The Graph of Maximal Intersecting Families of Sets

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Abstract

The vertices of the graph V_n [**?**] are maximal intersecting families of subsets of an n-element set (i.e. strong simple games). Two families $\mathcal{F}, \mathcal{G}$ are joined by an edge if $|\mathcal{F} \setminus \mathcal{G}| = 1$. Various properties of V_n are deduced here. $Aut(V_n) = S_n$. Thus, as a graph, V_n encodes all information about maximal intersecting families. The diameter of V_n is 2^{n-2}. Thus, no two strong simple games are negatively correlated in a random voting environment. Three operations are defined set theoretically, and characterized in terms of their graph theoretic properties: a unary inclusion $\iota : V_{n-1} \to V_n$, a binary choice function $\chi : V_{n-1} \times V_{n-1} \to V_n$, and a trinary median function $m : V_n \times V_n \times V_n \to V_n$.

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1 Introduction

A family $\mathcal{F}$ of subsets of an n-element set X is called *intersecting* if every two members have a nonempty intersection. An intersecting family can have at most 2^{n-1} members, since it may not contain both a set and its compliment. When one attains this bound, it is called a *maximal intersecting family.* Every intersecting family is an intersection of maximal intersecting families.

Every maximal intersecting family is an *upset.* That is to say, it is closed under taking supersets.

Strong simple games are also called voting schemes (or social choice functions) [**?**] since they allow a group of people to decide unambiguously between two choices, say 0 and 1. If $\mathcal{G}$ is a strong simple game and A is the set of voters who vote for 1, define the choice function g by $g(A) = 1$ if $A \in \mathcal{G}$ and 0 otherwise. This decision is a ipsodual monotonic function of the expressed preference of the voters in that $g(X \setminus A) = 1 - g(A)$ and $g(B) \geq g(A)$ when $B \supseteq A$. [**?**] Monotonicity guarantees that it is never in a voter's interest to falsify her true preferences, and duality guarantees that the result of the vote is well define and independent of which outcome we call 0 and which 1.

Each rediscovery of a theory gives birth to alternate notation and terminology. [**?**, **?**, **?**, **?**, **?**, **?**] We mainly use the language of voting schemes and games. X is viewed as a set of *players* or *voters.* A coalition $A \subseteq X$ is said to be *winning* if $A \in \mathcal{G}$, and it said to be blocking if $X \setminus A \notin \mathcal{G}$. We assume that if a coalition A wins, then all larger coalitions $B \supseteq A$ also win.

A game is considered *strong* if blocking implies winning. A game is considered *simple* if winning implies blocking. Thus, simple games are exactly intersecting upsets, and strong simple games are exactly maximal intersecting families of sets.

Strong simple games are also called voting schemes (or social choice functions) since they allow a group of people to decide unambiguously between two choices. This decision is a ipsodual monotonic function of the expressed preference of the voters. Monotonicity guarantees that it is never in a voter's interest to falsify her true preferences, and duality guarantees that the result of the vote is well defined.

Two important maximal intersecting families are the *dictatorship* $\mathcal{D}_a = \{A \subseteq X : a \in A\}$ and, when X has odd cardinality, the *democracy* $\mathcal{M}$ consisting of all sets containing a majority of the voters.

2 Graph

Two maximal intersecting families $\mathcal{F}$ and $\mathcal{G}$ are said to be *adjacent* if and only if $|\mathcal{F} \setminus \mathcal{G}| = 1$. We write $\mathcal{F} \sim \mathcal{G}$.

Each maximal intersecting family contains exactly one member of each pair of complementary subsets of X. Thus, the relation $\sim$ is symmetric, for if $\mathcal{F} \setminus \mathcal{G} = \{A\}$ then $\mathcal{G} \setminus \mathcal{F} = \{X \setminus A\}$.

In this situation, A is a minimal member of $\mathcal{F}$ (no subset of A is a member) and $X \setminus A$ is minimal in $\mathcal{G}$. Conversely, when A is minimal in $\mathcal{F}$, $\mathcal{F} \setminus \{A\} \cup \{X \setminus A\}$, the family obtained by *switching* A with its compliment, is a maximal intersecting family adjacent to $\mathcal{F}$ [**?**, Lemma 2.3]. So switching allows one to compute all the neighbors of a given maximal intersecting family. Repeated switching can transform any maximal intersecting family to any other.

Thus, the set of maximal intersecting families of subsets of $X = \{1, 2, \ldots, n\}$ together with the relation $\sim$ defines an undirected connected graph denoted V_n. (See table **??** for $n < 5$. See [**?**] for the key features of V_6.)

The distance between two maximal intersecting families is given by $|\mathcal{F} \setminus \mathcal{G}| = \frac{1}{2}|\mathcal{F} \oplus \mathcal{G}|$ where $\mathcal{F} \oplus \mathcal{G}$ denotes the symmetric difference of the sets $\mathcal{F}$ and $\mathcal{G}$. When viewed as voting schemes, the distances between maximal intersecting families of sets gives a measure of the size of "constitutional reforms." We will also see a probabilistic interpretation of the distance.

The graph V_n and the switching operation were first considered in [**?**].

In a graph G, let the *distance* $\operatorname{dist}(v, w)$ denote the length (number of edges) in the shortest path from vertex v to vertex w. Let the *interval* $I(v, w)$ denote the subgraph of G defined by the union of all paths of length $\operatorname{dist}(v, w)$ between v and w.

