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Labelle & Leroux

Identities ....

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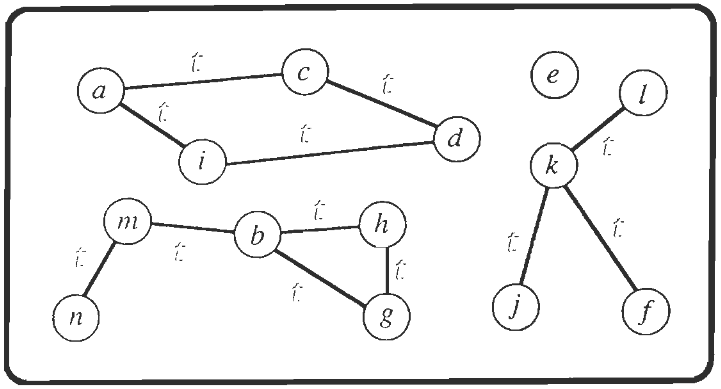
IDENTITIES AND ENUMERATION :

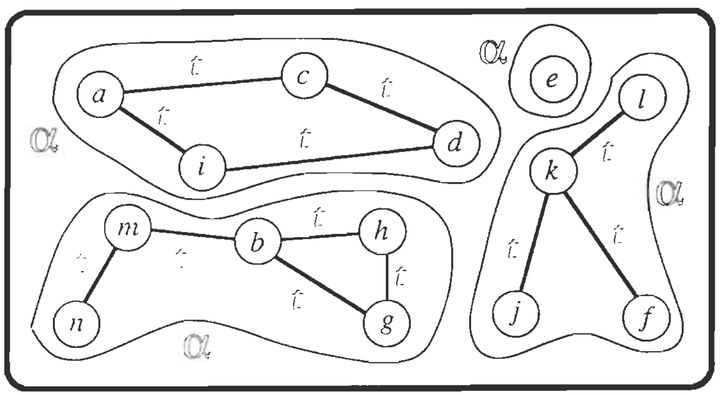
WEIGHTING CONNECTED COMPONENTS

Gilbert LABELLE* & Pierre LEROUX

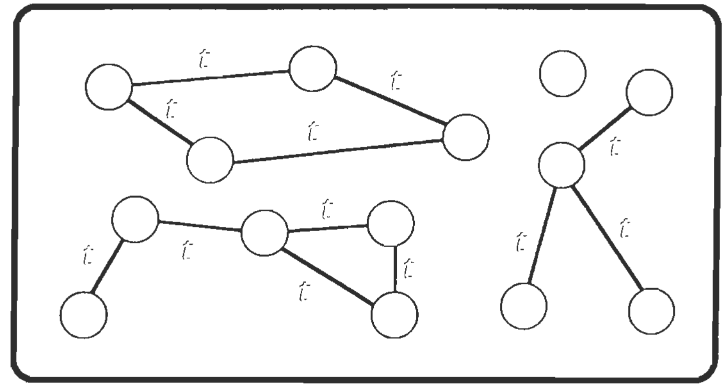
LACIM - UQAM



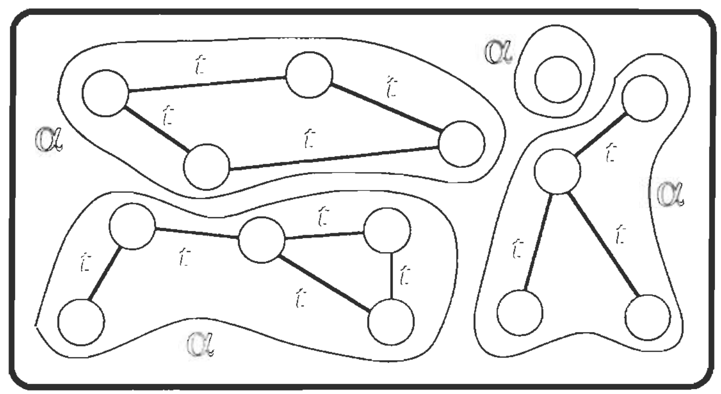
$$w = t^{12}$$



$$w^{(\alpha)} = \alpha^4 t^{12}$$



$$w = t^{12}$$



$$w^{(\alpha)} = \alpha^4 t^{12}$$

$$Gr_w = E(Gr_w^c) \longmapsto Gr_{w(\alpha)} = E(Gr_{\alpha w}^c)$$

More generally, we want to study

$$F_w = E(F_w^c) \longmapsto F_{w(\alpha)} = E(F_{\alpha w}^c)$$

for weighted species F_w

Informal definition :

A (weighted) species is a class of combinatorial structures that is closed under (weight-preserving) isomorphisms

SERIES ASSOCIATED TO SPECIES

André Joyal (≈ 1980), Gilbert Labelle (≈ 1990)

- ***Cycle index series*** **of a (weighted) species** F **:**

$$Z_F(x_1, x_2, x_3, \ldots) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} f_\sigma\, x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} \cdots$$

- ***Asymmetry index series*** **of** F **:**

$$\Gamma_F(x_1, x_2, x_3, \ldots) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} f_\sigma^*\, x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} \cdots$$

$$Z_F(x, 0, 0, \ldots) = \Gamma_F(x, 0, 0, \ldots) = F(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}$$

