6336

Pi-Mu Sequences

This rather silly name actually comes from the first initials of the words "plus" and "minus". The eta-like sequences which we here deal with are composed of exclusively +1's and -1's, whence "pi-mu". Like eta-sequences, they were not invented directly, but just appeared on the scene, in the context of another problem. That problem was this:

Write down "1". Now perform the operation of "reflection-plus-one", which means, copy every number you've so far written, but in the reverse order, and then add 1. Thus we get the sequence "1 2". And now, re-perform the reflection-plus-one. This time, we get "1 2 3 2". Again and again and again:

1 2 3 2 3 4 3 2 3 4 5 4 3 4 3 2 3 4 5 4 5 6 5 4 3 4 5 4 3 4 3 2 ....

The sequence always jumps up one, or down one. And those up's and down's are precisely the pi-mu sequence. They go like this:

\+ + - + + - - + + + - - + - - + + + - + + - - - + + - - + - - ....

Now you can perhaps find a pattern in this. Don't read on, if you want to have the pleasure yourself. The pattern I found, after a while, was this:

The odd-numbered elements alternate between plus and minus. That much is easy. What is left is the sequence below:

\+ + - + + - - + + + - - + - - ....

At first it looks as chaotic as the original sequence, and therefore of no interest; but all of a sudden it hits you that it IS the original sequence, which is quite another matter! The standard miracle has come to pass...

Let us give the name "u-sequence" to the original sequence, and "d-sequence" to the sequence of differences. Using the definition of reflection-plus-one, we can write down an equation which characterizes the u-sequence:

$$u(T + m) = 1 + u(T - m + 1)$$

where "T" stands for any power of two, and m is positive but no greater than T. Below, "T" will always represent a power of two.

Together with the initial condition $u(1) = 1$, this specifies the entire u-sequence. The definition of the d-sequence is:

$$d(n) = u(n+1) - u(n)$$

Suppose $0<s<T$. Then

$$\begin{aligned} d(T+s) &= u(T+s+1) - u(T+s) \\ &= 1 + u(T-s) - 1 - u(T-s+1) \\ &= -(u(T-s+1) - u(T-s)) \\ &= -d(T-s) \end{aligned}$$

This simple property is the basis for our proof. The proof itself consists of two parts, each of which utilizes mathematical induction. In the first part, we want to show that odd-numbered terms are alternately +1 and -1. That is, we want to show that

$$d(4k+1) = +1 \quad \text{and} \quad d(4k-1) = -1, \quad \text{for all } k$$

Certainly the proposition holds at the beginning, namely $d(1) = 1$ and $d(3) = -1$. Now, for the inductive step, assume that it holds for all odd numbers up to T, where T is a power of two which is at least 4. Let $4n+1$ be between T and $2T$; then

$$\begin{aligned} d(4n+1) &= d(T+4b+1) && \text{for some } b > 0 \\ &= -d(T-4b-1) && \text{using the above property} \\ &= -d(4k-1) && \text{where } 4k-1 < T \end{aligned}$$

Since $4k-1 < T$, we can use the inductive hypothesis: $d(4k-1) = -1$. But this gives $d(4n+1) = +1$. The analogous argument holds, starting with $4n-1$, and allows us to conclude that the proposition holds for all odd numbers up to $2T$. But of course this completes the inductive step, and hence the induction.

Now in the second part of the proof, we want to show that the even-numbered terms give the sequence back again, which means

$$d(2n) = d(n)$$

It holds at the beginning: $d(2) = d(1) = +1$. In fact, $d(T) = 1$ for all powers of 2, because of reflection-plus-one:

$$d(T) = u(2^n + 1) - u(2^n)$$

$$= 1 + u(2^n) - u(2^n) = 1$$

We now make the inductive assumption that $d(k) = d(2k)$ for all k less than T, where T is at least 2. Suppose $n = T+b$, where $0 < b < T$. Then

$$\begin{aligned} d(2n) &= d(2T+2b) \\ &= -d(2T-2b) && \text{by the above property} \\ &= -d(T-b) && \text{by the inductive hypothesis} \\ &= d(T+b) && \text{by the above property} \\ &= d(n) \end{aligned}$$

This completes the inductive step for numbers up to $2T$, which are not themselves powers of 2. But we've already done the work for powers of 2, which means that the second induction is complete, and that wraps up the proof that the d-sequence does indeed give itself back.