Note that the maximal intersecting intersecting families in the interval $I(\mathcal{F}, \mathcal{G})$ are exactly those that arise by starting at either $\mathcal{F}$ or $\mathcal{G}$ and never switching a set $A \in \mathcal{F} \cup \mathcal{G}$ with its complement.

3 Inclusion

Given $X \subseteq Y$, it is possible to extend a maximal intersecting family $\mathcal{F}$ on X to a maximal intersecting family $\iota(\mathcal{F})$ on Y.

$$\iota(\mathcal{F}) = \{B \subseteq Y : B \cap X \in \mathcal{F}\}.$$

Table 1: $V_n, (n = 1, 2, 3, 4, 5)$

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If we interpret $\mathcal{F}$ as a voting scheme, then $\iota(\mathcal{F})$ is a voting scheme with additional dummy voters who have no possible affect on the result. (Voters who do not belong to any minimal winning coalitions are called *dummies*.)

In particular, ι defines an injection between V_{n-1} and V_n. ι is not a homomorphism, since each edge of V_{n-1} is "bisected" in V_n; that is to say, if $\mathcal{F} \sim \mathcal{G}$ in V_{n-1}, then there is a unique maximal intersecting family $\mathcal{H}$ in V_n such that $\iota\mathcal{F} \sim \mathcal{H} \sim \iota\mathcal{G}$.

By viewing each member of X in turn as the "last" member, we can find n "bisected" copies of V_{n-1} in V_n. (One is highlighted in table **??**.) The intersection of each pair forms a doubly "bisected" copy of V_{n-2}.

Consider the collection of all intersecting upsets of subsets of $\{1, 2, ..., n\}$ (i.e., simple games). Define $\mathcal{F} \sim \mathcal{G}$ if their symmetric difference is a singleton. This gives a bipartite graph W_n with partition according to the parity of $|\mathcal{F}|$. The distance between two intersecting upsets is given by $|\mathcal{F} \oplus \mathcal{G}|$. The maximal intersecting families are in this graph but the distance between two in W_n is twice that in V_n.

In fact, W_{n-1} is isomorphic to V_n. The isomorphism is given by

$$\kappa(\mathcal{F}) = \mathcal{F} \cup n\mathcal{F}^*$$

where $n\mathcal{F} = \{n \cup A : A \in \mathcal{F}\}$ and $\mathcal{F}^* = \{\{1, 2, \ldots, n-1\} \setminus A : A \notin \mathcal{F}\}$ is the set of blocking coalitions for the simple game $\mathcal{F}$. $\iota = \kappa$ when restricted to maximal intersecting families, since $\mathcal{F} = \mathcal{F}^*$ for maximal intersecting families.

Note that $\iota : W_{n-1} \to W_n$.

4 Choice Function

A move in a sequential game is a choice by one of the players between two or more possibilities. Without loss of generality, we can consider only choice between two possibilities, since choices between a large number of possibilities can be made via iterated binary choices.

Thus, given two maximal intersecting families of sets $\mathcal{F}$ and $\mathcal{G}$ corresponding to games F and G, we ask what maximal intersecting family of sets $\mathcal{H} = \chi_a(\mathcal{F}, \mathcal{G})$ corresponds to the game in which the player a is presented with a choice between playing F and playing G. Obviously, $A \subseteq X$ wins H if it wins both F and G. Moreover, if $a \in A \subseteq X$, then A can win H even if it wins only one of F and G.

$$\chi_a(\mathcal{F}, \mathcal{G}) = (\mathcal{F} \cap \mathcal{G}) \cup \{A \in \mathcal{F} \cup \mathcal{G} : a \in A\}.$$

χ_a is a binary operator on V_n.

Proposition 1 *1.* $\chi_a(\mathcal{F}, \mathcal{G})$ *is a maximal intersecting family.*

2. $\chi_a(\mathcal{F}, \mathcal{G}) \in I(\mathcal{F}, \mathcal{G})$*. In other words,* $\chi_a(\mathcal{F}, \mathcal{G})$ *is on a minimal path from* $\mathcal{F}$ *to* $\mathcal{G}$*.*

$$dist(\mathcal{F}, \mathcal{G}) = dist(\mathcal{F}, \chi_a(\mathcal{F}, \mathcal{G})) + dist(\chi_a(\mathcal{F}, \mathcal{G}), \mathcal{G}) .$$

Moreover, $\chi_a(\mathcal{F}, \mathcal{G})$ *is the unique* $\mathcal{H} \in I(\mathcal{F}, \mathcal{G})$ *which minimizes* $dist(\mathcal{H}, \mathcal{D}_a)$*.*

3. Associativity

$$\chi_a(\chi_a(\mathcal{F}, \mathcal{G}), \mathcal{H}) = \chi_a(\mathcal{F}, \chi_a(\mathcal{G}, \mathcal{H})).$$

4. Commutativity

$$\chi_a(\mathcal{F}, \mathcal{G}) = \chi_a(\mathcal{G}, \mathcal{F}).$$

5. Idempotence

$$\chi_a(\mathcal{F}, \mathcal{F}) = \mathcal{F}.$$

Proof: 1. Let $a \in A$ and $B = X \setminus A$. If $A \notin \chi_a(\mathcal{F}, \mathcal{G})$, then $A \notin \mathcal{F}$ and $A \notin \mathcal{G}$. Thus, $B \in \mathcal{F}$ and $B \in \mathcal{G}$, so $B \in \chi_a(\mathcal{F}, \mathcal{G})$. Conversely, if $A \in \chi_a(\mathcal{F}, \mathcal{G})$, then $A \in \mathcal{F}$ or $A \in \mathcal{G}$. In either case, $B \notin \chi_a(\mathcal{F}, \mathcal{G})$.