$$Z_F(x, x^2, x^3, \ldots) = \tilde{F}(x) = \sum_{n \geq 0} \tilde{f}_n x^n$$

$$\Gamma_F(x, x^2, x^3, \ldots) = \bar{F}(x) = \sum_{n \geq 0} \bar{f}_n x^n$$

f_n = the number (or total weight) of the labelled *F-structures* on $[n] = \{1, 2, \ldots, n\}$,

$\tilde{f}_n$ = the number (or total weight) of the unlabelled *F-structures* on n nodes,

$\bar{f}_n$ = the number (or total weight) of the unlabelled asymmetric *F-structures* on n nodes.

q-SERIES ASSOCIATED TO SPECIES

Hélène Décoste (≈ 1990)

- *Canonical q-series* of a (weighted) species F :

$$F(x,q) = \sum_{n=0}^{\infty} f_n(q)\frac{x^n}{n!_q} = Z_F\left(\frac{(1-q)}{(1-q)}x, \frac{(1-q)^2}{(1-q^2)}x^2, \frac{(1-q)^3}{(1-q^3)}x^3, \ldots\right)$$

- *Canonical asymmetry q-series* of F :

$$F\langle x,q\rangle = \sum_{n=0}^{\infty} f_n\langle q\rangle\frac{x^n}{n!_q} = \Gamma_F\left(\frac{(1-q)}{(1-q)}x, \frac{(1-q)^2}{(1-q^2)}x^2, \frac{(1-q)^3}{(1-q^3)}x^3, \ldots\right)$$

$$n!_q = \frac{(1-q)}{(1-q)}\frac{(1-q^2)}{(1-q)}\frac{(1-q^3)}{(1-q)}\cdots\frac{(1-q^n)}{(1-q)}, \qquad n! = \lim_{q\to 1} n!_q$$

$$\lim_{q\to 1} F(x,q) = F(x), \quad \lim_{q\to 0} F(x,q) = \tilde{F}(x),$$

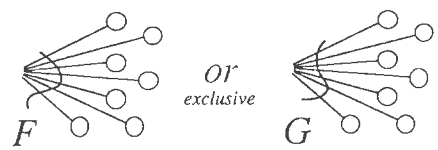
$$\lim_{q\to 1} F\langle x,q\rangle = F(x), \quad \lim_{q\to 0} F\langle x,q\rangle = \bar{F}(x).$$

$$f_n(q) \in \mathbb{N}_w[q], \qquad \deg f_n(q) \le n(n-1)/2,$$

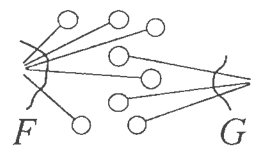
$$f_n\langle q\rangle \in \mathbb{Z}_w[q], \qquad \deg f_n\langle q\rangle \le n(n-1)/2.$$

