Scan A58929

Chae, Palmer, RWR

add to 5 segs

CHAE

A58929

stet

Computing ~~the number of~~ Claw-free Cubic Graphs with given Connectivity

A57848
A58930
A58931
A58932

Gab-Byung Chae
chaegabb@pilot.msu.edu

Edgar M. Palmer
palmer@math.msu.edu
Department of Mathematics
Michigan State University
East Lansing, MI, 48823

Robert W. Robinson
rwr@pollux.cs.uga.edu
Computer science Dept.
415 GSRC
University of Georgia
Athens, GA 30602-7404

October 31, 2000

Abstract

We use exponential generating functions to count claw-free cubic graphs with given connectivity. Tables are provided for connectivity 1, 2 and 3.

1 Introduction

A claw-free cubic graph G is a cubic graph which contains no induced subgraph isomorphic to $K_{1,3}$. Therefore these are precisely the cubic graphs whose vertices all belong to triangles. For convenience we refer to them as *cfc's.* For a study of the history of the enumeration of regular graphs the reader can consult Gropp [Gr92] and for more recent work on enumerating cubic and claw-free cubics see [PaRR0x], and [CPR0x].

In [PaRR0x] Palmer, Read, and Robinson studied claw-free cubic graphs and computed the number of claw-free cubic graphs with up to 52 vertices. Moreover, their paper contained a partial differential equation for the exponential generating function(egf) of labeled, general cubic graphs. In [CPR0x] this equation was used to derive recurrence relations for general cubic graphs with a specified number of multiple edges and loops by connectedness. There is another relevant paper with enumeration formulae for cfc's, namely [MPaRR0x]. Combining results of these papers makes it possible to count claw-free cubic graphs with given connectivity.

In the present paper, we will follow the terminology and method in [PaRR0x] to find the number of k-connected claw-free cubic graphs where $k = 1, 2$, and 3. In a claw-free cubic graph, every vertex belongs to a triangle. So the maximum number of triangles in which a vertex may lie is 3. Clearly, a vertex lies in 3 triangles if and only if it is a vertex of K_4. A *diamond* in a cfc is an induced subgraph isomorphic to $K_4 - e$. A vertex lies in exactly two triangles if and only if it is one of the vertices of degree 3 in a diamond. A *string of diamonds* is a induced subgraph in which diamonds are adjacent in series. A *ring of diamonds* is a connected component in which every vertex belongs to a diamond.

In [PaRR0x] there are two important operations which convert general cubic graphs to claw-free cubic graphs. One of them is the *expansion* operation which converts an edge of a general cubic graph to a string of diamonds. The inverse operation to *expansion* is called *reduction.* The other operation is *dilation* which inflates a vertex of a general cubic graph to a triangle. The inverse operation to *dilation* is called *contraction.* Consider a cfc with no component isomorphic to K_4 or a ring of diamonds. The reduction operation applied to all strings of diamonds in this cfc results in a general cubic graph with no loops but possibly some double edges. The two vertices of such a double edge are mutually adjacent to a third vertex. These vertices constitute a *trumpet*. Still, every vertex must belong to a triangle. Note that if e is an edge caused by a reduction, then such a triangle must be part of a trumpet in which e is a multiple edge. Now the contraction operation completes the conversion from claw-free cubic graphs to general cubic graphs.

For general graph theoretic terminology and notation we follow [CL96] and the basic knowledge of labeled enumeration techniques can be found in [HPa73].

2 Connected claw-free cubic graphs

We define

$$H(z) = \sum_{n=0}^{\infty} h_n \frac{z^n}{(2n)!}.$$

where h_n is the number of labeled claw-free cubic graphs on $2n$ vertices. Then $H(z^2)$ is the exponential generating function for these graphs. By applying expansion and dilation operations, Palmer, Read, and Robinson [PaRR0x] derived the following a differential equation whose formal power series solution is the egf for labeled claw-free cubic graphs on $2n$ vertices:

$$\begin{aligned} 0 &= (144z^8 + 288z^7 - 576z^4)H''(z) \\ &+ (-36z^{10} - 96z^9 + 24z^8 + 144z^7 + 576z^6 + 384z^5 \\ &- 576z^4 - 2880z^3 - 576z^2 + 1152)H'(z) \\ &+ (-15z^{11} - 74z^{10} - 130z^9 - 96z^8 + 144z^7 + 368z^6 + 336z^5 - 288z^4 \\ &- 240z^3 - 288z^2 - 96z)H(z). \end{aligned} \tag{1}$$

Equation (1) can be converted to a differential equation whose formal solution is the egf for the number of connected, claw-free cubic graphs by the substitution

$$H(z) = e^{H_1(z)}, \tag{2}$$

error!

