Number of binary matrices up to row and column permutations

Let $b_{m,n}$ be the number of m x n binary matrices, up to row and column permutations and $b_m(x)=\sum_{n=0}^{\infty} b_{m,n}x^n$. Also let $Z(S_n;x_1,x_2,...,x_n)$ denote the cycle index of symmetric group S_n of degree *n*. Then

$$b_{m,n}=\frac{1}{m!}\sum_{\pi(m)}\frac{m!}{k_1!1^{k_1}\,k_2!2^{k_2}...k_m!m^{k_m}}\,Z(S_n;c_1(\pi),c_2(\pi),...,c_n(\pi)),$$

where $\pi(m)$ runs through all partitions of *m* (i.e. nonnegative solutions of $k_1+2k_2+...+mk_m=m$) ;

$c_j(\pi)=2^{\sum_{i=1}^{m}(j,i)k_i}$, $1\le j\le n$, where $(j,i)=gcd\{j,i\}$.

Specially, for small values of m we have:

$$b_{0,n}=1$$
$$b_{1,n}=Z(S_n;2,2,...)$$
$$b_{2,n}=\frac{1}{2!}(Z(S_n;4,4,...)+Z(S_n;2,4,2,4,...))$$
$$b_{3,n}=\frac{1}{3!}(Z(S_n;8,8,...)+3Z(S_n;4,8,4,8,...)+2Z(S_n;2,2,8,2,2,8,...))$$
$$b_{4,n}=\frac{1}{4!}(Z(S_n;16,16,...)+8Z(S_n;4,4,16,4,4,16,...)+6Z(S_n;8,16,8,16,...)+3Z(S_n;4,16,4,16,...)+$$
$$+6Z(S_n;2,4,2,16,2,4,2,16,...))$$
$$b_{5,n}=\frac{1}{5!}(Z(S_n;32,32,...)+10Z(S_n;16,32,16,32,...)+15Z(S_n;8,32,8,32,...)+20Z(S_n;8,8,32,8,8,32,...)+$$
$$+20Z(S_n;4,8,16,8,4,32,4,8,16,8,4,32,...)+30Z(S_n;4,8,4,32,4,8,4,32,...)+24Z(S_n;2,2,2,2,32,2,2,2,2,32,...)),\ etc$$

But more efficient way to find the generating function $b_m(x)$ is to replace x_i by $\frac{1}{1-x^i}$ in the cycle index $Z(E_2^{S_m};x_1,x_2,...)$ of power group $E_2^{S_m}$, which can be calculated in the following way:

$Z(E_2^{S_m};x_1,x_2,...)=\frac{1}{m!}\sum_{\pi(m)}\frac{m!}{k_1!1^{k_1}\,k_2!2^{k_2}...k_m!m^{k_m}}\cdot\prod_{i|k}x_i^{l_i}$, where $\pi(m)$ runs through all partitions of *m* (i.e. nonnegative solutions of $k_1+2k_2+...+mk_m=m$) ;

$k=lcm\{i\mid k_i\ne 0\}$;

$l_i=l_i(\pi)=\frac{1}{i}\sum_{d|i}\mu(\frac{i}{d})\cdot 2^{\sum_{j=1}^{m}(j,d)k_j}$, where μ is Mobius function and $(j,d)=gcd\{j,d\}$.

From above formula we get:

$b_0(x) =$ 1/0!*(1/(1-x^1)^1)

$b_1(x) =$ 1/1!*(1/(1-x^1)^2)

$b_2(x) =$ 1/2!*(1/(1-x^1)^4+1/(1-x^1)^2/(1-x^2)^1)

$b_3(x) =$ 1/3!*(1/(1-x^1)^8+3/(1-x^1)^4/(1-x^2)^2+2/(1-x^1)^2/(1-x^3)^2)

$b_4(x) =$ 1/4!*(1/(1-x^1)^16+6/(1-x^1)^8/(1-x^2)^4+3/(1-x^1)^4/(1-x^2)^6+8/(1-x^1)^4/(1-x^3)^4+6/(1-x^1)^2/(1-x^2)^1/(1-x^4)^3)

$b_5(x) =$ 1/5!*(1/(1-x^1)^32+10/(1-x^1)^16/(1-x^2)^8+15/(1-x^1)^8/(1-x^2)^12+20/(1-x^1)^8/(1-x^3)^8+20/(1-x^1)^4/(1-x^2)^2/(1-x^3)^4/(1-x^6)^2+30/(1-x^1)^4/(1-x^2)^2/(1-x^4)^6+24/(1-x^1)^2/(1-x^5)^6)

$b_6(x) =$ 1/6!*(1/(1-x^1)^64+15/(1-x^1)^32/(1-x^2)^16+45/(1-x^1)^16/(1-x^2)^24+15/(1-x^1)^8/(1-x^2)^28+40/(1-x^1)^16/(1-x^3)^16+120/(1-x^1)^8/(1-x^2)^4/(1-x^3)^8/(1-x^6)^4+40/(1-x^1)^4/(1-x^3)^20+90/(1-x^1)^8/(1-x^2)^4/(1-x^4)^12+90/(1-x^1)^4/(1-x^2)^6/(1-x^4)^12+144/(1-x^1)^4/(1-x^5)^12+120/(1-x^1)^2/(1-x^2)^1/(1-x^3)^2/(1-x^6)^9), etc,

or for small values of *n*:

$$b_0(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + ...$$
$$b_1(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^5 + 6x^6 + ...$$
$$b_2(x) = 1 + 3x + 7x^2 + 13x^3 + 22x^4 + 34x^5 + 50x^6 + ...$$
$$b_3(x) = 1 + 4x + 13x^2 + 36x^3 + 87x^4 + 190x^5 + 386x^6 + ...$$
$$b_4(x) = 1 + 5x + 22x^2 + 87x^3 + 317x^4 + 1053x^5 + 3250x^6 + ...$$
$$b_5(x) = 1 + 6x + 34x^2 + 190x^3 + 1053x^4 + 5624x^5 + 28576x^6 + ..., etc.$$

Number of covers and minimal covers of an unlabeled n-set

Let $\overline{b}_m(x) = (1-x)\, b_m(x) = \sum_{n=0}^{\infty} \overline{b}_{m,n} x^n$, then $\overline{b}_{m,n}$ is the number of m x n binary matrices without zero columns up to row and column permutations or the number of m-covers (allowing empty sets and multiple sets) of an unlabeled n-set. The number $\overline{b}_{m,n-m}$, $n \geq m$, is the number of minimal m-covers of an unlabeled n-set, so if $\sum_{n=0}^{\infty} a_n(y)\, x^n = \sum_{n=0}^{\infty} x^n \overline{b}_n(x)\, y^n$ then

$a_0(y) = 1$

$a_1(y) = y$

$a_2(y) = y + y^2$

$a_3(y) = y + 2y^2 + y^3$

$a_4(y) = y + 4y^2 + 3y^3 + y^4$

$a_5(y) = y + 6y^2 + 9y^3 + 4y^4 + y^5$

$a_6(y) = y + 9y^2 + 23y^3 + 17y^4 + 5y^5 + y^6$, *etc*,

where coefficient of y^m, $0 \leq m \leq n$, in $a_n(y)$ is the number of minimal m-covers of an unlabeled n-set. As $a(x) = \sum_{n=0}^{\infty} a_n(1)\, x^n$ is the generating function for the number of minimal covers of an unlabeled n-set we get $a(x) = 1 + x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + 56x^6 + ...$, for small values of n.