MAIN OPERATIONS ON SPECIES

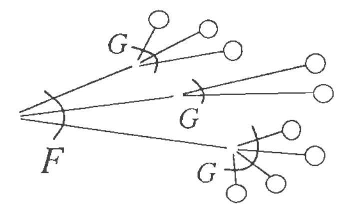
(F+G) – structure:



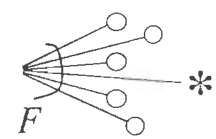
(F·G) – structure:



(F∘G) – structure:



F′ – structure:



IDENTITIES AND ENUMERATION

$$Z_{F+G} = Z_F + Z_G \qquad \Gamma_{F+G} = \Gamma_F + \Gamma_G$$
$$Z_{F\cdot G} = Z_F \cdot Z_G \qquad \Gamma_{F\cdot G} = \Gamma_F \cdot \Gamma_G$$
$$Z_{F\circ G} = Z_F \circ Z_G \qquad \Gamma_{F\circ G} = \Gamma_F \circ \Gamma_G$$
$$Z_{F'} = \frac{\partial Z_F}{\partial x_1} \qquad \Gamma_{F'} = \frac{\partial \Gamma_F}{\partial x_1}$$

$$(f_w \circ g_v)(x_1, x_2, x_3, \ldots) := f_w(g_v(x_1, x_2, x_3, \ldots),\ g_{v^2}(x_2, x_4, x_6, \ldots),\ g_{v^3}(x_3, x_6, x_9, \ldots), \ldots)$$

A combinatorial identity between species

$F,\ G,\ H,\ \ldots$

gives — gratis —

an analytical identity between the series

$$Z_F, Z_G, Z_H, \ldots, \qquad \Gamma_F, \Gamma_G, \Gamma_H, \ldots,$$
$$F(x),\ G(x),\ H(x),\ \ldots,$$
$$\tilde{F}(x),\ \tilde{G}(x),\ \tilde{H}(x),\ \ldots, \qquad \bar{F}(x),\ \bar{G}(x),\ \bar{H}(x),\ \ldots,$$
$$F(x,q),\ G(x,q),\ H(x,q),\ \ldots, \qquad F\langle x,q\rangle,\ G\langle x,q\rangle,\ H\langle x,q\rangle,\ \ldots.$$

THEOREM (Weighting connected components)

$\exists!$ (universal virtual weighted) species, $\Lambda^{(\alpha)}$, such that

$$F_{w^{(\alpha)}} = \Lambda^{(\alpha)} \circ F_w^{+}.$$

Moreover,

1) $$\Lambda^{(\alpha)}(x) = (1+x)^{\alpha},$$

2) $$Z_{\Lambda^{(\alpha)}} = \prod_{n\geq 1} (1+x_n)^{\lambda_n(\alpha)},$$

3) $$\widetilde{\Lambda^{(\alpha)}}(x) = \prod_{n\geq 1} (1+x^n)^{\lambda_n(\alpha)},$$

4) $$\Lambda^{(\alpha)}(x,q) = \prod_{n\geq 1} \left(1 + \frac{(1-q)^n}{(1-q^n)} x^n\right)^{\lambda_n(\alpha)},$$

5) $$\Gamma_{\Lambda^{(\alpha)}} = \prod_{n\geq 1} (1+x_n)^{\gamma_n(\alpha)},$$

6) $$\overline{\Lambda^{(\alpha)}}(x) = \prod_{n\geq 1} (1+x^n)^{\gamma_n(\alpha)},$$

