \leq p-1$. Dropping any zero terms in S and, if necessary, rearranging the remaining terms in nonincreasing order, we obtain a

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new partition π' of $2q-2d_1$. Hakimi's theorem asserts that π is graphical if and only if π' is graphical. Thus one may decide whether or not π is graphical by iterating the construction. If, at any stage, the derived partition has its leading term $> p-1$, the original partition π is not graphical; if, on the other hand, a partition is reached that is recognized as graphical, then so is π. For example, all partitions of the form $(2^a, 1^{2b})$ are graphical by construction provided that when $b = 0$, $a \neq 1,2$. Similarly for partitions of the form $(k, 1^k)$, etc.

In the following seven sections we enumerate graphical partitions in the simplest cases, namely when $p \geq q-6$. Proofs are given for $p \geq q-3$, but for $p = q-4$, $q-5$, and $q-6$, the results are stated without proof. The final section is devoted to listing numerical results, many of which fall outside the scope of the theorems given here.

2. A basic mapping. Let π be a partition of $2q$ into p positive parts, with largest part $\ell(\pi)<p$. Let $\{\pi\}$ be the set of all such partitions with given parameters p and q.

Lemma 1. The set $\{\pi\}$ is 1-1 with the set $\{\lambda\}$ of partitions of $2q-p$, restricted as follows:

(a) λ has largest part $\ell(\lambda) \leq p$

(b) The number of parts $\nu(\lambda)$ of λ satisfies $\nu(\lambda) \leq p-2$.

To prove the lemma, we define a mapping $M(\lambda)\to\pi$ by the following construction. With $\lambda = (\lambda_1, \lambda_2, \ldots)$, let $\lambda_a \equiv (p, \lambda_1, \lambda_2, \ldots)$; λ_a is then a partition of $2q$. We now take the conjugate of λ_a — denoted by $\tilde{\lambda}_a$ — and set $\pi = \tilde{\lambda}_a$. Clearly, π is a partition of $2q$ into p parts; its largest part $\ell(\pi) < p$, because $\nu(\lambda_a) \leq p-1$. Since the construction is uniquely reversible, M is a bijection. This proves the lemma.

3. Here and in the rest of the paper we shall write $g_{p,q}$ for the number of partitions, with parameters p,q, which are graphical. We deal first with the case $p = q+1$. The result here is particularly simple in view of

Theorem 1. Every partition π of $2q$ into $q+1$ parts is the degree sequence of at least one tree.

This theorem is evidently very well-known [2]. One proof goes as follows. We take the parts of π which are greater than 1 — $d_1, d_2, \ldots, d_k$, say — and construct a tree with these d's as the degrees of the internal nodes; this can generally be done in many ways. We then verify (by counting) that the number of terminal nodes is $q+1-k$. Since we have a tree by construction, the number of lines is q, and the theorem is proved.

Theorem 2. $g_{q+1,q} = P(q-1)$, the number of partitions of $q-1$.

Proof. Use the bijection of lemma 1 with λ a partition of $q-1$. The cardinality of $\{\lambda\}$ is $P(q-1)$, and since all $\pi = M(\lambda)$ are graphical by theorem 1, theorem 2 is proved.

Remark: Theorem 2 may be stated in more picturesque language, viz: the number of possible degree sequences for trees with q lines is the number of partitions of $q-1$.

The preceding result is generalized in

Theorem 3. $g_{q+j,q} = P(q-j)$, $1 \le j \le q$.

For $j = 1$, this is theorem 2. The general case follows from the observation that every partition of $2q$ into $q+j$ parts, $2 \le j \le q$, is the degree sequence of a disconnected graph consisting of a tree with $q - j+1$ lines and $j-1$ copies of the (unique) tree with one line. Thus every such partition is graphical. The proof is completed by invoking the bijection of lemma 1.

4. The case $p = q$.

Theorem 4. $g_{q,q} = P(q) - [q/2]-1$,

where the notation $[x]$ means integral part of x.

