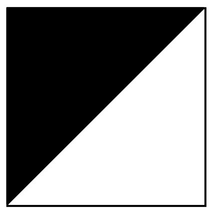
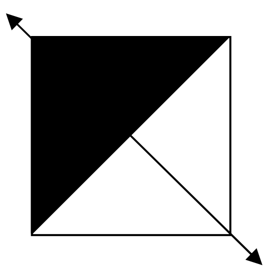
Unique symmetrical triangle quilt patterns along the diagonal of an nxn square

Tom Young (tyoung@isd16.org)

One can create a quilt square by taking a square piece of cloth and making a shaded triangle over one corner of that original square cloth. The basic shape of such a quilt square could look like:



Looking at this 1x1 square in terms of its diagonal symmetry (as shown below), there is only one unique way to construct the shaded triangle, pointing upwards as is shown. The shaded triangle could be the down pointing triangle, but that is not unique. It is the same as the upward pointing triangle through a 180 degree rotation.

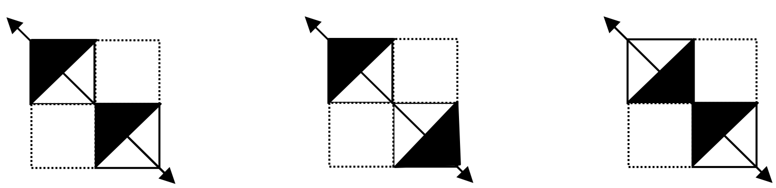


Introducing notation, the shaded triangle can either point up (U) or point down (D). In the 1x1 square the shaded triangle can have two states:

U or D

These states are identical through a 180 degree rotation. Another way of looking at this is to "Read the state backwards, changing the 'polarity' of each sub-state." So if you read D backwards and change the 'polarity' of D you get U!

This becomes clearer when looking at higher n. Examine the 2x2 square and its 2 smaller squares along its diagonal.



UU UD DU

One would think that there should be four configurations: UU, UD, DU, and DD. However, DD is the same as UU. It is obvious when you draw it out and rotate it but since you can read DD backwards and change the 'polarity' (D to U) as you read, you can see that it is the same as UU. DD and UU are duals of each other.

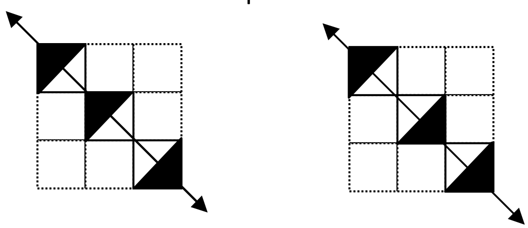
This "read backwards and change polarity" doesn't work for the other two. UD read backwards and changing polarity is still UD. It is its own dual. Similarly, DU read backwards and changing polarity is still DU. It also is its own dual and therefore unique.

The idea of duals becomes plainer with increasing n.

Looking at the 3x3 squares, there are 4 unique configurations. These can be found by examining the 2^3 = 8 possible configurations: UUU, UUD, UDU, DUU, DDU, DUD, UDD, DDD.

UUU is the dual of DDD
UUD is the dual of UDD reading UUD backwards and changing polarity gives UDD
UDU is the dual of DUD
DUU is the dual of DDU

This is demonstrated with the pictures of UUD and UDD:



UUD same as UDD through rotation

Note: none of the 8 possible configurations are duals of themselves. **This can only happen if there are an even number of U-D sub-states and an equal number of U and D states**. This can be shown in more detail with the 4x4 case.

There are 2^4 = 16 possible configurations of the 4x4 quilt square diagonal. They are:

UUUU, DDDD these are duals of each other
UUUD, UDDD
UUDU, DUDD
UDUU, DDUD
DUUU, DDDU
DUUD, UDDU
UUDD these are duals of themselves
UDUD
DUDU
DDUU

The 'read backwards and change polarity' rule establishes the existence of the dual. For a pattern to have its own dual, each U-D sub-state at the beginning has to have the opposite sub-state at the end. For instance, look at creating the UUDD state.