7) $$\Lambda^{(\alpha)}\langle x,q\rangle = \prod_{n\geq 1} \left(1 + \frac{(1-q)^n}{(1-q^n)} x^n\right)^{\gamma_n(\alpha)},$$

where

$$\lambda_n(\alpha) = \frac{1}{n}\sum_{d\mid n} \mu(n/d)\,\alpha^d = \begin{cases}\text{number of Lyndon words}\\ \text{of length } n \text{ over "}\alpha\text{" letters,}\end{cases}$$

$$\gamma_n(\alpha) = -\lambda_n(-\alpha) - \lambda_{n/2}(-\alpha) - \lambda_{n/4}(-\alpha) - \cdots.$$

PROOF (Sketch). Let

X_α = the species of singletons having weight α,
E = the species of sets,
$E^+ = E - 1$ = the species of non-empty sets,
$(E^+)^{<-1>}$ = the inverse of E^+ under substitution,
(Joyal, SLN 1234)

$$F_w = E(F_w^c) = 1 + E^+(F_w^c),$$
$$F_w^+ = F_w - 1 = E^+(F_w^c).$$

Define
$$\Lambda^{(\alpha)} := E \circ X_\alpha \circ (E^+)^{<-1>}.$$
Then
$$F_{w(\alpha)} = E(F_{\alpha w}^c) = E \circ X_\alpha \circ F_w^c$$
$$= E \circ X_\alpha \circ (E^+)^{<-1>} \circ F_w^+$$
$$= \Lambda^{(\alpha)} \circ F_w^+ .$$

The other formulas follow from computations, using

$$Z_{(E^+)^{<-1>}} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1 + x_n), \qquad \text{etc} .$$

CONSEQUENCES of $F_{w(\alpha)} = \Lambda^{(\alpha)} \circ F_w^+$.

1) $$F_{w(\alpha)}(x) = F_w(x)^\alpha,$$

2) $$Z_{F_{w(\alpha)}}(x_1, x_2, \ldots) = \prod_{n \geq 1} Z_{F_{w^n}}(x_n, x_{2n}, \ldots)^{\lambda_n(\alpha)},$$

3) $$\widetilde{F_{w(\alpha)}}(x) = \prod_{n \geq 1} \widetilde{F_{w^n}}(x^n)^{\lambda_n(\alpha)},$$

4) $$F_{w(\alpha)}(x, q) = \prod_{n \geq 1} F_{w^n}\left(\frac{(1-q)^n}{(1-q^n)} x^n, q^n\right)^{\lambda_n(\alpha)},$$

5) $$\Gamma_{F_{w(\alpha)}}(x_1, x_2, \ldots) = \prod_{n \geq 1} \Gamma_{F_{w^n}}(x_n, x_{2n}, \ldots)^{\gamma_n(\alpha)},$$

6) $$\overline{F_{w(\alpha)}}(x) = \prod_{n \geq 1} \overline{F_{w^n}}(x^n)^{\gamma_n(\alpha)},$$

7) $$F_{w(\alpha)}\langle x, q\rangle = \prod_{n \geq 1} F_{w^n}\left\langle \frac{(1-q)^n}{(1-q^n)} x^n, q^n \right\rangle^{\gamma_n(\alpha)} .$$

SETS : $Z_E = \exp(\sum_{n\geq 1} x_n/n)$, $\Gamma_E = \exp(\sum_{n\geq 1} (-1)^{n-1} x_n/n)$.

CHOOSE $F_w = E$ **in** $F_{w^{(\alpha)}} = \Lambda^{(\alpha)} \circ F_w^+$.

1) $$e^{\alpha x} = (e^x)^\alpha ,$$

2) $$e^{\sum_{k=1}^{\infty} \alpha^k x_k / k} = \prod_{n\geq 1} e^{\lambda_n(\alpha) \sum_{k=1}^{\infty} x_{kn}/k} ,$$

3) $$\frac{1}{1-\alpha x} = \prod_{n\geq 1} \left(\frac{1}{1-x^n}\right)^{\lambda_n(\alpha)} ,$$

(cyclotomic identity)