Proof. Repeat the construction of theorem 1, i.e. construct a tree with internal nodes of degree $d_1, d_2, \ldots, d_k$, $d_i > 1$, $i=1,2,\ldots k$. It is easy to see that the tree so constructed will have $q+1$ lines (hence $q+2$) points). Choose any two terminal nodes which are directly connected to two distinct internal nodes. Connect these two internal nodes by a line, and delete the two chosen terminal nodes together with their corresponding lines. The result is a graph with one cycle which has π as its degree sequence. This construction is possible if and only if π has at least 3 parts greater than 1; if there were only two internal nodes in the tree, the structure produced would be a multigraph (if there is only one internal node, the construction is obviously impossible). But those partitions π with less than 3 parts > 1 correspond, under the mapping M, to partitions $\lambda \vdash q$ with $\ell(\lambda) \leq 2$. For, if $\ell(\lambda) = 2$ ($\ell(\lambda) = 1$ is impossible), π will end in 1^{q-2}, and since $\nu(\pi) = q$, π must contain only 2 parts > 1. Thus to set up a correspondence between a subset of the $\{\lambda\}$ of lemma 1 and the subset of $\{\pi\}$ which consists of graphical partitions, we need only exclude from $\{\lambda\}$ those partitions with $\ell\{\lambda\}\leq 2$. Now the number of partitions $\mu \vdash q$ with $\ell(\mu)\leq 2$ is $1+[^q/2]$. Thus the cardinality of the required subset of $\{\lambda\}$ is $P(q)-[^q/2]-1$. (Note that all $\mu \vdash q$ with $\nu(\mu)>q-2$ have $\ell(\mu)\leq 2$). The theorem follows.

5. The cases $p = q-r$, $r = 1,2,3$.

5.1. We now introduce some further notation which will be used in the sequel.

Def: If $\pi = M(\lambda)$ is graphical, λ will be called <u>legal</u>; in the contrary case, λ is <u>illegal</u>.

Def: Let $\{L^*(\lambda)\}_j^i$ denote, for given p and q, the set of partitions λ which satisfy the conditions of lemma 1, ordered by the usual lexicographical rule, the largest being some given partition $\lambda^{(i)}$, the smallest $\lambda^{(j)}$. The cardinality of this set will be written $|L^*(\lambda)|_j^i$.

Remark: The notation $L^*(\lambda)$ implies what it is convenient to call "lex* order"; this differs from the usual lex order only in excluding those partitions λ for which $\nu(\lambda) > p-2$.

Def: Let π' be the partition derived from π by a single application of the Hakimi algorithm (see Section 1); it is clear that π' always exists, since $\ell(\pi) < p$. If π has parameters p,q, those of π' will be denoted by p',q'.

Def: Let $\lambda' = M^{-1}(\pi')$ be the partition (of 2q'-p') obtained by applying to π' the inverse of the mapping defined in Section 2.

.2. Lemma 2. Let $\lambda = (m_1, m_2, \ldots)$ be a partition with parameters p, q, satisfying the conditions of lemma 1, and suppose that $p < q$, so that $r \equiv q-p > 0$. Write $\nu = \nu(\lambda)$. Then

(a) π has the form

$$\pi = \left(\nu+1, s_1, s_2, \ldots, s_{m_2-1}, 2^{m_1-m_2}, 1^{q-r-m_1}\right), \quad 3 \leqslant s_i \leqslant \nu + 1.$$

(b) If $\nu+1 \leqslant m_2-1$, π' has parameters $q' = q-\nu-1$, $p' = q-r-1$, $r' = r-\nu$, while $\ell(\lambda') = m_1-1$.

(c) If $\nu+1 = m_2-1 + k$, $1 \leqslant k \leqslant m_1-m_2$, the parameters of π' are the same as in (b) above, but $\ell(\lambda') = m_1-1-k$.

(d) If $\nu+1 > m_1-1$, π' has parameters $q' = q-\nu-1$ (as before), $p' = q+m_1-r-\nu-3$, $r' = r-m_1+2$, with $\ell(\lambda') = m_2-1$.