Table 1: Boundary conditions.

$h_1(1) = 0$	$h_1(7) = 13621608000$
$h_1(2) = 12$ 1	$h_1(8) = 8009505504000$
$h_1(3) = 60$	$h_1(9) = 3123380227968000$
$h_1(4) = 2520$	$h_1(10) = 1832279324908032000$
$h_1(5) = 453600$	$h_1(11) = 2054813830468439040000$
$h_1(6) = 59875200$	$h_1(12) = 1665031453088810526720000$

where $H_1(z^2)$ is the egf for connected, claw-free cubic graphs, *i.e.*

$$H_1(z) = \sum_{n=0}^{\infty} h_1(n) \frac{z^n}{(2n)!},$$

and $h_1(n)$ is the number of connected, labeled cfc graphs with $2n$ vertices. After substitution in equation (1) of $H(z)$ and its derivatives from equation (2), we have the following differential equation for $H_1(z)$:

$$\begin{aligned} 0 = &(144z^8 + 288z^7 - 576z^4)H_1'(z)H_1'(z) \\ &+ (144z^8 + 288z^7 - 576z^4)H_1''(z) \\ &+ (-36z^{10} - 96z^9 + 24z^8 + 144z^7 + 576z^6 + 384z^5 \\ &- 576z^4 - 2880z^3 - 576z^2 + 1152)H_1'(z) \\ &+ (-15z^{11} - 74z^{10} - 130z^9 - 96z^8 + 144z^7 + 368z^6 + 336z^5 - 288z^4 \\ &- 240z^3 - 288z^2 - 96z). \end{aligned} \tag{3}$$

The recurrence relation for the number of connected, claw-free cubic graphs can be found by extracting the coefficient of $\frac{z^n}{(2n)!}$ from both sides of (3). The relation is supported by the boundary conditions in Table 1:

For $n \geq 13$, we have:

$$\begin{aligned}
h_1(n) = &-144\frac{(2n)!}{1152n}\sum_{k=1}^{n-7}\frac{kh_1(k)(n-7-k)h_1(n-7-k)}{(2k)!(2n-14-2k)!} \\
&-288\frac{(2n)!}{1152n}\sum_{k=1}^{n-6}\frac{kh_1(k)(n-6-k)h_1(n-6-k)}{(2k)!(2n-12-2k)!} \\
&+576\frac{(2n)!}{1152n}\sum_{k=1}^{n-3}\frac{kh_1(k)(n-3-k)h_1(n-3-k)}{(2k)!(2n-6-2k)!} \\
&-144\frac{(2n)!}{(1152n)(2n-14)!}(n-7)(n-8)h_1(n-7) \\
&-288\frac{(2n)!}{(1152n)(2n-12)!}(n-6)(n-7)h_1(n-6) \\
&+576\frac{(2n)!}{(1152n)(2n-6)!}(n-3)(n-4)h_1(n-3) \\
&+36\frac{(2n)!}{(1152n)(2n-20)!}(n-10)h_1(n-10) \\
&+96\frac{(2n)!}{(1152n)(2n-18)!}(n-9)h_1(n-9) \\
&-24\frac{(2n)!}{(1152n)(2n-16)!}(n-8)h_1(n-8) \\
&-144\frac{(2n)!}{(1152n)(2n-14)!}(n-7)h_1(n-7) \\
&-576\frac{(2n)!}{(1152n)(2n-12)!}(n-6)h_1(n-6) \\
&-384\frac{(2n)!}{(1152n)(2n-10)!}(n-5)h_1(n-5) \\
&+576\frac{(2n)!}{(1152n)(2n-8)!}(n-4)h_1(n-4) \\
&+2880\frac{(2n)!}{(1152n)(2n-6)!}(n-3)h_1(n-3) \\
&+576\frac{(2n)!}{(1152n)(2n-4)!}(n-2)h_1(n-2).
\end{aligned} \tag{4}$$

By applying *Mathematica* to this recurrence relation, we calculated the numbers of 1-connected cfc graphs shown in Table 2. Actually the boundary conditions are not found only by the equation (3) directly. It just gives us partial values of them. In order to find exact values, we have to add the contribution from the constants in equation (3) to the values which come from the output of above recurrence relation (4) for n up to 12.