4) $$E(\alpha x, q) = E(x,q)^\alpha \cdot E\left(\tfrac{(1-q)^2}{(1-q^2)} x^2, q^2\right)^{\lambda_2(\alpha)} \cdot E\left(\tfrac{(1-q)^3}{(1-q^3)} x^3, q^3\right)^{\lambda_3(\alpha)} \cdot \cdots ,$$

(q-cyclotomic identity)

5) $$e^{\sum_{k=1}^{\infty} (-1)^{k-1} \alpha^k x_k / k} = \prod_{n\geq 1} e^{\gamma_n(\alpha) \sum_{k=1}^{\infty} (-1)^{k-1} x_{kn}/k} ,$$

6) $$(1+\alpha x) = \prod_{n\geq 1} (1+x^n)^{\gamma_n(\alpha)} ,$$

(cocyclotomic identity)

7) $$E\langle \alpha x, q\rangle = E\langle x,q\rangle^\alpha \cdot E\left\langle \tfrac{(1-q)^2}{(1-q^2)} x^2, q^2\right\rangle^{\gamma_2(\alpha)} \cdot E\left\langle \tfrac{(1-q)^3}{(1-q^3)} x^3, q^3\right\rangle^{\gamma_3(\alpha)} \cdot \cdots ,$$

(q-cocyclotomic identity).

COROLLARY (Weighting connected components by -1)

$\exists!$ (universal virtual weighted) species, $\Lambda^{(-1)}$, such that

$$F_{w^{(-1)}} = \Lambda^{(-1)} \circ F_w^+ .$$

Moreover,

1) $$\Lambda^{(-1)}(x) = 1/(1+x),$$

2) $$Z_{\Lambda^{(-1)}} = (1+x_2)/(1+x_1),$$

3) $$\widetilde{\Lambda^{(-1)}}(x) = (1+x^2)/(1+x),$$

4) $$\Lambda^{(-1)}(x,q) = \left(1+\left(\tfrac{1-q}{1+q}\right)x^2\right)/(1+x),$$

5) $$\Gamma_{\Lambda^{(-1)}} = \frac{1}{(1+x_1)\cdot(1+x_2)\cdot(1+x_4)\cdot(1+x_8)\cdot \cdots},$$

6) $$\overline{\Lambda^{(-1)}}(x) = 1-x,$$

7) $$\Lambda^{(-1)}\langle x,q\rangle = 1 \Big/ \prod_{k\geq 0}\left(1+\frac{(1-q)^{2^k}}{(1-q^{2^k})} x^{2^k}\right).$$

Proof :

$$\lambda_n(-1) = \begin{cases} -1, & \text{if } n=1, \\ 1, & \text{if } n=2, \\ 0, & \text{if } n\geq 3, \end{cases} = \begin{cases} \text{number of Lyndon words} \\ \text{of length } n \text{ over " } -1 \text{ " letters,} \end{cases}$$

$$\gamma_n(-1) = -\chi(n \text{ is a power of } 2).$$

CONSEQUENCES of $F_{w^{(-1)}} = \Lambda^{(-1)} \circ F_w^+$.

1) $$F_{w^{(-1)}}(x) = 1 / F_w(x),$$

2) $$Z_{F_{w^{(-1)}}}(x_1, x_2, \ldots) = \frac{Z_{F_{w^2}}(x_2, x_4, x_6, \ldots)}{Z_{F_w}(x_1, x_2, x_3, \ldots)},$$

3) $$\widetilde{F_{w^{(-1)}}}(x) = \frac{\widetilde{F_{w^2}}(x^2)}{\widetilde{F_w}(x)},$$

4) $$F_{w^{(-1)}}(x, q) = \frac{F_{w^2}((\frac{1-q}{1+q})x^2, q^2)}{F_w(x, q)},$$

5) $$\Gamma_{F_{w^{(-1)}}}(x_1, x_2, \ldots) = 1 / \prod_{k \geq 0} \Gamma_{F_{w^{2^k}}}(x_{2^k}, x_{2 \cdot 2^k}, x_{3 \cdot 2^k}, \ldots),$$

