A triangle for calculating the Salié numbers A005647.

Peter Bala, April 24, 2017

Oste and van der Jeugt [1, Section 7] show that a continued fraction of the form

$$\cfrac{1}{1-xd_0-\cfrac{xh_1}{1-xd_1-\cfrac{xh_2}{1-xd_2-\cfrac{xh_3}{1-xd_3-\cdots}}}} \tag{1}$$

is the generating function for 2-Motzkin paths weighted by the integers d_i and h_i. This combinatorial interpretation allows one to rapidly calculate the terms of a sequence whose generating function can be expressed as a continued fraction of the form (1). The results are conveniently displayed in the form of a lower triangular array, where the $d's$ occur as multiplication factors along the diagonals of the array and the $h's$ as horizontal multiplication factors along the rows of the array. In the particular case of the Salié numbers A005647, the generating function can be expressed as the continued fraction

$$1/(1-x/(1-2x/(1-5x/(1-8x/(1-\cdots-(2n^2-2n+1)x/(1-2n^2x/(1-...))))))).$$

So in this case the $d's$ are all zero and the horizontal multiplication factors are given by the sequence $a(2n) = 2n^2, a(2n+1) = 2n^2 - 2n + 1$. The Salié number sequence is the leading diagonal of the following lower triangular array:

1										
↓										
1	— x1 —>	**1**								
↓		↓								
1	— x2 —>	**3**	— x1 —>	**3**						
↓		↓		↓						
1	— x5 —>	8	— x2 —>	**19**	— x1 —>	**19**				
↓		↓		↓		↓				
1	— x8 —>	**16**	— x5 —>	**99**	— x2 —>	**217**	— x1 —>	**217**		
↓		↓		↓		↓		↓		
1	— x13 —>	**29**	— x8 —>	**331**	— x5 —>	**1872**	— x2 —>	**3961**	— x1 —>	**3961**
⋮										

References

[1] R. Oste and J. Van der Jeugt, Motzkin paths, Motzkin polynomials and recurrence relations, Electronic Journal of Combinatorics 22(2) (2015), #P2.8. Section 7