2. $\chi_a(\mathcal{F}, \mathcal{G})$ contains all coalitions which are members of both $\mathcal{F}$ and $\mathcal{G}$. When $\mathcal{F}$ contains a coalition, and $\mathcal{G}$ contains its compliment, $\chi_a(\mathcal{F}, \mathcal{G})$ contains the coalition to which a belongs.

3. In either case, we have the maximal intersecting family which takes from either complementary pair the one containing a unless the complement is in all three of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{G}$.□

Due to the last three properties, we can think of χ_a as operating on sets of distinct games. From each complementary pair of coalitions, it takes the one containing a unless the other coalition wins every one of the games. If we take $a = n$ and $\kappa : W_{n-1} \to V_n$ as before then

$$\chi_n\{\mathcal{F}_1, \ldots, \mathcal{F}_k\} = \kappa \left(\bigcap_{i=1}^{k} \kappa^{-1} \mathcal{F}_i \right)$$

If we focus our attention on maximal intersecting families in which $a = n$ is a dummy, then we have a map from sets of strong simple games on $\{1, 2, \ldots, n-1\}$ into V_n. Suppose $S \subseteq V_{n-1}$ is a set of strong simple games. Then

$$\chi(S) = \left(\bigcap_{\mathcal{F} \in S} \mathcal{F} \right) \bigcup \left(\bigcup_{\mathcal{F} \in S} n\mathcal{F} \right) = \chi_n \left(\{\iota\mathcal{F} : \mathcal{F} \in S\} \right).$$

Also, $\chi\{\mathcal{F}\} = \iota\mathcal{F}$, and as a binary operation $\chi : V_{n-1} \times V_{n-1} \to V_n$,

$$\chi(\mathcal{F}, \mathcal{G}) = (\mathcal{F} \cap \mathcal{G}) \cup n\mathcal{F} \cup n\mathcal{G}.$$

Not only is $\chi(\mathcal{F}, \mathcal{G})$ on a path between $\iota(\mathcal{F})$ and $\iota(\mathcal{G})$ as implied by Proposition **??**, but indeed, it is the mid-point of such a path.

Proposition 2 $\mathit{dist}(\mathcal{F}, \mathcal{G}) = \mathit{dist}(\iota(\mathcal{F}), \chi(\mathcal{F}, \mathcal{G})) = \mathit{dist}(\chi(\mathcal{F}, \mathcal{G}), \iota(\mathcal{G}))$.

Proof: $\chi(\mathcal{F}, \mathcal{G}) \setminus \iota(\mathcal{G}) = \mathcal{F} \setminus \mathcal{G}$.□

In the case of $\mathcal{F} \sim \mathcal{G}$, $\chi(\mathcal{F}, \mathcal{G})$ is the unique "bisection" point between $\iota(\mathcal{F})$ and $\iota(\mathcal{G})$. However, in general, this is not the case. (See for example V_4 in table **??**.)

5 Median

We now will define a trinary operator $m : V_n \times V_n \times V_n \to V_n$ There are several possible motivations for the definition of this operator.

First, let $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ be maximal intersecting families of sets and interpret them as voting schemes. Now, vote once using each. The three results are then combined by majority rule, best two out of three. The resulting voting scheme is called the median $m(\mathcal{F}, \mathcal{G}, \mathcal{H})$, since it corresponds to the median operation of lattice theory when $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are thought of as ipsodual elements of the free distributive lattice. In other words,

$$m(\mathcal{F}, \mathcal{G}, \mathcal{H}) = (\mathcal{F} \cap \mathcal{G}) \cup (\mathcal{F} \cap \mathcal{H}) \cup (\mathcal{G} \cap \mathcal{H}) = (\mathcal{F} \cup \mathcal{G}) \cap (\mathcal{F} \cup \mathcal{H}) \cap (\mathcal{G} \cup \mathcal{H}).$$

Proposition 3 *If $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are maximal intersecting families, then so is their median $m(\mathcal{F}, \mathcal{G}, \mathcal{H})$.*

Proof: Clearly this is a family which contains one from each pair of complementary sets. It is intersecting since any two members are both members of at least one of the three families $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$.□

Alternate Proof: As above, let $*$ denote the natural duality relation in the free distributive lattice. Namely, $\mathcal{F}^* = 2^X \setminus \{X \setminus A : A \in \mathcal{F}\}$ — the family of blocking coalitions for $\mathcal{F}$. Then a maximal intersecting family is exactly a self-dual upset — an *ipsodual* member of the free distributive lattice.

$$\begin{aligned} m(\mathcal{F}, \mathcal{G}, \mathcal{H})^* &= ((\mathcal{F} \cap \mathcal{G}) \cup (\mathcal{F} \cap \mathcal{G}) \cup (\mathcal{G} \cap \mathcal{H}))^* \\ &= (\mathcal{F}^* \cup \mathcal{G}^*) \cap (\mathcal{F}^* \cup \mathcal{G}^*) \cap (\mathcal{G}^* \cup \mathcal{H}^*) \\ &= m(\mathcal{F}^*, \mathcal{G}^*, \mathcal{H}^*) \\ &= m(\mathcal{F}, \mathcal{G}, \mathcal{H}).\square \end{aligned}$$

Finally, we have the following graph theoretic characterization of the median.