Step one: U_ _ _
Step Two: U_ _ D the end must be opposite polarity of the beginning
Step Three: UU_D
Step Four: UUDD the third sub-state is the opposite of the second

Each of the first two sub-states has its opposite in the 4th and 3rd places. There are, therefore, only 2^2 states that have their own duals

Looking at the 5x5 case shows again that there are 2^5 = 32 possible configurations but only 16 actual symmetrical quilt patterns.

1. UUUUU, DDDDD these are duals of each other
2. UUUUD, UDDDD
3. UUUDU, DUDDD
4. UUDUU, DDUDD

5. UDUUU, DDDUD
6. DUUUU, DDDDU
7. UUUDD, UUDDD
8. UUDUD, UDUDD
9. UDUUD, UDDUD
10. DUUUD, UDDDU
11. UUDDU, DUUDD
12. UDUDU, DUDUD
13. DUUDU, DUDDU
14. UDDUU, DDUUD
15. DUDUU, DDUDU
16. DDUUU, DDDUU

Examining the 6x6 case illustrates the “read backwards and change polarity” rule and again shows that a state can be its own dual only if there are an even number of U-D sub-states and an equal number of U-D sub-states.

There are 2^6 = 64 possible configurations, but some of these are duals of themselves.

1. UUUUUU, DDDDDD
2. UUUUUD, UDDDDD
3. UUUUDU, DUDDDD
4. UUUDUU, DDUDDD
5. UUDUUU, DDDUDD
6. UDUUUU, DDDDUD
7. DUUUUU, DDDDDU
8. UUUUDD, UUDDDD
9. UUUDUD, UDUDDD
10. UUDUUD, UDDUDD
11. UDUUUD, UDDDUD
12. DUUUUD, UDDDDU
13. UUUDDU, DUUDDD
14. UUDUDU, DUDUDD
15. UDUUDU, DUDDUD
16. DUUUDU, DUDDDU
17. UUDDUU, DDUUDD
18. UDUDUU, DDUDUD
19. DUUDUU, DDUDDU
20. UDDUUU, DDDUUD
21. DUDUUU, DDDUDU
22. DDUUUU, DDDDUU
23. UDDUDU, DUDUUD
24. UDUUDD, UUDDUD
25. DUUUDD, UUDDDU
26. UUDDDU, DUUUDD
27. UDUDDU, DUUDUD
28. DUDDUU, DDUUDU
29. DUUDDU
30. DDDUUU
31. UDUDUD
32. DUDUDU
33. DDUDUU
34. UDDUUD
35. UUUDDD
36. UUDUDD

There are 2^3 = 8 states that are their own duals so the total number of symmetric triangle quilt patterns along the diagonal is 36.

The pattern so far is summarized in the following table:

N	1	2	3	4	5	6
Total number of symmetric triangle quilt states	1	3	4	10	16	36

The pattern can be formulized by:

If n = odd, then number of states can be generalized as the total number of possibilities cut in half. Or (2^n)/2 = 2^(n-1)

If n = even, then the total number of states can be generalized by:

1. Take the total number of combinations and subtract off the combinations with equal U-D sub-states
2. Take half of this. So far, all duals with unequal numbers of U-D sub-states have been counted
3. Then take the combinations with equal U-D sub-states and subtract off those that are duals of themselves
4. Take half of this. Now we've counted those equal U-D sub-states that are not duals of themselves
5. Add in the equal U-D sub-states that are duals
6. (2^n – C(n, n/2))/2 + (C(n, n/2) – 2^(n/2))/2 + 2^(n/2) which =
7. 2^(n-1) + 2^((n-2)/2)

An example of even n, using the case where n = 6

1. There are 2^6 = 64 possible UD combinations. There are C(6,3) = 20 combinations with 3 U substates and 3 D substates. There are then 64 – 20 = 44 UD states with 0, 1, 2, 4, 5, or 6 U sub-states (and complementary 6, 5, 4, 2, 1, 0 D sub-states)
2. Half of 44 = 22. This eliminates the duals from the total count
3. There are C(6,3) = 20 combinations with 3 Us and 3 Ds. However, 2^3 = 8 of these are duals of themselves. Do not count them yet. Therefore we have 20 – 8 = 12 possible combinations to talk about
4. But these 12 have duals. 12/2 = 6 combinations that have duals
5. Now add in the 8 combinations that are duals of themselves 22 + 6 + 8 = 36

This is the same formula and pattern of the OEIS pattern A051437