	♠ 6 5 ♥ A K 10 8 2 ♦ 10 4 ♣ K J 7 3	
♠ 3 ♥ Q 7 5 4 ♦ Q 9 7 3 ♣ Q 10 8 5		♠ A 4 ♥ J 9 6 ♦ J 8 5 ♣ A 9 6 4 2
	♠ K Q J 10 9 8 7 2 ♥ 3 ♦ A K 6 2 ♣ —	

Solutions

The following are solutions to problems published in July/August, 1973:

J/A1 South has won a contract of six spades with the hands shown at the top of this page. West's opening lead is ♠3, taken by ♠A. East returns ♠4. How can South make the contract?

The following is from Ed Gershuny:

South drops a high spade under the ♠A and, by playing his ♠2, wins the second trick in dummy. Dummy leads ♣K, East must cover (or South would pitch a small diamond and the other one would be discarded under the second high heart), and South trumps. South then runs the rest of his trumps, coming down to four diamonds and a heart. Dummy must retain ♥A, ♥K, ♥10, a diamond, and ♣J. West must retain ♣Q and either three hearts or three diamonds (his fifth card doesn't matter), while East protects the other red suit. If West has saved hearts, declarer plays two rounds of diamonds. On the second diamond: if East discards the ♣Q, dummy discards the ♥10, winning the last three tricks with two high hearts and the ♣J; if East discards one of his hearts, dummy discards the club and wins the last three tricks with ♥A, ♥K, and ♥10. If West has saved diamonds, declarer will lead a heart and play two rounds. On the second round of hearts, West is also squeezed. He cannot discard his ♣Q, so he must throw one of the diamonds. Declarer then wins the last three tricks with ♦A, ♦K, and ♦6.

Also solved by John G. Connine, John W. Dawson, Charles Estes, Ralph H. Evans, Stanley A. Horowitz, Michael Kay, Thomas Mauther, John Maynard, Avi Ornstein, Michael Padlijosky, Fred Price, Smith D. Turner, David W. Ulrich, and the proposer, Winslow H. Hartford.

J/A2 Each of four dogs, located at the four corners of a square field, simultaneously spots the dog in the corner to his right and runs towards that dog, always pointing directly toward him. All the dogs run at exactly the same speed and thus finally meet in the center of the field. How far did each dog travel?

The following solution is from Lydall Morrill, who writes that he first encountered this problem as an undergraduate at M.I.T., when the characters were ants, not dogs; and it was sometimes stipulated that the occupants of alternate corners of the square were of the opposite sex, he writes, "thus supplying a modicum of motivation for the ensuing chase." Mr. Morrill's solution:

Since each pursued ant/dog is always moving at right angles to the line between him and his pursuer, his own motion contributes nothing to diminish the distance between them. Thus only the pursuer's motion accounts for the steady shortening of that distance. The pursuer always moves directly toward the pursued and must traverse the distance that separated them at the start—namely, the side of the original square field. The rotating motion of the square formed by the four ants/dogs of course causes their trajectories to be spiral in shape, but this is charmingly irrelevant to the distance each pursuer has to travel.

Note that by considerations of symmetry the four dogs must remain on the vertices of a square (of decreasing side length, of course).—Ed.

Also solved by 20 readers, for whose names there simply isn't space this month.

J/A3 A circular table is pushed into the corner of a rectangular room. A coin resting on the edge of the table is 9" from one wall and 8" from the other. What is the diameter of the table?

The following solution is from Phelps Meaker; he complains that "most of your puzzles and problems are way above my waning capabilities." But when he turned the page upside down while studying **J/A3**, "the x and y coordinates fairly shouted for attention":

$(R - x)^2 + (R - y)^2 = R^2$
$(R - 8)^2 + (R - 9)^2 = R^2$
$R^2 - 34R + 145 = 0$; $R = 29$ or 5;
$D = 58''$.

Mr. Meaker continues: "I became curious about the root 5, a positive whole number. Then it came to me that a smaller circle passing through the same point might be crowded into the corner, and a radius of 5 might fit. So I wrote a new equation:

$(8 - R)^2 + (9 - R)^2 = R^2$.

The quadratic was identical to the first, and I had a 10" circle in the corner! It would be nice to hear how Mr. McKinnon created this problem, with nice whole numbers."

Also solved by 35 other readers—a list which is simply too long to print.

J/A4 If you have just been given the dice at a crap table, what are the odds against your winning at least once?