Proposition 4 *Suppose $\mathcal{F}, \mathcal{G}, \mathcal{H} \in V_n$. Then $m(\mathcal{F}, \mathcal{G}, \mathcal{H})$ is the unique $\mathcal{I} \in V_n$ such that $dist(\mathcal{F}, \mathcal{I}) + dist(\mathcal{G}, \mathcal{I}) + dist(\mathcal{H}, \mathcal{I})$ is minimal.*

Proof: Note that the choice we make from each complimentary pair contributes 0, 1, 2, or 3 to the sum independent of the other complimentary pairs. To minimize the sum, it suffices to minimize each contribution independently.□

Note that the median operation defined above correspond to the usual graph theoretic median operation. The classical median of three vertices $s, t, u \in G$ is defined to be the following set which may be empty or may have several members $M(s, t, u) = I(s, t) \cap I(s, u) \cap I(t, u)$.

Proposition 5 *Let $\mathcal{F}, \mathcal{G}, \mathcal{H} \in V_n$. Then $M(\mathcal{F}, \mathcal{G}, \mathcal{H})$ is equal to the one element set $\{m(\mathcal{F}, \mathcal{G}, \mathcal{H})\}$.*

Proof: All graphs obey the triangle inequality $\text{dist}\,(u, z) + \text{dist}\,(z, v) \leq \text{dist}\,(u, v)$ with equality if and only if $z \in I(u, v)$. Hence, $2(\text{dist}\,(u, z) + \text{dist}\,(v, z) + \text{dist}\,(w, z)) \leq \text{dist}\,(u, v) + \text{dist}\,(u, w) + \text{dist}\,(v, w)$ with equality if and only if $z \in M(u, v, w)$. The result now follows from proposition **??**.□

A permutation π of X can be naturally associated with the following automorphism of V_n

$$\pi\mathcal{F} = \{\pi A : A \in \mathcal{F}\}$$

where $\pi A = \{\pi a : a \in A\}$. Such automorphisms are called inner automorphisms, since they result in a natural way from automorphisms of the underlying set.

The non-existence of "outer" automorphisms is a non-trivial question. For example, the symmetric group of order six does in fact possess outer automorphisms [**?**]. However, the graph of maximal intersecting families of sets does not have outer automorphisms.

Theorem 6 *There are exactly $n!$ automorphisms of the graph V_n. They are the automorphisms associated with the $n!$ permutations of the set X.*

Proof: $\mathcal{F} \in V_n$ is of degree one (i.e. it has one neighbor) if and only if it has a unique minimal member. The dictatorships $\mathcal{D}_a = \{A \subseteq X : a \in A\}$ are thus the only maximal intersecting families of degree 1. They are permuted in all $n!$ ways by the inner automorphisms of V_n.

It suffices to show that the identity is the only automorphism of V_n fixing all dictatorships. By proposition **??**, the median is a graph theoretic property. Moreover, by Monjardet's Theorem [**?**, **?**] all maximal intersecting families can be written in terms of the median operation m and the n dictatorships $\mathcal{D}_a$.□

Alternate Proof: By part 2 of proposition **??**, χ_a can be defined graph theoretically. Moreover, all maximal intersecting families can be represented as multi-player games. (For example, players take turns voting yes or no, and the first player on the unique winning coalition is declared "winner.") Its binary game-tree representation indicates how the family can be written in terms of χ_a and $\mathcal{D}_a$. [**?**] □

The first proof is in reality a special case of the second given the following relationship between χ_a and m:

Proposition 7 $\chi_a(\mathcal{F},\mathcal{G}) = m(\mathcal{F},\mathcal{G},\mathcal{D}_a)$.

Proof:

$$\begin{aligned} m(\mathcal{F},\mathcal{G},\mathcal{D}_a) &= (\mathcal{F}\cap\mathcal{G})\cup(\mathcal{F}\cap\mathcal{D}_a)\cup(\mathcal{G}\cap\mathcal{D}_a) \\ &= (\mathcal{F}\cap\mathcal{G})\cup a\mathcal{F}\cup a\mathcal{G}.\Box \end{aligned}$$

6 Probability

Distances in the graph V_n admit a probabilistic interpretation [**?**, §6].

Interpret all maximal intersecting subsets $\mathcal{F}$ of X as voting schemes. Suppose additionally that all voters $a \in X$ vote independently "yes" or "no" each with probability 1/2. That is, choose a coalition at random using the uniform distribution on the 2^n possibilities. Consider $\mathcal{F}$ ($\sim\mathcal{F}$) to be the event that the chosen coalition is winning (losing) in the game $\mathcal{F}$. Since $\mathcal{F}$ has 2^{n-1} winning coalitions, the probability of "success" is $\mathbf{P}(\mathcal{F}) = 1/2$.