For example, when $n = 12$ the number $15*24!/(1152*12) = 673229602575129600000$ comes from the constants in equation (3) and the value 1664358223486235397120000 comes from the recurrence relation (4) by using the previous boundary values for $n < 12$. The sum of these two numbers gives us the boundary condition $h_1(12) =$ 1665031453088810526720000.

3 2-connected Claw-free cubic graphs

A 2-connected general cubic graph can be converted to a 2-connected cfc graph by the expansion operation, which converts an edge of a general cubic graph to a string of diamonds, and the dilation operation, which inflates a vertex of a general cubic graph to a triangle. The smallest 2-connected general cubic graph is K_4. By the dilation operation K_4 is converted to a 2-connected cfc graph of order 12. Therefore these operations produce 2-connected cfc graphs of order at least 12. However, there are 2-connected cfc graphs that can not be produced in this way. In fact there are three types of such graphs:

(a) The triangular prism of order $2n = 6$.

(b) The rings of two or more diamonds of order $2n \geq 8$.

(c) The graphs of order $2n \geq 10$ obtained by expanding with diamonds the edges of the triangular prism that do not belong to a triangle.

Next we need the egf's for these three families. Of course, K_4 is counted by $z^2/4!$. The rings of diamonds are counted by

$= \frac{1}{16}z^4 + \frac{1}{48}z^6 + \frac{1}{128}z^8 + \cdot$

all 1's!

$$\sum_{m=2}^{\infty} \frac{z^{2m}}{2m2^m} = -\frac{z^2}{4} - \frac{1}{2} ln(1 - z^2/2). \qquad (5)$$

And 2-connected cfc graphs obtained by expanding with diamonds the edges of

Table 2: Number of 1-connected cfc graphs with $2n \leq 60$.

new

A57848

n	$h_1(n)$
1	0
2	1
3	60
4	2520
5	453600
6	59875200
7	13621608000
8	8009505504000
9	3123380227968000
10	1832279324908032000
11	2054813830468439040000
12	1665031453088810526720000
13	1925086583971531588608000000
14	3552833935369312965955584000000
15	5046746501122027301952608256000000
16	9861817424365745824355502612480000000
17	27365975784025025428617030645350400000000
18	61323963707903030791402423349300428800000000
19	183552463622453002911375211047071799705600000000
20	720052647634369560568722076458423100794470400000000
21	2368360586025317757755816851785727694336557056000000000
22	10146784583491669585186721242630185440440056545280000000000
23	53795556323350118084055188978516784012472039393198080000000000
24	246424470779562683529001284375203235733354613795916349440000000000
25	1438454000284443072212393572236725837273648853200978470502400000000000
26	995856269134237859148875229290819201228673905380011308954419200000000000 00
27	610405473680692784571394583659731682395837369994603392192447774720000000 000000
28	467635877086568967899889446658084395803760155824110361591021274398720000 0000000000
29	411416340553637766635303196337295226242448905055930592322323048160034816 0000000000000
30	326975339782632378275301295587570674991301069597930348752701542511412038 86080000000000000

the triangular prism that do not belong to a triangle has egf

$$z^3b^3/12$$

where

$$b(z) = \sum_{k=0}^{\infty}(\frac{z^2}{2})^k = (1 - z^2/2)^{-1} \tag{6}$$

is the egf of the strings of diamonds.

Let $\Phi(z^2)$ be the egf of all three types above. Then

$$\Phi(z) = z^2/24 + \sum_{m=2}^{\infty}\frac{z^{2m}}{2m2^m} + \sum_{k=0}^{\infty}\frac{(k+1)(k+2)}{24}\frac{z^{2k+3}}{2^k}.$$

Let $G_2(x,y)$ be the egf

$$G_2(x,y) = \sum_{s,d} g_2(2m,d)\frac{x^s y^d}{(2n)!}$$

where $g_2(2m,d)$ is the number of 2-connected labeled general cubic graphs of order $2m$ with s single edges, d double edges and $2m = \frac{2s+4d}{3}$. We define $f_2(2n,d)$ to be the number of cfc's of order $2n$ built from 2-connected general cubic graphs with s single edges, d double edges and no loops by dilating vertices and expanding edges. Then we have the following formula which is simpler then the one in [MPaRR0x] because we do not have loops in 2-connected cfc's.

