Notes on for A103885

Peter Bala, March 2020

The terms of A103885 are defined by means of the binomial sum

$$a(n) = \sum_{k=1}^{n} 2^k \binom{n}{k} \binom{2n-1}{k-1}. \tag{1}$$

We give several alternative representations of $a(n)$ and show that the supercongruence A103885(p) $\equiv$ A103885(1) $\left(\bmod\ p^3\right)$ holds for all prime $p \geq 5$.

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$$a(n) = \frac{1}{2}\,[x^n]\left(\frac{1+x}{1-x}\right)^{2n} \tag{2}$$

$$a(n) = \left[x^{2n}\right]\left(\frac{1+x}{1-x}\right)^{n} \tag{3}$$

Proof. If we expand the binomials on the right side of (2) (resp. (3)) and extract the coefficient of x^n (resp. x^{2n}) we obtain the binomial sum representation (4) (resp. (5)) below.

$$a(n) = \frac{1}{2}\sum_{k=0}^{n}\binom{2n}{n-k}\binom{2n+k-1}{k} \tag{4}$$

$$a(n) = \sum_{k=0}^{n}\binom{n}{k}\binom{2n+k-1}{n-1} \tag{5}$$

We can verify (4) and (5) (and hence (2) and (3)) by using Zeilberger's algorithm to show that both these sums satisfy the same linear recurrence as the defining sum (1), namely,

$$\begin{aligned}(n+1)(2n+1)(5n^2-5n+1)a(n+1) &= 2\left(55n^4-34n^2+3\right)a(n) + \\ &\quad (n-1)(2n-1)\left(5n^2+5n+1\right)a(n-1),\end{aligned}$$

and checking that the sums have the same initial values. □

Another representation for $a(n)$ involves the o.g.f. $S(x)$ of the sequence of large Schröder numbers A006318:

$$\mathrm{a}(n) = [x^n]\, S(x)^n \qquad (6)$$

Proof. The proof of (6) uses the Lagrange–Bürmann inversion formula, which we take in the following form: Let $G(x)$ be an arbitrary formal power series. Let $f(x) = \sum_{n\geq 1} f_n x^n$ with $f_1 \neq 0$ be a formal power series with compositional inverse denoted by $\bar{f}(x)$. Then

$$\left[x^N\right] G\left(f(x)\right) = \frac{1}{N}\left[x^{N-1}\right] G'(x)\left(\frac{x}{\bar{f}(x)}\right)^N.$$

Now the generating function of the sequence of large Schröder numbers $S(x) = \dfrac{1 - x - \mathrm{sqrt}\left(1 - 6x + x^2\right)}{2x}$ satisfies the quadratic equation

$$(1-x)S(x) - xS(x)^2 = 1.$$

Hence

$$\frac{1 + xS(x)}{1 - xS(x)} = S(x).$$

Thus the series reversion

$$\overline{\left(\frac{x(1-x)}{1+x}\right)} = xS(x).$$

Applying the Lagrange–Bürmann formula inversion formula with $N = 2n$, $G(x) = x^n$ and $f(x) = xS(x)$ yields

$$\begin{aligned}\left[x^{2n}\right] x^n S(x)^n = [x^n]\, S(x)^n &= \frac{1}{2n}\left[x^{2n-1}\right] n x^{n-1}\left(\frac{1+x}{1-x}\right)^{2n} \\ &= \frac{1}{2}[x^n]\left(\frac{1+x}{1-x}\right)^{2n} \\ &= a(n) \quad \text{by (2).}\end{aligned}$$

□

Supercongruences. Given an integer sequence $s(n)$, there exists a formal power series $G(x) = 1 + g_1x + g_2x^2 + \cdots$, with rational coefficients, such that

$$s(n) = [x^n]\, G(x)^n \quad \text{for } n \geq 1. \tag{7}$$

$G(x)$ is given by

$$G(x) = \frac{x}{\text{Rev }(xE(x))}, \tag{8}$$

where we now use Rev to denote the series reversion (inversion) operator, and the power series $E(x) = \exp\left(\sum_{n\geq 1} s(n)\frac{x^n}{n}\right)$. See [Stan'99, Exercise 5.56 (a), p. 98, and its solution on p. 146] or [Bal'15].

We can invert (8) to express $E(x)$ in terms of $G(x)$:

$$E(x) = \frac{1}{x}\text{Rev}\left(\frac{x}{G(x)}\right). \tag{9}$$

One simple consequence of (8) and (9) is the following:

the power series $G(x)$ is integral $\Longleftrightarrow$ the power series $E(x)$ is integral.

Given a sequence $s(n)$, the condition that the power series $E(x) = \exp\left(\sum_{n\geq 1} s(n)\frac{x^n}{n}\right)$ is integral is known to be equivalent to the statement that the Gauss congruences

$$s\left(mp^k\right) \equiv s\left(mp^{k-1}\right) \pmod{p^k}$$

hold for all prime p and positive integers m, k [Stan'99, Ex. 5.2 (a), p. 72, and its solution on p. 104].