Now, consider two maximal intersecting families of sets $\mathcal{F}$ and $\mathcal{G}$. What is the conditional probability of vote $\mathcal{G}$ succeeding given that $\mathcal{F}$ occurs? By definition,

$$\begin{aligned} \mathbf{P}(\mathcal{G}|\mathcal{F}) &= \mathbf{P}(\mathcal{F}\cap\mathcal{G})/\mathbf{P}(\mathcal{F}) \\ &= 2^{1-n}|\mathcal{F}\cap\mathcal{G}| \\ &= 1-2^{1-n}|\mathcal{F}\setminus\mathcal{G}| \\ &= 1-2^{1-n}\mathrm{dist}\,(\mathcal{F},\mathcal{G}) \end{aligned}$$

Thus, maximal intersecting families which are close to each other in V_n are highly correlated. (In the extreme, $\mathbf{P}(\mathcal{F}|\mathcal{F}) = 1$, since $\mathrm{dist}\,(\mathcal{F},\mathcal{F}) = 0$.)

By the above, $\mathbf{P}(\mathcal{F}|\mathcal{G}) = \mathbf{P}(\mathcal{G}|\mathcal{F})$. Also, $\mathbf{P}(\mathcal{F},\mathcal{G}) = \mathbf{P}(\iota(\mathcal{F}),\iota(\mathcal{G}))$. Thus, $1-\mathbf{P}(\mathcal{F},\mathcal{G})$ (unlike $\mathrm{dist}\,(\mathcal{F},\mathcal{G})$) is a measure of proximity preserved by the embeddings ι. Of course, $1 - \mathbf{P}(\mathcal{F}, \mathcal{G}) = \text{dist}\,(\mathcal{F}, \mathcal{G})/2^{n-1}$ can be viewed as a normalized distance.

We shall now show that there are no negatively correlated maximally intersecting families of sets. In other words, $\mathbf{P}(\mathcal{F}|\mathcal{G}) \geq \mathbf{P}(\mathcal{F}) = 1/2$, or equivalently, $\text{dist}\,(\mathcal{F}, \mathcal{G}) \leq 2^{n-2}$ or $|\mathcal{F} \cap \mathcal{G}| \geq 2^{n-2}$. One proof is a consequence of [**?**, **?**, p. 104]. An alternate proof due to Kleitman [**?**, **?**, p. 87] can be adapted to identify exactly when two maximal intersecting families of sets are independent. In other words, when does $\mathbf{P}(\mathcal{F}|\mathcal{G}) = 1/2$?

Theorem 8 *Let $\mathcal{F}$ and $\mathcal{G}$ be upsets in the algebra of subsets of $X = \{1, 2, \ldots, n\}$. Then*

$$|\mathcal{F}||\mathcal{G}| \leq 2^n |\mathcal{F} \cap \mathcal{G}|$$

with equality in the case that every element of X is a dummy in one or both of of $\mathcal{F}$ and $\mathcal{G}$.

Proof: By induction on n. Decompose

$$\mathcal{F} = n\mathcal{F}_1 \cup \mathcal{F}_2 \text{ and } \mathcal{G} = n\mathcal{G}_1 \cup \mathcal{G}_2$$

where $\mathcal{F}_i$ and $\mathcal{G}_i$ are upsets in the algebra of subsets of $\{1, 2, \ldots, n-1\}$ and $\mathcal{F}_2 \subseteq \mathcal{F}_1$ and $\mathcal{G}_2 \subseteq \mathcal{G}_1$. Now, let $\alpha_i = |\mathcal{F}_i|$ and $\beta_i = |\mathcal{G}_i|$. Let $x = \alpha_1 - \alpha_2 \geq 0$, and $y = \beta_1 - \beta_2 \geq 0$.

$$\begin{aligned}
|\mathcal{F}||\mathcal{G}| &= (\alpha_1 + \alpha_2)(\beta_1 + \beta_2) \\
&= 2(\alpha_1\beta_1 + \alpha_2\beta_2) - xy \\
&\leq 2(\alpha_1\beta_1 + \alpha_2\beta_2) \\
&\leq 2(2^{n-1}|\mathcal{F}_1 \cap \mathcal{G}_1| + 2^{n-1}|\mathcal{F}_2 \cap \mathcal{G}_2|) \\
&= 2^n(|n\mathcal{F}_1 \cap n\mathcal{G}_1| + |\mathcal{F}_2 \cap \mathcal{G}_2|) \\
&= 2^n|\mathcal{F} \cap \mathcal{G}|.
\end{aligned}$$

For equality to hold, xy must be 0, and we must have equality at $n-1$. Thus, each of $1, 2, \ldots, n-1$ must be a dummy in one or both upsets. $x = 0$ means that n is a dummy in $\mathcal{F}$. $y = 0$ means that n is a dummy in $\mathcal{G}$.□

By induction, we can characterize when equality holds in the following inequality conjectured by Erdös and proved by Kleitman. [**?**, **?**, p. 90]

Corollary 9 *The intersection of k maximal intersecting families of X has at least 2^{n-k} members with equality only when their sets of non-dummies are disjoint.*□

7 Radius and Diameter

The *diameter* of a graph G is the greatest distance attainable in the graph,

$$\delta_G = \max_{v,w \in G} \operatorname{dist}(v, w).$$

Its perimeter is the set of points v attaining this maximum for some w.

Corollary 10 *For $n > 1$, $\delta_{V_n} = 2^{n-1}$. The perimeter of V_n consists of those maximal intersecting families of sets with at least one dummy.*$\square$

Thus, $\iota(V_{n-1})$ lies on the perimeter of V_n, and conversely each family on the perimeter of V_n is isomorphic to a family $\iota(\mathcal{F})$ where $\mathcal{F} \in V_{n-1}$.