The following is from Winslow H. Hartford:

If you throw a 7 or 11 you win; $P_w = 8/36$. If you throw a 2, 3, or 12, you lose; $P_l = 4/36$. Thus the following table:

Throw	4	5	6	8	9	10
P_{Throw}	3/36	4/36	5/36	5/36	4/36	3/36
P_{Win}	3/9	4/10	5/11	5/11	4/10	3/9

The total probability of a win is 8/36 + 1/36 + 16/360 + 25/396 + 25/396 + 16/360 + 1/36 = 1952/3960. So the probability of a loss is 2008/3960.

Also solved by Archie Gann, Jack Parsons, Henry Randall, Frank Rubin, John T. Rule, Smith D. Turner, George Wynne, Harry Zaremba, and the proposer, Jerry Blum.

J/A5 Find the greatest common divisor of $a^m - 1$ and $a^n - 1$, where a is a positive integer.

The following technical solution from S. D. Comer is the only one which seems to be a completely rigorous proof:

The greatest common divisor (gcd) is $a^{gcd(m,n)} - 1$. Proof: Let $z_n = e^{2\pi i/n}$. $F_n(x)$, the nth cyclotomic polynomial, is $\pi_{\substack{1 \leq h \leq n \\ (h,n)=1}} (x - z_n^h)$, where $(h, n) = gcd(h, n)$. $F_n(x)$ is irreducible in $Z[x]$ and its roots (in C) are the primitive nth roots of unity. (1) If k/n, then $(a^k-1)/(a^n-1)$. k/n implies $x^n - 1 = \pi_{d/n} F_d(x) = \pi_{d/k} F_d(x) \cdot \pi_{\substack{d/n \\ d/k}} F_d(x) = (x^k - 1) \cdot \pi_{\substack{d/n \\ d/k}} F_d(x)$.

Now let x = a.

(2) Therefore $a^{gcd(m,n)}$ divides both $a^m - 1$ and $a^n - 1$.

(3) $d \neq d'$ implies $F_d(x) \neq F_{d'}(x)$ because every root of $F_d(x)$ generates a cyclic group of order d and every root of $F_{d'}(x)$ generates a cyclic group of order d'.

(4) $F_d(x)$ divides $x^n - 1$ implies d/n. The unique factorization into irreducible factors of $x^n - 1 = \pi_{d'/n} F_{d'}(x)$ so $F_d(x)$ dixides $x^n - 1$ implies $F_d(x) = F_{d'}(x)$ for d'/n. Thus d = d' divides n.

(5) Therefore if $F_d(x)$ divides $x^n - 1$ and $x^m - 1$, then d divides gcd(m, n).

(6) Thus,
$s(x) = \pi_{\substack{d/n \\ d/(m,n)}} F_d(x)$ and $t(x) = \pi_{\substack{d/m \\ d/(m,n)}} F_d(x)$ are relatively prime.

(7) Thus there exists q(x) and r(x) in Z[x] such that
$1 = q(x)s(x) + r(x)t(x)$.
Now multiply both sides by $a^{(m,n)} - 1$:

(8) $x^{(m,n)} - 1 = q(x)s(x)(x^{(m,n)} - 1) + r(x)t(x(x^{(m,n)} - 1) = q(x)(x^n - 1) + r(x)(x^m - 1)$.

Let x = a:

(9) $a^{(m,n)} - 1 = q(a)(a^n - 1) + r(a)(a^m - 1)$. So any factor of $a^n - 1$ and $a^m - 1$ also divides $a^{gcd(m,n)}$. From (2) and (9) we get the desired result.

Also solved by Winslow H. Hartford, Thomas Kauffman, John Maynard, Avi Ornstein, John E. Prussing, and Robert Wallingford.

Solutions to Speed Department

The following solutions are from the proposers of this month's Speed Department problems:

SD 1 During each dt seconds dog #1 moves v dt toward dog #2, but dog #2 moves "away" from dog #1 at an angle of 360°/N. Thus the velocity of closure is $v[1 - \cos(2\pi/N)]$. If the initial separation was L, the time for closure is $L/(v[1 - \cos(2\pi/N)]$.

SD 2 The first player puts his penny in the exact center of the table. Thereafter, after the second player places a penny, the first player places his on the opposite side of the table so that the center penny is the mid-point of the line connecting player #1's penny and the last penny of player #2. Player #1 must place the last penny.

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