So far, we've only seen one pi-mu sequence. We can generalize the notion in a natural way. A pi-mu sequence must satisfy three requirements:

(1) It must be composed exclusively of +1's and -1's.

(2) Its odd-numbered terms must alternately be +1 and -1; however, it does not matter which one comes first.

(3) Its even-numbered terms must form a pi-mu sequence.

(Let us make the convention that if a sequence satisfies requirements (1) and (2), then the sequence composed of its even-numbered terms is called its "derivative". This is not the same as the earlier notion of derivative, but it is similar in spirit.)

Now -- how would this definition of pi-mu-ness ever help us to determine if a given sequence were pi-mu? If we attempted to apply it, we would merely be forced, by requirement (3), into checking the pi-mu-ness of the derivative; and then around the bush we would run, around and around and around. To ameliorate the situation, let us say that

If a sequence forces us into infinite regress, then it qualifies as pi-mu. However, if the regress is halted by the failure of some sequence to pass requirement (1) or (2), then of course none of the sequences involved is pi-mu.

Under this definition, the d-sequence is pi-mu, since it equals its own derivative, which clearly forces us into infinite regress. Equally clearly, any sequence which equals any finite-order derivative of itself is pi-mu, because it forces infinite regress. But there are sequences which are pi-mu without equaling any of their own derivatives.

To help describe such sequences, it will be useful to define the "signature" of a pi-mu sequence. To specify the odd-numbered terms in a pi-mu sequence, you need only give the sign of the first term; from there on out they alternate. So let us call the first term's sign the "sign of the sequence". Now each derivative, being pi-mu as well, has a sign. The "signature" of a pi-mu sequence, is, then, the sequence of signs of the pi-mu sequence and all its derivatives. For instance, the sign of the d-sequence is "+", since it begins with +1; and since it equals its own derivative, the rest of the signature must be composed of "+" only. Thus, the d-sequence has signature "+ + + + + ...". We have said this before, in different words. It was when we observed that d(T) = 1, where T is any power of two. In fact, the signature of a pi-mu sequence is precisely the sequence d(1), d(2), d(4), d(8), d(16), ..., a fact which you can check for yourself.

Now there is an obvious question which comes up: are there pi-mu sequences whose signatures are themselves pi-mu sequences? The answer is "certainly". This is because you can specify the signature of a pi-mu sequence as you wish; in other words, there are no constraints on the signature of a pi-mu sequence (other than being composed of plus and minus signs). The reason is this: setting the sign of the top level determines every odd term, but leaves the derivative totally undetermined. Now setting the sign of the derivative determines every second term of the derivative -- and this means, every fourth term in the top level, namely terms number 2, 6, 10, 14, etc.. Setting the sign of the second derivative determines every eighth term in the top level, namely terms number 4, 12, 20, etc.. And so it goes. None of the assigments conflicts with any of the others, so the signature is choosable by you, and in fact completely determines the pi-mu sequence. It is easy as pi-mu to make a pi-mu sequence whose signature is pi-mu; just choose its signature to be the d-sequence. This sequence looks as follows; its signature is indicated as well.

```
++--+--+++-++--+++--+---++-++---++--+--+++-++---++--+---++-++---++--+
++ -   +       +               -                               -
12 4   8       16              32                              64
```

This is very nice, but we can get more intricately pi-mu if we try. What I mean is, the signature of this sequence is the d-sequence; so the signature of the signature is just a boring, bland old "+ + + ...". It would spice things up to find a pi-mu sequence whose signature is not only pi-mu, but which also has a signature -- which is not only pi-mu, but which also... In short, we want a sequence with the property that the operation of "taking the signature" can be done over and over again.

Well, this can be arranged. We just make the sequence equal its own signature. An example, with its signature, is shown below.

```
++--+---++-++--+++--+--+++-++---++--+---++-++---++--+--+++-++---++--+...
++ -   -       +               -                               -   ...
12 4   8       16              32                              64  ...
```

Now a pi-mu sequence need not equal its own signature in order for it to be "infinitely signaturizable". (Incidentally, let us coin a less atrocious term than "signaturizable". Since signatures remind us a little bit of derivatives, we could say they are "derivatives of the second type". Then instead of "signaturizable", we could say "differentiable in the second way".)