The proof of this lemma is a matter of straightforward bookkeeping. The principal tool is the well-known fact:

If $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$, then

$$\tilde{\gamma} = \left(k^{\gamma_k}, (k-1)^{\gamma_{k-1}-\gamma_k}, \ldots, 2^{\gamma_2-\gamma_3}, 1^{\gamma_1-\gamma_2}\right).$$

This is easily established by using the Ferrers-Sylvester graph of γ.

5.3. Theorem 5. $g_{q-1,q} = |L^*(\lambda)|_2^1$, $q \geq 5$,

with $\lambda^{(1)} = (q-1,2)$, and $\lambda^{(2)} = (4,2,1^{q-5})$

Remark: Since for any graph with p points the maximum number of lines is $q = \binom{p}{2}$, q=5 is indeed the smallest value for which r=1 is possible.

Corollary 1. $g_{q-1,q} = p(q+1,q-1) - p(q+1,3) - 1$,

where, as usual, $p(n,m)$ is the number of partitions of n with largest part $\leq m$.

Proof of Theorem 5. Let $\lambda \vdash q+1$ with $\nu(\lambda) \leq q-3$ and $\ell(\lambda) \leq q-1$. Clearly, $\nu \geq 2$. In accordance with the results given in lemma 2, only the case $\nu+1 > \ell(\lambda)-1$ need be considered, since for smaller ν, $r'=1-r$ is negative, so that the corresponding λ' is legal by theorems 2 and 3. Therefore we take $\nu+1 > \ell(\lambda)-1$, so that $r'=r-\ell(\lambda)+2 = 3-\ell(\lambda)$. Now if $\ell(\lambda) \geq 4$, $r'< 0$ and π' is graphical. In other words, all λ with

$\ell(\lambda) \geq 4$ which satisfy the conditions of lemma 1 are legal (the case $\lambda = (4,1^{q-3})$ has $\nu = q-2$, and is therefore excluded). Now take $\ell(\lambda) = 3$, so that $r'=0$ and $p'=q'$. But π is of the form $\pi = (\nu+1,a,b,1^{q-4})$, whence π' has at most two parts > 1. As shown in the proof of theorem 4, this means that π' is not graphical. Similarly for $\ell(\lambda) = 2$ (π' has at most one part > 1). The theorem follows.

Proof of Corollary 1. If we exclude $\lambda = (4,1^{q-3})$, the number of legal λ given by the theorem is indeed one less than the difference $p(q+1),q-1) - p(q+1,3)$; in fact, $p(q+1),q-1)$ is the number of partitions of $q+1$ with largest part $\leq q-1$, while the second term removes those partitions of $q+1$ with largest part ≤ 3. The only partition λ in this range which is illegal by length is $\lambda = (4,1^{q-3})$, so subtracting 1 from the difference we obtain the correct count.

5.4. Theorem 6. $g_{q-2,q} = |L^*(\lambda)|_2^1$, $q \geq 6$,
with $\lambda^{(1)} = (q-2,4)$ and $\lambda^{(2)} = (4^2, 1^{q-6})$.

Corollary 2. $g_{q-2,q} = p(q+2,q-2) - p(q+2,3) - p(q-2,3)-3$, $q\geq 8$.

The proof of theorem 6 is similar to that of theorem 5, though naturally somewhat more elaborate. According to lemma 1, we take $\lambda \vdash q+2$, $\ell(\lambda) \leq q-2$, $\nu(\lambda) \leq q-4$. Clearly $\nu \geq 2$.

(a) If q=6, $\{L^*(\lambda)\}$ consists of the single partition $\lambda = \left(4^2\right)$, and $\pi = M(\lambda) = \left(3^4\right)$, the degree sequence of the complete graph on 4 points. Thus q=6 is disposed of, and we may take $q \geqslant 7$.