Let $a \in X$ be a voter. The power $d_{\mathcal{F}}(a)$ of a in $\mathcal{F}$ is defined to be the number of sets $A \in \mathcal{F}$ such that $a \in A$. It is a good measure of power, since it is directly related to the distance to the dictatorship by a,

$$d_{\mathcal{F}}(a) = 2^{n-1} - \operatorname{dist}(\mathcal{F}, \mathcal{D}_a).$$

The probability of success, given that a votes "yes", increases linearly (from 1/2 to 1) with the power of a, $\mathbf{P}(\mathcal{F}|\mathcal{D}_a) = 2^{1-n} d_{\mathcal{F}}(a)$. Dictatorship by a corresponds to the maximal power 2^{n-1}, whereas minimal power 2^{n-2} is achieved if a is a dummy.

The partity of $d_{\mathcal{F}}(a)$ yields a proper coloring of V_n. Thus, V_n is bipartite.

A voting scheme $\mathcal{F}$ is defined to be *regular* if every voter has the same strength. A sufficient condition for regularity is that the scheme be *transitive*—i.e., that a transitive group of permutations fix $\mathcal{F}$.

The *radius* of a graph G is

$$\rho_G = \min_{v \in G} \max_{w \in G} \operatorname{dist}(v, w).$$

The center of a graph is the set of vertices v attaining this minimum.

The scope of our next main result (theorem **??**) depends in part on the validity of Chvátal's conjecture [**?**, **?**].

Conjecture 11 *Let $\mathcal{S}$ be a downset and consider all intersecting subfamilies $\mathcal{T} \subseteq \mathcal{S}$. Among those with the largest cardinality is a family $\mathcal{T}$ which is the intersection of $\mathcal{S}$ and a dictatorship.*

Or equivalently, for every upset $\mathcal{U}$ there is a dictatorship among the maximum intersecting families $\mathcal{F}$ which minimize $|\mathcal{U} \cap \mathcal{F}|$.

For short, we will say that Chvátal's conjecture holds for $\mathcal{U}$ when there is a dictatorship among the maximal intersecting families with minimal size intersection with it. Thus, by theorem **??**, Chvátal's conjecture is true for every intersecting family with a dummy. Furthermore, Chvátal's conjecture holds in the following special case:

Proposition 12 *Chvátal's conjecture is true for a democracy $\mathcal{M}$.*

Proof: We must to show that $\operatorname{dist}(\mathcal{M},\mathcal{F})$ attains its maximum in V_n if $\mathcal{F}$ is a dictatorship; that no maximal intersecting family is further from democracy than a dictatorship. Let $\mathcal{F}$ be a maximal intersecting family, and let $\mathcal{F}_i$ be the intersecting family of i-element coalitions in $\mathcal{F}$. Then

$$\operatorname{dist}(\mathcal{F},\mathcal{M}) = |\mathcal{F} \setminus \mathcal{M}| = \sum_{i=0}^{(n-1)/2} |\mathcal{F}_i|.$$

Furthermore, [**?**] for $i \leq n/2$, there are no intersecting families $\mathcal{F}_i$ of over $\binom{n-1}{i-1}$ i-element subsets of $X = \{1, 2, \ldots, n\}$. So

$$\operatorname{dist}(\mathcal{F},\mathcal{M}) \leq \sum_{i=0}^{(n-1)/2} \binom{n-1}{i-1}.$$

This bound is attained by all dictatorships. Thus, no family is further from democracy than a dictatorship. (In fact, only dictatorships are maximally distant from $\mathcal{M}$, since only in the case of dictatorship does $\mathcal{F}_1 \neq \emptyset$.) □

When n is even but not a power of two, define a *near democracy* to be a regular maximal intersecting family which includes all the majority coalitions. In other words, an intersecting family consisting of all the majority coalitions and, for each a, half of the coalitions of size $n/2$ which contain a, that is $\frac{1}{2}\binom{n-1}{n/2-1}$ of them. When n is a power of two this last number is half an odd number. In this case, define a *near democracy* to be a maximal intersecting family with all the majority coalitions and for each a, $\frac{1}{2}\left(\binom{n-1}{n/2-1} \pm 1\right)$ of the coalitions of size $n/2$ containing a.

Lemma 13 *There are near democracies for every even n.*

Proof: Let $n = 2m$, $X = \{1, 2, \ldots, 2m\}$ and $Y = \{1, 2, \ldots, 2m-1\}$. Consider the cyclic group generated by the permutation σ which fixes $2m$ and cyclically permutes the elements of Y. The orbits of the action of this group on the m element subsets of Y all have $2m-1$ members. There are thus $\frac{1}{2m-1}\binom{2m-1}{m}$ such orbits.

When n is not a power of two, this integer is even. Choose half these orbits and form a maximal intersecting family of subsets of X by taking all majority coalitions, the m element sets from the chosen orbits, and the complements (in X) of the m element sets in the unchosen orbits. The result is a near democracy.

When n is a power of two there are an odd number of orbits of m element subsets of Y. Temporarily ignore the orbit containing $S = \{1, 2, \ldots, m\}$ and perform the construction above with the remaining orbits. From the orbit of S take $S, \sigma^2 S, \sigma^4 S, \ldots, \sigma^{2m-2} S$ and the complements of $\sigma S, \sigma^3 S, \ldots, \sigma^{2m-3} S$.□

Theorem 14 1. *For n odd, the radius of V_n is given by*

$$\rho_{V_n} = 2^{n-2} - \frac{1}{2}\binom{n-1}{(n-1)/2}.$$

The unique maximal intersecting family of sets in the center of V_n is the n-voter democracy $\mathcal{M}$.