6) $$\overline{F_{w^{(-1)}}}(x) = 1 / \prod_{k \geq 0} \overline{F_{w^{2^k}}}(x^{2^k}),$$

7) $$F_{w^{(-1)}}\left\langle x, q \right\rangle = 1 / \prod_{k \geq 0} F_{w^{2^k}}\left\langle \frac{(1-q)^{2^k}}{(1-q^{2^k})} x^{2^k}, q^{2^k} \right\rangle.$$

PERMUTATIONS : $Z_S = \prod_{n \geq 1} \frac{1}{(1 - x_n)}$, $\Gamma_S = \frac{(1 - x_2)}{(1 - x_1)}$.

CHOOSE $F_w = S$ in $S_{w^{(-1)}} = \Lambda^{(-1)} \circ S_w^+$.

1) $$S_{w^{(-1)}}(x) = 1 - x,$$

2) $$Z_{S_{w^{(-1)}}}(x_1, x_2, x_3, \ldots) = (1 - x_1)(1 - x_3)(1 - x_5) \cdots ,$$

3) $$\widetilde{S_{w^{(-1)}}}(x) = (1 - x)(1 - x^3)(1 - x^5) \cdots ,$$

4) $$S_{w^{(-1)}}(x, q) = (1 - x)(1 - \tfrac{(1-q)^3}{(1-q^3)} x^3)(1 - \tfrac{(1-q)^5}{(1-q^5)} x^5) \cdots ,$$

5) $$\Gamma_{S_{w^{(-1)}}}(x_1, x_2, x_3, \ldots) = 1 - x_1,$$

6) $$\overline{S_{w^{(-1)}}}(x) = 1 - x,$$

7) $$S_{w^{(-1)}}\left\langle x, q \right\rangle = 1 - x.$$

ANALYSIS of $\Lambda^{(\alpha)}$

$$F_{w^{(\alpha)}} = \Lambda^{(\alpha)} \circ F_w^+ , \qquad \Lambda^{(\alpha)} = E \circ X_\alpha \circ (E^+)^{<-1>}$$

1) $$\Lambda^{(\alpha)} \circ \Lambda^{(\beta)} = \Lambda^{(\alpha\beta)},$$

$$\left((1+x)^\alpha\right)^\beta = (1+x)^{\alpha\beta}.$$

2) $$\Lambda^{(\alpha)} \cdot \Lambda^{(\beta)} \neq \Lambda^{(\alpha+\beta)},$$

Although $(1+x)^\alpha (1+x)^\beta = (1+x)^{\alpha+\beta}$.

COROLLARY of 1) : The number of Lyndon words of length n over $\alpha\beta$ letters satisfies

$$\lambda_n(\alpha\beta) = \sum_{i \cdot j = n} \lambda_i(\alpha^j)\, \lambda_j(\beta).$$

REASON for 2) : $\lambda_n(\alpha+\beta) \neq \lambda_n(\alpha) + \lambda_n(\beta)$, if $n \geq 2$.

Note [Metropolis and Rota 1983] :

$$\lambda_n(\alpha\beta) = \sum_{[i,j]=n} (i,j)\, \lambda_i(\alpha)\, \lambda_j(\beta).$$

3) Molecular decomposition of $\Lambda^{(\alpha)}$ versus its generating series $\Lambda^{(\alpha)}(x)$:

$$\Lambda^{(\alpha)} = E \circ X_\alpha \circ (E^+)^{<-1>}$$

(Computed up to degree 7 by Y. Chiricota and Maple.)