(b) Suppose $\nu = 2$. We have $\lambda = (q-2-j, j+4)$, with $0 \leqslant j \leqslant [\frac{q-6}{2}]$. π is therefore of the form $\pi = \left(3^j, 2^{q-2-2j}, 1^j\right)$ (note that this is certainly graphical for $j = 0,1$). Since $\ell(\lambda) \equiv m_1 > 4$, we have $\nu+1 = 3 < m_1-1$ and $r' = r-\nu = 0$ by lemma 2. But $\nu+1 = 3 < j+4$, so $\ell(\lambda') = m_2-1 = j+3 \geqslant 3$. Therefore π' is graphical by theorem 4.

(c) Take $\nu \geqslant 3$. Then by lemma 2 we need only consider the case $\nu+1 > m_1-1$, since in the contrary case $r' = r-\nu = 2-\nu < 0$, implying that π' is graphical. But for $\nu+1 > m_1-1$ we have $r' = r-m_1+2 = 4-m_1$. Since this is negative for $m_1 > 4$, we have shown that all λ with $\ell(\lambda) > 4$ are legal.

(d) If $\lambda = (4,4,\ldots)$ we shall have $r' = 0$, $\ell(\lambda') = 3$, implying that π' is graphical and λ legal. Therefore we have shown that the set $\{L^*(\lambda)\}_2^1$ with $\lambda^{(1)}$, $\lambda^{(2)}$ specified by theorem 6, consists of legal partitions.

(e) It remains to show that all $\lambda < \lambda^{(2)} = \left(4^2, 1^{q-6}\right)$ are illegal. First we take $\lambda = (4,m_2,\ldots)$ with $m_2 < 4$. Then, with $m_2 = 3$, $\nu \geqslant 3$, we have $\pi = \left(\nu+1, a,b,2,1^{q-6}\right)$. π' will have $r'=r-m_1+2 = 0$, but there will be at most two parts > 1, so π' (and hence π) is not graphical. Similarly for $m_2 = 2$.

Now let $m_1 < 4$. Since $\nu \geq 3$, we have $r' = 1$, $\ell(\lambda') = m_2-1 \leq 2$, and λ' is illegal by theorem 5. Thus we have shown that all $\lambda < (4^2, 1^{q-6})$ are illegal, and the proof of theorem 6 is complete.

Corollary 2 is straightforward except for the subtractive constant 3, which enumerates the partitions in the interval which have $\nu > q-4$; these are $(6,1^{q-4})$, $(5,1^{q-3})$, and $(5,2,1^{q-5})$.

5.5 Theorem 7. $g_{q-3,q} = |L^*(\lambda)|_2^1$, $q \geq 9$,

with $\lambda^{(1)} = (q-3,6)$ and $\lambda^{(2)} = (5,4,2,1^{q-8})$.

Corollary 3. $g_{q-3,q} = p(q+3,q-3) - p(q+3,4) - p(q-2,3) - 8$, $q \geq 11$.

In this case $q = 9$ is not the minimum q for which $r = 3$; we can have $p = 5$, $q = 8$, in which case it is easily verified that $g_{5,8} = 2$. The proof of theorem 7 is in every way similar to that of the two previous theorems. First we show that all λ with $\ell(\lambda) > 5$ are legal, then that all with $\ell(\lambda) = 5$ which are $\geqslant \lambda^{(2)}$ are legal, then that $\lambda = (5,m_2,\ldots)$, $m_2 < 4$, is illegal, and finally that all λ with $\ell(\lambda) < 5$ are illegal. The details are unilluminating. Actually, the legality of λ for $\ell(\lambda) > 5$ and its illegality for $\ell(\lambda) < 5$ follow from two general theorems which we prove in the next section.

The corollary needs little comment. For those interested, we list the 8 partitions which violate $\nu(\lambda) \leqslant q-5$:

$(8,1^{q-5})$, $(7,1^{q-4})$, $(7,2,1^{q-6})$, $(6,1^{q-3})$, $(6,2,1^{q-5})$, $(6,2^2,1^{q-7})$, $(6,3,1^{q-6})$, and $(5,4,1^{q-6})$.

