Stochastic aspects of the Josephus formula

Josephus problem: n numbered persons (1, 2,…, n) are placed clockwise in a circle. Starting with number 1, every n^{th} person is removed (killed) except the survivor with number a(n).
For which index n is the survival number a(n) equal to a given value m?
Let b(m,k), k=1,2,.., be the sequence defined by a(b(m,k)) = m.

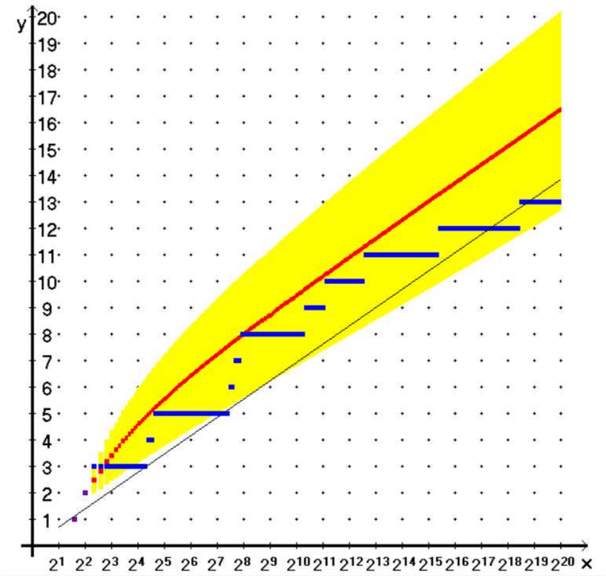
Examples:
b(1,k)= 1, 2, end
b(2,k)= 3, 4, 5, 20, 24, 173, 197, 236, 1264, 2138, 6042, 42547, 353069, ..
b(3,k) is empty.
b(4,k)= 6, 8, 15, 28, 40, 192, 1536, 2211, 2222, 29017, 33965, 154483, 251402, 326675, 346606, ..
b(26,k) = 27, 29, 33, 55, 113, 165, 206, 248, 527, 782, 3092, 4425, 15812, 17227, 107288, 235892, 301615, 348207, 355889, 415733, 916682, ..

Visualization:

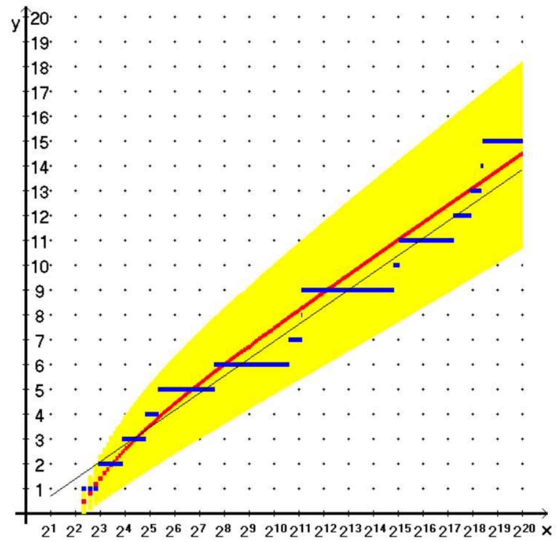
In the diagrams (blue lines)
x is the number of persons on a logarithmic scale
f(m,x) is the frequency of survival cases a(n)=m with n≤x

$$f(m,x)= \begin{cases} 0 & x \le m \\ f(m,x-1)+1 & a(x)=m \\ f(m,x-1) & \text{else} \end{cases}$$

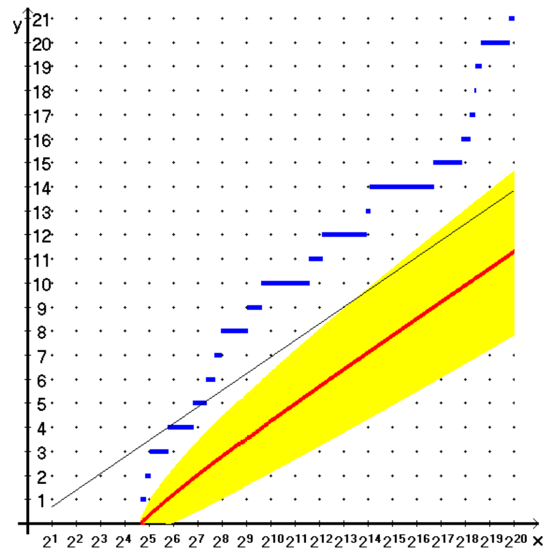
b(m,k) is the sequence of jump discontinuities on the x-axis.
Straight line→ log(x)
See stochastic model:
red line→ expected frequency g(m,x)
yellow area→ g(m,x)± σ(m,x))



sequence b(2,k)



sequence b(4,k)



sequence b(26,k)

Stochastic model

The sequence of victims in the removing process with n persons starts with n,1,3,6,.., $\frac{1}{2}r(n)(r(n)-1) < n$. Example n=9: 9,1,3,6, .. with r(9)=4. r(n) is the number of easily predictable victims.
Instead of continuing the removing process we randomly choose one of the other persons, say number m, to survive. Before the choice, the probability of survival is $p(n)= \frac{1}{n-r(n)}$.