$$\begin{aligned} &= 1 + X_\alpha \\ &\quad + (E_2)_{\alpha^2} - (E_2)_\alpha \\ &\quad + (E_3)_{\alpha^3} - (E_3)_\alpha + (XE_2)_\alpha - (XE_2)_{\alpha^2} \\ &\quad + (E_4)_{\alpha^4} - (E_4)_\alpha + (XE_3)_\alpha - (XE_3)_{\alpha^2} \\ &\quad + (X^2E_2)_{\alpha^2} - (X^2E_2)_\alpha + (E_2^2)_{\alpha^2} - (E_2^2)_{\alpha^3} \\ &\quad + (E_2 \circ E_2)_\alpha - (E_2 \circ E_2)_{\alpha^2} \\ &\quad + \cdots . \end{aligned}$$

$$\Lambda^{(\alpha)}(x) = e^{\alpha \log(1+x)} = (1+x)^\alpha$$

$$= 1 + \alpha x + \alpha(\alpha-1)\frac{x^2}{2!} + \alpha(\alpha-1)(\alpha-2)\frac{x^3}{3!}$$
$$+ \alpha(\alpha-1)(\alpha-2)(\alpha-3)\frac{x^4}{4!} + \cdots .$$

4) In fact, $\Lambda^{(\alpha)}$ is a weighted "set-like" species.

m_n = number of *molecular* set-like species of degree n.

a_n = number of *atomic* set-like species of degree n.

$$\begin{cases} \displaystyle\sum_{n\geq 0} m_n x^n = \prod_{k\geq 1} \frac{1}{(1-x^k)^{a_k}} \\ \displaystyle\sum_{n\geq 1} \frac{a_n}{n^s} = 1 + (\zeta(s)-1)\cdot \sum_{n\geq 1} \frac{m_n}{n^s} \end{cases}$$

For $n \geq 2$,
$$\begin{cases} \displaystyle m_n = \frac{1}{n}\sum_{k=1}^{n} \Big(\sum_{d\mid k} d a_d\Big) m_{n-k} \\ \displaystyle a_n = \sum_{\substack{k\mid n \\ k<n}} m_{n/k} \end{cases}$$

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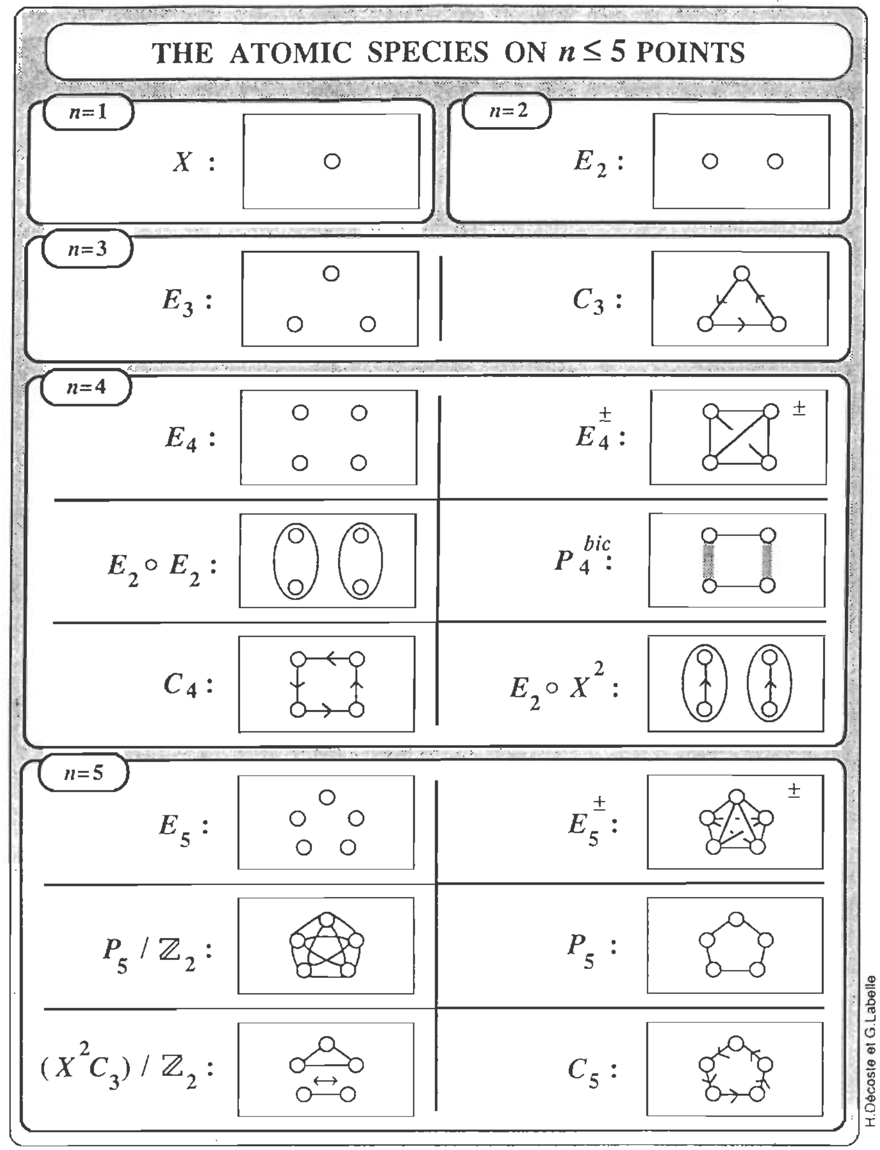
m(n) for n = 0 .. 40 :