6. Two general theorems.

Theorem 8. For $r \geq 0$, λ is legal if $\ell(\lambda) \equiv m_1 > r + 2$.

In the proof of this theorem and the next, all the results of lemma 2 will be assumed without further comment.

Proof of theorem 8. We assume that the theorem is true for all $r' \leq r-2$; as we have seen, it is true for $r = 0,1,2$.

(a) Let $\nu+1 \leqslant m_2-1$. Then $r' = r - \nu \leqslant r-2$, and $\ell(\lambda') = m_1-1 \equiv m_1'$, so that the result follows from the induction hypothesis.

(b) $\nu+1 = m_2-1 + k$, $1 \leqslant k \leqslant m_1-m_2$. Here again $r' = r-\nu$, but $\ell(\lambda') \equiv m_1' = m_1 - (k+1)$. Thus m_1 is reduced by k+1, while r is reduced by $\nu = m_2-2+k = m_2-3+k+1$. Thus $m_1' > r'+2$ if $m_2 \geqslant 3$, in which case the result follows by induction. If $m_2=2$, we have $\pi = \left(\nu+1, a, 2^{m_1-2}, 1^{q-r-m_1}\right)$. Since $\nu+1 \leqslant m_1-1$, π' will have q-r-1 terms; thus π' will be graphical if $\ell(\pi') = a-1 \leqslant q-r-2$. But $a \leqslant \nu+1 \leqslant m_1-1 \leqslant q-r-1$, so the condition is satisfied. Thus π' is graphical, and λ is legal.

(c) Finally, if $\nu+1 > m_1-1$, $r' = r-m_1+2 < 0$ because $m_1 > r+2$ by hypothesis. Thus π' is graphical and λ is legal.

Theorem 9. Take $m > 2$ and let $\lambda \vdash q+r$ satisfy the conditions of lemma 1 with $\ell(\lambda) = m$. Then if $r \geq \binom{m-1}{2}$, λ is illegal.

Proof by Induction. We have seen that the theorem is true for $r = 0,1,2$. Assume it is true for $r' < r$. Suppose first that $m \nmid q+r$, with $2 \leq m < q+r$. Then

$$\nu \geq 1 + \left[\frac{q+r}{m}\right] > 1 + \left[\frac{2r}{m}\right] \geq 1 + \left[\frac{m^2-3m+2}{m}\right] = m-2.$$ Therefore $\nu > m-2$, whence π' has $r' = r-m+2$, and $\ell(\lambda') = m-1 \equiv m'$. But $r' \geq \binom{m'-1}{2}$ because $r \geq \binom{m-1}{2}$, and the illegality of λ follows from that of λ' (induction hypothesis). The same proof goes through if $m|q+r$, where we can take $\nu > \left[\frac{m^2-2m}{m}\right] = m-2$.

7. For general $r = q-p$ there is little of interest that can be said which goes beyond the results of theorems 8 and 9. For $r \geq 4$ the set of legal λ is the sum of disjoint intervals $\{L^*(\lambda)\}_2^1$, $\{L^*(\lambda)\}_4^3$,... the specification of the boundaries $\lambda^{(i)}$, $\lambda^{(j)}$ of each interval being of ever-increasing complexity. This is illustrated in the next three theorems, which we state without proof.

Theorem 10. $g_{q-4,q} = |L^*(\lambda)|_2^1 + |L^*(\lambda)|_4^3$, $q \geq 10$,

with $(\lambda \vdash q+4)$

$\lambda^{(1)}$ = first λ with $\ell(\lambda) = q-4$

$\lambda^{(2)} = (6,4,2^2,1^{q-10})$

$\lambda^{(3)}$ = first λ with $\ell(\lambda) = 5$

$\lambda^{(4)} = (5^2,3,1^{q-9})$.

Corollary 4. With $q \geq 15$, $g_{q-4,q} = p(q+4,q-4) + p(q-1,5)$

$- \{p(q+4,5) + p(q-1,4) + p(q-2,3) + p(q-6,2) + 17\}$.