2. *For n even, the radius of V_n is bounded by*

$$2^{n-2} - \frac{1}{2}\binom{n-1}{(n-2)/2} \le \rho_{V_n} \le 2^{n-2}.$$

If n is not a power of two then the lower bound is attained exactly when Chvátal's conjecture holds for some of the near democracies. In this case the center consists of exactly those near democracies.

If n is a power of two then the lower bound plus $\frac{1}{2}$ is attained exactly when Chvátal's conjecture holds for some of the near democracies. In this case the center includes those near democracies.

In particular, if Chvátal's conjecture is true then the center includes all near democracies.

One necessary condition for $\mathcal{F}$ to be in the center of V_n is that all voters a have power $d_{\mathcal{F}}(a)$ at least $2^{n-1} - \rho_{V_n}$; otherwise, the maximal intersecting family $\mathcal{F}$ would be too far from $\mathcal{D}_a$.

Let $\mathcal{F}$ be a maximal intersecting family and define

$$d = d_{\mathcal{F}} = \sum_{a \in X} d_{\mathcal{F}}(a) = \sum_{A \in \mathcal{F}} |A|.$$

Table 2: Radius and Diameter of V_n

n	Center of V_n	Transitive?	ρ_{V_n}	δ_{V_n}
1	Unique element	Yes	0	0
2	2 dictatorship	No	1	1
3	Democracy	Yes	1	2
4	4 constitutional monarchies	No	3	4
5	Democracy	Yes	5	8
6	60 "icosahedral" families	Yes	11	16
7	Democracy	Yes	22	32
8	?	No	$\geq$ 47	64
9	Democracy	Yes	93	128
10	?	?	$\geq$ 193	256
11	Democracy	Yes	386	512
12	?	No	$\geq$ 793	1024
13	Democracy	Yes	1586	2048
14	?	?	$\geq$ 3238	4096
15	Democracy	Yes	6476	8192
16	?	No	$\geq$ 13167	16384
17	Democracy	Yes	26333	32768
18	?	?	$\geq$ 53381	65536
19	Democracy	Yes	106762	131072
20	?	?	$\geq$ 215955	262144

Then $d_{\mathcal{F}}$ is maximized for n odd by the democracy which takes the larger of each complementary pair and for n even by each of the $2^{\binom{n}{n/2}/2}$ games with a majority win and some rule for breaking tie votes. Easy binomial sums thus give

$$d \leq n\left[2^{n-2} + \binom{n-1}{(n-1)/2}/2\right]$$

for n odd and

$$d \leq n\left[2^{n-2} + \binom{n-1}{(n-2)/2}/2\right]$$

for n even with equality exactly in the cases described.

Suppose that $\mathcal{F}$ is in the center of V_n.

$$\begin{aligned}\rho_{V_n} &\geq \operatorname{dist}(\mathcal{F}, \mathcal{D}_i)\\ &= 2^{n-1} - d_{\mathcal{F}}(i)\\ &\geq 2^{n-1} - d/n\\ &\geq 2^{n-2} - \binom{n-1}{\lfloor (n-1)/2 \rfloor}/2\end{aligned}$$

The radius is bounded above by the diameter $\rho_{V_n} = 2^{n-2}$. We can subtract 1 from the diameter for $n > 2$ since there are the families with no dummies.

Proof of theorem **??***:*

(1) Let $\mathcal{M}$ denote the n voter democracy. As noted above, total voter power $d_{\mathcal{F}}$ attains its maximum for $\mathcal{F} = \mathcal{M}$. Moreover, by symmetry, all voters have equal power $d_{\mathcal{M}}(a) = D_{\mathcal{M}}/n = 2^{n-2} + \binom{n-1}{(n-1)/2}/2$. Thus, democracy is at distance $2^{n-1} - D_{\mathcal{M}}/n = 2^{n-2} - \binom{n-1}{(n-1)/2}/2$ from all dictatorships $\mathcal{D}_a$. All other maximal intersecting families are further away from some dictatorship. As we saw in the proof of proposition **??**, $\operatorname{dist}(\mathcal{M}, \mathcal{F})$ attains its maximum if and only if $\mathcal{F}$ is a dictatorship.

(2) A family achieves the lower bound for the average distance to dictatorships exactly if it has all majority coalitions and half of the coalitions of size $n/2$. This average is also the maximum distance to a dictatorship exactly when the family is regular, that is, a near democracy on an even number of voters which is not a power of two. If Chvátal's conjecture holds, no other family is further from that near dictatorship. For powers of two, the average is half an odd integer so the radius is bounded by the next largest integer. The near democracies are that far from some dictatorships and, if Chvátal's conjecture is true, no further from anything else. □

Open Questions: An open question (aside from Chvátal's conjecture!) is to pursue is the characterization of the center of V_n for n even. (See table **??**.) For n a power of two can there be families in the center which include a coalition of size $n/2 - 1$?

Notice that the center is not necessarily connected, nor is it closed with respect to the binary choice and trinary median functions defined above. (See for example V_4.) Indeed, when the families described in theorem **??** comprise the center, they are all in the same half of the bipartite graph V_n and hence are an independent set.

What is the average distance $\operatorname{dist}(\mathcal{F}, \mathcal{G})$ between two maximal intersecting families $\mathcal{F}$ and $\mathcal{G}$? That is to say, what is the expected correlation $\mathcal{P}(\mathcal{F}|\mathcal{G})$ between two maximal intersecting families?