Lemma 3.1 *For fixed n, d, we have*

$$f_2(2n,d) = \sum_{m,j} g_2(2m,d)\binom{2n}{6m}\frac{\binom{6m}{3,\cdots,3}}{(2m)!}(3!)^{2m-2d}(3^2\cdot 2)^d\left(\binom{3m}{j}\right)\binom{4j}{4,\cdots,4}(12)^j, \tag{7}$$

where j is the number of diamonds and $2n \geq 12$.

Proof. Suppose G is a 2-connected labeled general cubic graphs counted by $g_2(2m,d)$. First we choose $6m$ labels from $2n$ available and arrange them in $2m$ unordered groups of three each for triangles. Then

$$\binom{2n}{6m}\frac{\binom{6m}{3,\cdots,3}}{(2m)!}$$

is the number of ways to do this. In triangles, the number of ways to label the vertices according to the adjacencies is

$$(3!)^{2m-2d}(3^2 \cdot 2)^d.$$

Since there are $3m$ original edges in the 2-connected labeled general cubic graph G, they can be expanded by j diamonds using combinations with repetition we find that the number of ways to do this is

$$\left(\!\binom{3m}{j}\!\right) = \binom{3m+j-1}{j}.$$

The number of ways to arrange the remaining labels for the diamonds and the the number of ways to assign labels to individual diamonds is

$$\binom{4j}{4, \cdots, 4}(4!/2)^j.$$

□

Note that the number of 2-connected cfc graphs which can be obtained by (7) depends on the number of double edges in 2-connected labeled general cubic graphs $g_2(2m, d)$ which were already computed in [CPR0x]. Define

$$B_2(z^2) = \sum_{n=0}^{\infty} b_2(2n)\frac{z^{2n}}{(2n)!}$$

be the egf of 2-connected cfc graphs which can be obtained by (7), then $b_2(2n) = \sum_d f_2(2n, d)$. Let

$$H_2(z^2) = \sum_{n=0}^{\infty} h_2(n)\frac{z^{2n}}{(2n)!}$$

be the egf of 2-connected cfc graphs. Then we have

$$H_2(z^2) = B_2(z^2) + \Phi(z^2).$$

But the computing the number of 2-connected cfc graphs by using this egf is not quite simple. To get the $b_2(2n)$, we need to compute the number $f_2(2n, d)$ and sum them up according to the number of vertices $2n$. And then extract the coefficients of three egf's in $\Phi(z^2)$. By adding, finally, the above numbers from each egf's according to the number of vertices, we can have the numbers of 2-connected cfc graphs as in Table 3.

Table 3: Number of 2-connected cfc graphs with $2n \leq 36$.

A58929

n	$h_1(n)$
2	1
3	60
4	2520
5	453600
6	59875200
7	10897286400
8	6701831136000
9	2623194782208000
10	1338096104497152000
11	1633313557551836160000
12	1324107982344764897280000
13	1408369399403068118016000000
14	2818005386051236981856256000000
15	3984871608553561924638375936000000
16	7418092561827244386962686894080000000
17	22027134615845465196052794703872000000000
18	49003622223231250364949254126429798400000000

4 3-connected Claw-free cubic graphs

Let $G_3(x, y)$ be the egf

$$G_3(x) = \sum_s g_3(2m) \frac{x^s}{(2m)!}$$

where $g_3(2m)$ is the number of 3-connected labeled general cubic graphs of order $2m$ with s single edges and $2m = \frac{2s}{3}$. Then we define $f_3(2n)$ to be the number of cfc's of order $2n$ built by dilating vertices in general cubic graphs with s single edges, no double edges and no loops. Suppose G is a 3-connected labeled general cubic graph counted by $g_3(2m)$. Then the contribution of G to $f_3(2n)$, with $(2m) \cdot 3 = 2n$, is determined by by arranging the $2n$ labels in $2m$ unordered groups of three vertices each for triangles. Here is the simple relationship between $f_3(2n)$ and $g_3(2m)$.