Theorem 11. $g_{q-5,q} = |L^*(\lambda)|_2^1 + |L^*(\lambda)|_4^3 + |L^*(\lambda)|_6^5$, $q \geq 12$,

with $(\lambda \vdash q+5)$

$\lambda^{(1)}$ = first λ with $\ell(\lambda) = q-5$

$\lambda^{(2)} = (7,4,2^3,1^{q-12})$

$\lambda^{(3)}$ = first λ with $\ell(\lambda) = 6$

$\lambda^{(4)} = (6,5,3,2,1^{q-11})$

$\lambda^{(5)}$ = first λ with $\ell(\lambda) = 5$

$\lambda^{(6)} = (5^3,1^{q-10})$

Corollary 5. For $q \geq 17$, $g_{q-5,q} = p(q+5,q-5) + p(q-1,6) + p(q-10,5)$

$- \{p(q+5,6) + p(q-1,4) + p(q-2,3) + p(q-6,2) + 36\}$

<u>Theorem 12.</u> $g_{q-6,q} = |L^*(\lambda)|_2^1 + |L^*(\lambda)|_4^3 + |L^*(\lambda)|_6^5 + |L^*(\lambda)|_8^7$, $q \geqslant 14$,

with $(\lambda \vdash q+6)$

$\lambda^{(1)}$ = first λ with $\ell(\lambda) = q-6$

$\lambda^{(2)} = (8,4,2^4,1^{q-14})$

$\lambda^{(3)}$ = first λ with $\ell(\lambda) = 7$

$\lambda^{(4)} = (7,5,3,2^2,1^{q-13})$

$\lambda^{(5)}$ = first λ with $\ell(\lambda) = 6$

$\lambda^{(6)} = (6^2,3^2,1^{q-12})$

$\lambda^{(7)}$ = first λ of the form $(6,5,\ldots)$

$\lambda^{(8)} = (6,5^2,2,1^{q-12})$.

<u>Corollary 6.</u> For $q \geq 20$, $g_{q-6,6} = p(q+6,q-6) + p(q-1,7) + p(q-6,6) + p(q-10,5) - \{p(q+6,7) + p(q-1,4) + p(q-2,3) + 2\,p(q-6,2) + p(q-9,2) + 70\}$.

In these theorems "first" refers to lex^* order. All cases with $r = 4,5$, and 6 are covered except for a few small values of q where the full interval structure is not realized. The values of $g_{q-r,q}$ for these cases may be found in Table I.

If one is interested in deriving theorems like those above for $r > 6$, the boundaries of the various intervals can be found by computer search, and the theorems themselves can be established by "straightforward but tedious" techniques. We remark that the use of such theorems for the calculation of $g_{p,q}$ has great computational advantage over direct application of the Hakimi algorithm.

Once the interval structure has been determined for some given value of r, "explicit" formulae (like those in the corollaries) may be written down immediately, save for the subtractive constant which enumerates the number of partitions of $q+r$ (in the legal intervals) which violate $\nu \leq q-r-2$. But since this number is, in fact, a constant for sufficiently large q, it can be found by comparing the explicit formula with the actual values of $g_{q-r,q}$ calculated by other means (e.g., by use of the Hakimi algorithm). Of course, this would not be necessary if we used the more general partition function $p(n,m,s)$, the number of partitions of n with largest part $\leq m$ <u>and</u> with at most s parts. Since, however, $p(n,m,s)$ is not extensively tabulated (in contrast to $p(n,m)$ [3]), little would be gained thereby.

8. Numerical Results.

8.1. In Table I we give the values of $g_{p,q}$, $q \leq 27$, calculated by direct iteration of the Hakimi algorithm applied to $\pi \vdash 2q$, $\nu(\pi) = p$, with $\ell(\pi) < p$. In order to save space, we have omitted the values with $p > q$; it follows from theorems 2 and 3 that these may be read off directly from a table of $P(N)$, the number of partitions of N. Table I may be used for checking corollaries 1 through 6; in addition, many values of $g_{p,q}$ are given which are not covered by the theorems presented here.