8 Degree

The *degree* of a maximal intersecting family $\mathcal{F}$ in V_n is the number of vertices adjacent to $\mathcal{F}$ in V_n. This is also the number of minimal winning coalitions it has since switching the minimal winning coalitions of $\mathcal{F}$ yields a distinct adjacent maximal intersecting families and all adjacent vertices are obtained in this way.

What is the maximal degree in V_n? To answer this question, we need the following lemma.

Lemma 15 *1. Let $i \leq n/2$, and let $\mathcal{A}$ be an antichain of sets of cardinality at most i. Define the shade*

$$\nabla^{(i)}\mathcal{A} = \{A \subseteq X : |A| = i \text{ and } A \text{ contains some member of } \mathcal{A}.\}$$

Then $|\mathcal{A}| \leq |\nabla^{(i)}\mathcal{A}|$ with equality if and only if $\mathcal{A}$ consists exclusively of i element sets.

2. Let $i \geq n/2$, and let $\mathcal{A}$ be an antichain of sets of cardinality at least i. Define the shadow

$$\Delta^{(i)}\mathcal{A} = \{A \subseteq X : |A| = i \text{ and } A \text{ is contained by some member of } \mathcal{A}.\}$$

Then $|\mathcal{A}| \leq |\Delta^{(i)}\mathcal{A}|$ with equality if and only if $\mathcal{A}$ consists exclusively of i element sets.

Proof: Follows immediately by induction from a result of Sperner [**?**, **?**, p. 11].□

Theorem 16 *1. If n is odd, the maximal degree in V_n is $\binom{n}{(n+1)/2}$. This maximum is attained only for the democratic maximal intersecting family $\{A \subseteq X : |A| > n/2\}$.*

2. If n is even, the maximal degree in V_n is $\binom{n}{n/2-1}$. This maximum is attained only for the n maximal intersecting families $\mathcal{M}_a$ for $a \in X$ where $A \in \mathcal{M}_a$ if $|A| > n/2$, or if $|A| = n/2$ and $a \in A$.

Proof: **(1)** Apply Sperner's theorem [**?**] to the set of minimal winning coalitions of $\mathcal{F}$.

(2) Let $\mathcal{A}$ consist of those minimal winning coalitions of $\mathcal{F}$ with at most $n/2$ elements and let $\mathcal{B}$ consist of those minimal winning coalitions with over $n/2$ elements. Clearly, $\nabla^{(n/2)}\mathcal{A}$ is the set of winning $n/2$ element coalitions. By definition, $\nabla^{(n/2)}\mathcal{A}$ has $\binom{n}{n/2}/2$ elements—one from each complementary pair. By part 1 of Lemma **??**, $\mathcal{A}$ has at most $\binom{n}{n/2}/2$ members with equality if and only if there are no minority winning coalitions.

Let $\mathcal{D}$ be the collection of $n+1$ element coalitions which do *not* cover $\nabla^{(n/2)}\mathcal{A}$.

$$\mathcal{D} = \{D \subseteq X : |D| = n/2 + 1 \text{ and } \forall A \in \mathcal{A}, A \not\subseteq D\}.$$

By transitivity, $\Delta^{(n/2+1)}\mathcal{B} \subseteq \mathcal{D}$. Moreover, by part 2 of Lemma **??**, $|\Delta^{(n/2+1)}\mathcal{B}| \geq |\mathcal{B}|$.

The compression lemma [**?**, p.119] implies that $\binom{n-1}{i-1}$ i-element subsets of an n-element set of contained by at least $\binom{n-1}{i}$ $i+1$-element subsets with equality if and only if all of the given subsets contain a common element.

Thus, $\mathcal{D}$ contains at most $\binom{n}{n/2+1} - \binom{n}{n/2}/2 = \binom{n-1}{n/2+1}$ members. In all, the cardinality of the set of minimal winning coalitions $\mathcal{A} \cup \mathcal{B}$ is at most $\binom{n}{n/2}/2 + \binom{n-1}{n/2+1} = \binom{n}{n/2+1}$. Moreover, the equalities hold only if there are no minority wins and all n-element wins contain a common voter.

The only such maximal intersecting family is $\mathcal{M}_a$. $\mathcal{M}_a$ is the voting scheme in which voter a is given two votes and everyone else is given one vote; $A \in \mathcal{F}$ if $|A| > n/2$ or if $|A| = n/2$ and $a \in A$. There are thus $\binom{n-1}{n/2} = \binom{n-1}{n/2-1}$ minimal coalitions not involving a, and $\binom{n-1}{n/2-2}$ minimal coalitions involving a for a total of $\binom{n}{n/2+1}$ minimal winning coalitions.$\square$

Open Question: What is the average degree of a maximal intersecting family? That is to say, what is the expected number of minimal elements of $\mathcal{F}$? Given an asymptotic formula for the average degree, we could deduce an asymptotic formula for the number of edges in V_n, or visa versa.

Table 3: Average Degree

n	0	1	2	3	4	5	6	7
number of vertices in V_n	0	1	2	4	12	81	2 646	1 422 564
number of edges in V_n	0	0	1	3	16	185	10 886	10 552 451
approx. avg. degree	-	0.000	1.000	1.500	2.667	4.568	8.228	14.836
minimal degree	-	0	1	1	1	1	1	1
maximal degree	-	0	1	3	4	10	15	35

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