It therefore follows from (6) and the above remarks that the sequence $a(n) = \text{A103885}(n)$ satisfies the Gauss congruences. In fact, calculation suggests that A103885 satisfies the stronger supercongruences

$$a\left(mp^k\right) \equiv a\left(mp^{k-1}\right) \pmod{p^{3k}} \tag{10}$$

for prime $p \geq 5$ and all positive integers m and k. We prove a particular case.

Proposition 1. *The supercongruence $a(p) \equiv 2 \pmod{p^3}$ holds for prime $p \geq 5$.*

Proof. Let $p \geq 5$ be prime. We make use of the binomial sum representation (5) for $a(p)$. We rewrite the sum by separating out the first ($k = 0$) summand and last ($k = p$) summand and adding together the k-th and $(p-k)$-th summands for $1 \leq k \leq \frac{p-1}{2}$ to obtain

$$a(p) = \binom{2p-1}{p-1} + \binom{3p-1}{p-1} + \sum_{k=1}^{\frac{p-1}{2}} \binom{p}{k} \left(\binom{2p+k-1}{p-1} + \binom{3p-k-1}{p-1} \right).$$

Now by Wolstenholme's theorem [Mes'11, p. 3] and by [Mes'11, equation 15]

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$$

and

$$\binom{3p-1}{p-1} \equiv 1 \pmod{p^3},$$

both congruences holding for all prime $p \geq 5$.

Hence

$$a(p) \equiv 2 + \sum_{k=1}^{\frac{p-1}{2}} \binom{p}{k} \left(\binom{2p+k-1}{p-1} + \binom{3p-k-1}{p-1} \right) \pmod{p^3}, \quad \text{prime } p \geq 5. \tag{11}$$

To establish the Proposition we will show that each summand on the right side of (11) is divisible by p^3. Clearly, the first factor $\binom{p}{k}$ in each summand is divisible by p for k in the range of summation. Therefore, to prove the Proposition, it is enough to show that the second factor $\binom{2p+k-1}{p-1} + \binom{3p-k-1}{p-1}$ is divisible by p^2 for all values of k in the range of summation. To show this, we write the second factor as a product of two terms, each of which is divisible by p.

One easily checks that

$$\binom{2p+k-1}{p-1} + \binom{3p-k-1}{p-1} = \frac{(2p+k-1)!}{(2p-k)!(p-1)!} \left\{ \frac{(3p-k-1)!}{(2p+k-1)!} + \frac{(2p-k)!}{(p+k)!} \right\}. \tag{12}$$

The first factor on the right side of (12) is a rational number divisible by p for k in the range $1 \ldots \frac{p-1}{2}$ since its numerator is exactly divisible by p^2 and its denominator is exactly divisible by p. To show that the second factor on the right side of (12) is also divisible by p we first set $r = p - 2k \geq 1$ and $s = p - 2k - 1 \geq 0$. Then we have

$$\begin{aligned} \frac{(3p-k-1)!}{(2p+k-1)!} + \frac{(2p-k)!}{(p+k)!} &= (3p-k-1)\cdots(3p-k-r) + (2p-k)\cdots(2p-k-s) \\ &\equiv (-1)^r (k+1)\cdots(k+r) + (-1)^{s+1} k \cdots (k+s) \pmod{p} \\ &\equiv 0 \pmod{p} \end{aligned}$$

since $r = s + 1$ and $k + r \equiv -k \pmod{p}$.

We have shown that $\binom{2p+k-1}{p-1} + \binom{3p-k-1}{p-1}$ is divisible by p^2 for $1 \le k \le \frac{p-1}{2}$, thus completing the proof of the Proposition. □

A generalisation. We define a two parameter family of sequences $a_{(r,s)}(n)$ by

$$a_{(r,s)}(n) = [x^{rn}]\, S(x)^{sn} \quad r \in \mathbb{N}, s \in \mathbb{Z}. \tag{13}$$

So by (6), $a_{(1,1)}(n) = \text{A103885}(n)$. We conjecture that the supercongruences

$$a_{(r,s)}\left(mp^k\right) \equiv a_{(r,s)}\left(mp^{k-1}\right) \pmod{p^{3k}} \tag{14}$$

hold for prime $p \ge 5$ and all $r \in \mathbb{N}$ and $s \in \mathbb{Z}$.

References

[Bal'15] Representing a sequence as [x$\hat{\{\}}$n] G(x)$\hat{\{\}}$n, uploaded to A066398

[Mes'11] R. Mestrovic, Wolstenholme's theorem: Its Generalizations and Extensions in the last hundred and fifty years (1862-2011), arXiv:1111.3057 [math.NT], 2011.

[Stan'99] R. P. Stanley, Enumerative Combinatorics, Volume 2, Cambridge University Press, 1999