1, 1, 2, 3, 7, 9, 20, 26, 54, 74, 137, 184, 356, 473, 841, 1154, 2034,
2742, 4740, 6405, 10874, 14794, 24515, 33246, 54955, 74380, 120501,
163828, 263144, 356621, 567330, 768854, 1212354, 1644335, 2567636,
3478873, 5403223, 7314662, 11265825, 15258443, 23363143

a(n) for n = 0 .. 40 :

0, 1, 1, 1, 3, 1, 6, 1, 10, 4, 12, 1, 33, 1, 29, 13, 64, 1, 100, 1, 156,
30, 187, 1, 443, 10, 476, 78, 877, 1, 1326, 1, 2098, 188, 2745, 36,
5203, 1, 6408, 477, 11084

These sequences are *not* in Sloane's book (1973).



MOLECULAR DECOMPOSITION OF THE SUBSTITIONAL INVERSE OF THE SPECIES OF NON-EMPTY SETS

1, X

2, $-E2$

3, $X\,E2 - E3$

4, $-X^2\,E2 + X\,E3 - E4 + E2(E2)$

5, $-X\,E2^2 + E2\,E3 + X^3\,E2 - X^2\,E3 + X\,E4 - E5$

6, $2\,X^2\,E2^2 - 2\,X\,E2\,E3 + E2\,E4 - E2(E2)\,E2 - X^4\,E2 + X^3\,E3 - X^2\,E4 + X\,E5 - E6$
$+ E3(E2) - E2(X\,E2) + E2(E3)$

7, $X\,E2^3 - E2^2\,E3 - 2\,X^3\,E2^2 + 3\,X^2\,E2\,E3 - 2\,X\,E2\,E4 + E2\,E5 - X\,E3^2 + E3\,E4$
$+ X^5\,E2 - X^4\,E3 + X^3\,E4 - X^2\,E5 + X\,E6 - E7$

8, $-2\,X^2\,E2^3 + 3\,X\,E2^2\,E3 - E2^2\,E4 - E2\,E3^2 + E2(E2)\,E2^2 + 2\,X^4\,E2^2 - 4\,X^3\,E2\,E3$
$+ 3\,X^2\,E2\,E4 - 2\,X\,E2\,E5 + E2\,E6 + 2\,X^2\,E3^2 - 2\,X\,E3\,E4 + E3\,E5 - E3(E2)\,E2$
$- X^6\,E2 + X^5\,E3 - X^4\,E4 + X^3\,E5 - X^2\,E6 + X\,E7 - E8 + E4(E2) + E2(E4)$
$- E2(X\,E3) + E2(X^2\,E2) - E2(E2(E2))$

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