Table for 3≤n≤8:

n	3	4	5	6	7	8
r(n)	2	3	3	3	4	4
possible m	2	2	2, 4	2, 4, 5	2, 4, 5	2, 4, 5, 7
p(n)	1	1	½	⅓	⅓	¼

The true function f(m,x) is modeled by a function of expected values $g(m,x)= \sum_{n=m+1}^{x} p(n)$ with the variance

$$v(m,x) = \sum_{n=m+1}^{x} p(n)\cdot(1-p(n)) \text{ and the standard deviation } \sigma(m,x)= \sqrt{v(m,x)}.$$

Discussion

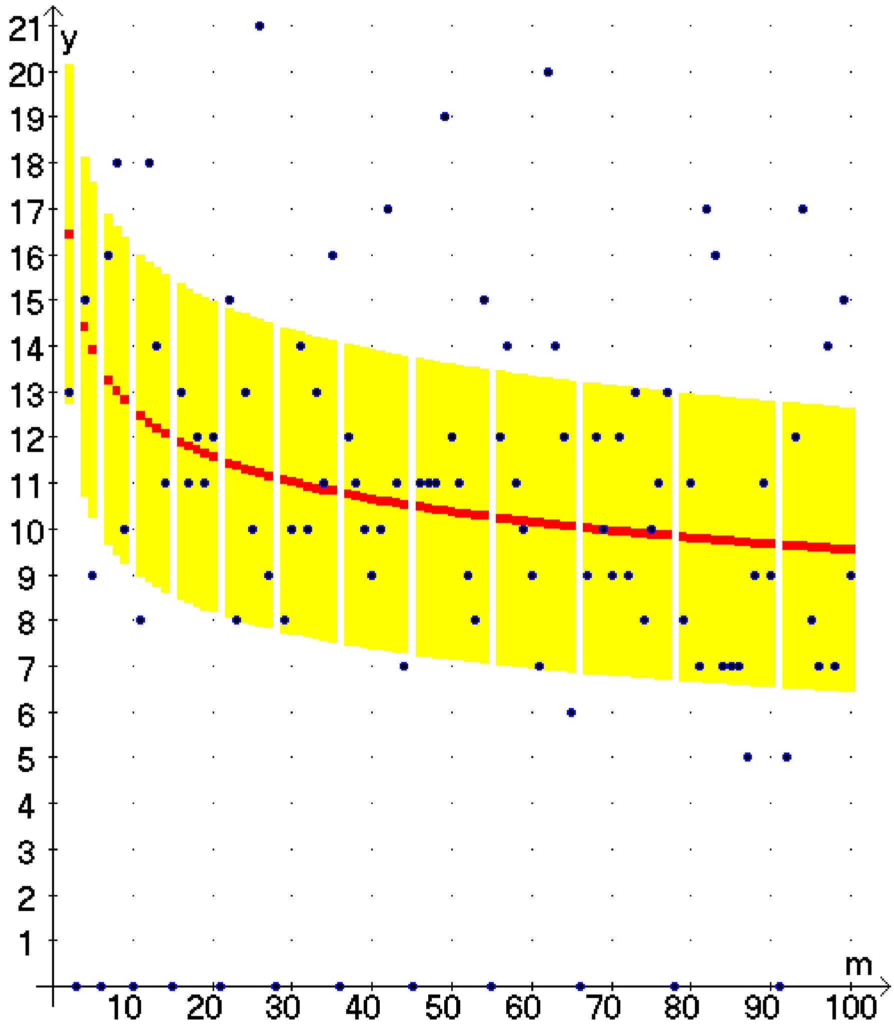
m = 2, 4
The true frequency deviates not too much from the expected one. If this local trend holds for x→∞ then f(x)→∞ because g(x) tends to the harmonic series (with an additional constant) and so: g(x)=O(log(x)).
Therefore b(m,k) is likely to be extendable for each k>0.

m =26
This number is chosen as the “local winner” with 21 terms ($f(m,2^{20})=21$):
A Monte Carlo simulation (1000 runs) based on the stochastic model produced m_{stoch} = 125±524, which simply means that the winner number varies a lot, but the corresponding number of terms was 21±2, in good accordance with the true maximum.

m=½j*(j-1), j>2
m is a “global” victim number. A person with such a number never survives: b(m,k) is empty. This has been considered in the stochastic model.

m = 1699
This is the smallest “local” victim number: If b(m,1) exists, it is > 2^{20} - equivalent to $f(m,2^{20})=0$. Monte Carlo yielded m_{stoch} = 1368±698.

m = 2,3,4, .. , 100 (general view)
The last diagram shows y(m) = f(m,2^20), i.e. how many terms of the sequence b(m,k) exist with b(m,k) ≤ 2^20 (blue spots). Note the global victims on the m-axis. The red spots mark the expected values of y(m) and the yellow areas the standard deviation. Distribution of the 99 blue spots: 12 on the m-axis, 22 outside the yellow areas and 65 inside, which is a fraction $^{65}/_{87}$ = 74.7%. This is a value one would roughly expect if the blue spots were randomly scattered around the red line.



y = f(m,2^20)

Summary: The stochastic model seems to be reasonable.
Where does this come from?

Formula: a(n) = t(n,n) + 1 with the recurrence t(k,n) = (t(k-1,n) + n) mod k; t(1,n) = 0. Each step of the recurrence can be thought of as a simple linear congruential generator for pseudo-random numbers. t(n,n) is created by a chain of such generators with an increasing modulus k and this chain seems to be a good generator which simulates, depending on n, the same survival probability for any number x ≠ n,1,3,6,10,15,..(<n).