Table 4: Number of 3-connected cfc graphs with $2n \leq 90$.

n	$h_1(n)$
2	1
3	60
6	19958400
9	622452999168000
12	25852016738884976640000
15	6752895722718697367782686720000000
18	7393367369949286697176489031997849600000000
21	2627800504609683185243971405741688045646643200000000000
24	254276754658521110407033535459811588630840304679780352000000000000
27	5888995718306949420002641051088116070701509588327006047777587200000000 0000000
30	2958043254219256263308821276826065586114318405885952562068085100983678 40256000000000000000
33	2980890152919008019109186878589815790225188844356031585068994472885037 07828453008998400000000000000000
36	5656777720266027005731188874543000223254820151681873549147483906896091 164814511454342829299466240000000000000000000
39	1918038978375086995781974747217183468020461111349932500125442693126328 60958978793819181561437812810831626240000000000000000000
42	1111591539254229856723916110508301896483504051919049651086896747934939 649957201714150533128051831216937487176742993920000000000000000000000
45	1060057598064401612674900420305917000710146946719537685894497626633654 849683785877426968779460035111370065575453993044941123616768000000000 0000000000000

A58930

Lemma 4.1 *For fixed n, we have*

$$f_3(2n) = g_3(2m)\frac{(6m)!}{(2m)!}, \tag{8}$$

where $2n = 6m$.

The smallest 3-connected general cubic graph is K_4. Therefore the dilation operation produces 3-connected cfc graphs of order at least 12. But it will produce every 3-connected cfc graph except the triangular prism of order $2n = 6$. The numbers $g_3(2m)$

Table 5: Number of unlabeled and labeled cfc with $\kappa(G) = 1$

2n	# of unlabeled cfc with $\kappa(G) = 1$	# of labeled cfc with $\kappa(G) = 1$
14	1	2724321600
16	1	1307674368000
18	3 A58932	500185445760000
20	5	494183220410880000
22	11	421500272916602880000
24	20	340923470744045629440000

can be found by using Wormald's recurrence relation [W79c]. We have

$$g_3(2m) = (2m)!\frac{r(m)}{3m2^m}$$

where

$$r(m) = (3m-2)(r(m-1) + \sum_{i=2}^{m-2} r(i)r(m-i).$$

We used this method to compute the numbers $g_3(2m)$, $i.e$ the number of 3-connected cfc graphs, shown in 4. This complete the enumeration of cfc graphs with given connectivity.

5 Conclusion

The values of $h_1(n)$ and $h_3(n)$ were checked for $n \leq 12$ by calculating the order of the automorphism graphs of the small connected cfc's.

The numbers $h_2(n)$ were also checked for small value of n by finding the diagrams of the unlabeled cfc's with connectivity 1. These are the graphs that contribute to the difference between $h_1(n)$ and $h_2(n)$. For example, there are 20 unlabeled cfc's with $2n = 24$ vertices and $\kappa(G) = 1$. And the number of ways to label these is 340923470744045629440000. And then we compare this to the number which is the difference between the $h_1(12)$ and $h_2(12)$ which are found in the Tables 2 and 3, respectively as follows:

$$340923470744045629440000 = h_1(12) - h_2(12)$$

The numbers on the right side of the Table 5 are exactly the differences between $h_1(n)$ and $h_2(n)$.

Finally, we note that almost 80% of cfc's are 2-connected when the number of vertices is 34 or 36. This is consistent with the observation in [MPaRR0x] that almost all cfc's are 2-connected.

References

[CPR0x] G. Chae, E.M. Palmer, & R.W. Robinson, Computing the number of labeled general cubic graphs, to appear.

[CL96] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Chapman & Hall, 2-6 Boundary Row, London (1996).

[Gr92] Harald Gropp, Enumeration of regular graphs 100 years ago, *Discrete Mathematics* **101** (1992) 73-85.

[HPa73] F. Harary and E.M. Palmer, *Graphical Enumeration*, Academic, New York (1973).

[MPaRR0x] B.D. McKay, E.M. Palmer, R.C. Read, & R.W. Robinson, The asymptotic number of claw-free cubic graphs, *Proc. Eighth QuadrennialInternat. Conf. on Graph Theory, Combinatorics, Algorithms, and Applications* (Kalamazoo, 1996), to appear.

[PaRR0x] E.M. Palmer, R.C. Read, & R.W. Robinson, Counting claw-free cubic graphs, to appear.

[RW80] R.C. Read and N.C. Wormald, Number of labeled 4-regular graphs, *Journal of Graph Theory,* **4** (1980) 203-212.

[W79c] N.C. Wormald, Enumeration of labelled graphs II : Cubic graphs with a given connectivity, *J. London Math.Soc. (2)* **20** (1979) 1-7.