8.2. Let $T_{p,q}$ be the number of graphical partitions when zero parts (i.e., isolated points) are allowed. Clearly, $T_{p,q} = \sum_{i \leqslant p} g_{i,q}$; in other words, we get $T_{p,q}$ by summing the columns of Table I. If isolated points are allowed, any simple graph with p points is in 1-1 correspondence with the "conjugate" graph (in which lines and non-lines are interchanged). As a consequence of this we have $T_{p,q} = T_{p,\overline{q}}$, $\overline{q} \equiv \binom{p}{2} - q$, so that $T_{p,q}$ is symmetric in q. This suggests an alternative scheme for the calculation of $g_{p,q}$. If $g_{i,q}$ is known for all $i < p$ and all corresponding values of q, the new values $g_{p,q}$ need be calculated only for $q \leqslant \left[\frac{1}{2}\binom{p}{2}\right]$. It is not obvious, however, that such a scheme has much to recommend it.

Let $T(p) = \sum_{q=0}^{\binom{p}{2}} T_{p,q}$, and let $L(p)$ be the total number of distinct linear graphs on p points [4]. These two sets of numbers are compared in Table II (to get $T(12)$, the calculation of $g_{p,q}$ was extended through $q = 33$ for $p \leqslant 12$). The table shows just how poor the degree sequence is as a tool for classifying linear graphs.

We wish to thank M. L. Stein for the calculation of Table 1 as well as for other, indispensible, exploratory calculations. We also thank N. Metropolis for many useful discussions, and in particular for elucidating the interval structure in the case r = 6 (theorem 12).

TABLE I

$g_{p,q}$ for $p \leq q$

p \ q	3	4	5	6	7	8	9	10	11	12	13	14	15
3	1	0	0	0	0	0	0	0	0	0	0	0	0
4		2	1	1	0	0	0	0	0	0	0	0	0
5			4	4	4	2	1	1	0	0	0	0	0
6				7	9	11	11	9	7	5	2	1	1
7					11	15	22	26	29	29	26	23	18
8						17	25	38	49	63	74	81	83
9							25	37	58	81	110	142	174
10								36	55	87	124	176	239
11									50	77	123	179	261
12										70	108	172	253
13											94	146	233
14												127	198
15													168

p \ q	16	17	18	19	20	21	22	23	24	25	26	27
3	0											
4	0											
5	0											
6	0											
7	13	8	5	2	1	1						
8	84	77	69	57	44	34	24	15	9	5	2	1
9	201	224	245	250	253	241	223	197	169	138	109	83
10	311	387	470	548	627	692	745	780	790	782	746	705
11	365	492	647	825	1021	1229	1461	1677	1903	2100	2270	2399
12	373	530	736	991	1310	1685	2119	2612	3164	3754	4390	5038
13	345	513	740	1039	1431	1932	2556	3308	4214	5256	6485	7847
14	314	465	695	1008	1429	1990	2730	3669	4851	6317	8082	10200
15	261	413	615	919	1 39	1910	2685	3715	5060	6778	8976	11689
16	222	345	543	806	1205	1757	2516	3550	4947	6785	9183	12287
17		288	447	701	1041	1555	2271	3253	4610	6447	8897	12114
18			375	580	903	1339	1995	2909	4172	5915	8295	11481
19				480	741	1149	1702	2530	3687	5282	7500	10526
20					616	947	1460	2155	3195	4646	6650	9433
21						781	1196	1835	2706	3998	5805	8296
22							990	1511	2304	3386	4988	7222
23								1243	1890	2869	4207	6177
24									1562	2363	3567	5215
25										1945	2932	4405
26											2422	3635
27												2996

Δ A259873

A4250

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TABLE II

p	T(p)	L(p)
1	1	1
2	2	2
3	4	4
4	11	11
5	31	34
6	102	156
7	342	1044
8	1213	12346
9	4361	274668
10	16016	12005168
11	59348	1018997864
12	222117	165091172592

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