A Multisection of q-Series

Michael Somos 30 Jan 2017
ms639@georgetown.edu

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1 Introduction to Ramanujan theta functions

Ramanujan used an approach to q-series which is general and is suggestive and can help to find new results, but somehow seems to be practically unknown. I have been using it for a few years and decided to give a few examples of how to apply it in practice. I have tried to keep this as brief as possible and yet contain enough of the details so that it suggests how to continue in the same direction. Details are available on request.

There is no precise general definition of q-series that I know of. However, it is usually an expression involving the variable q and perhaps other variables in an infinite sum or product. I will give the definition of the most important q-series which is as fundamental to the theory of q-series as the exponential is to differential calculus. Euler applied q-series to the theory of partitions of integers. Later, Jacobi introduced q-series as a basis for his theory of theta and elliptic functions with applications to number theory.

The power series here are given in terms of the variable q instead of the customary x. Ramanujan himself preferred the use of the plain x, however there are some connections with elliptic function theory where $q = e^{\pi i z}$ or else $q = e^{2\pi i z}$ is commonly used. Jacobi introduced q-series in 1829 but did not name them. By 1867, q was called the "nome" by Ole-Jacob Broch in his book "Traite Elementaire des Fonctions Elliptiques". In the theory of q-hypergeometric series q is sometimes called the "base".

To motivate the definition, recall the simplest converging infinite series which is the geometric power series, the sum of all nonnegative powers of x summing to $1/(1-x)$. We would like a multiplicative analog of this series. One possibility is to use $1 - x^n$ as the factors in the infinite product, but starting at $n = 1$ because $1 - x^0 = 1 - 1 = 0$ would cause the product to vanish immediately. Therefore we have the following definition.

Definition 1. The Ramanujan f function is defined by

$$f(-q) = (1-q)(1-q^2)(1-q^3)(1-q^4)...$$

where $|q| < 1$ is required for convergence. Often, we use formal power series and so all we need is that q^n converges to zero as n goes to infinity. The reason for the $f(-q)$ instead of the simpler $f(q)$ comes from Ramanujan's symmetric two variable theta function which is

Definition 2. The general two variable Ramanujan f function as used in his notebooks is defined by him using the following series

$$f(a,b) = 1 + (a+b) + (ab)(a^2+b^2) + (ab)^3(a^3+b^3) + (ab)^6(a^4+b^4) + ...$$

where $|ab| < 1$ is required for convergence. The idea behind Ramanujan's function is that it is a two-way infinite sum of terms in which the quotient of consecutive terms is a geometric progression. Thus we have

$$f(a,b) = ... + a^6b^{10} + a^3b^6 + ab^3 + b + 1 + a + a^3b + a^6b^3 + a^{10}b^6 + ...$$

This can be written with summation notation using an index variable n summing $a^{n(n+1)/2}b^{n(n-1)/2}$ over all integer n. Further, and, surprisingly, it has an infinite product expression as follows

$$f(a,b) = (1+a)(1+b)(1-q)(1+aq)(1+bq)(1-q^2)(1+aq^2)(1+bq^2)(1-q^3)...$$

where $q = ab$. This is equivalent to Jacobi's triple product identity, thus $f(a,b)$ is an example of a q-series that has both infinite sum and infinite product expressions.

In terms of $f(a,b)$ Ramanujan's one variable theta function is

$$f(-q) = f(-q, -q^2),$$

which perhaps explains why Ramanujan used $f(-q)$ instead of the more obvious $f(q)$. Since this can be confusing, I will use my own ad hoc notation for this important function, writing it as

$$y(q) = f(-q).$$

Euler used $1/y(q)$ as the generating function for the sequence of partitions of non-negative integers. He found a striking pattern to the power series expansion of $y(q)$ which is

$$y(q) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + ...$$

It appears that each term is a power of q with coefficients plus one or minus one. The signs appear to alternate with two minus signs followed by two plus signs and so on.

The pattern of the exponents may not be immediately obvious. Euler found that they are the generalized pentagonal numbers. To make the pattern even more obvious we can use the following method. We inflate the exponents by replacing q by q^{24} and multiply by a factor q. The result is

$$qy(q^{24}) = q - q^{25} - q^{49} + q^{121} + q^{169} - q^{289} + ...$$

It appears that each exponent is a square number with the general form of the exponent being $(6k+1)^2$ where k is any integer including negative integers, and the coefficient is $(-1)^k$. Thus, we can write it as a two-way infinite series

$$qy(q^{24}) = ... - q^{(-17)^2} + q^{(-11)^2} - q^{(-5)^2} + q^{(1)^2} - q^{(7)^2} + q^{(13)^2} + ...$$

Returning to $y(q)$ we note that the exponents of q are both even and odd but do not strictly alternate. They go through a pattern of length eight. The exponents modulo three go through a pattern of length six and all three residues occur. The exponents modulo five go through a pattern of length ten and these results generalize.

2 Bisection, Trisection and Multisection of power series

This suggests that it might be interesting to split up the power series into parts depending on the residue of the exponent modulo a fixed modulus. For modulus two this is called bisection, and in the general case it is called multisection. Note carefully that there is another closely related version of multisection known as dissection in which the sections need to be multiplied by powers of the variable and the argument replaced by the n-th power of the variable in order to become equal to multisection sections.

It is already interesting to try bisecting known series. For example, bisecting the exponential power series gives the series for hyperbolic sine and cosine. The general n-section case is given by Abraham Ungar in a 1982 American Mathematical Monthly article. Note that there is an algebraic relation between the hyperbolic sine and cosine and also in Ungar's general case. Also note that multisections can be expressed using n-th roots of unity.

For the series $A = y(q)$, applying bisection gives nothing obvious, so we trisect the series into three others and find that

$$A_0 = 1 - q^{12} - q^{15} + q^{51} + q^{57} - q^{117} - q^{126} + ...,$$
$$A_1 = -q + q^7 + q^{22} - q^{40} - q^{70} + q^{100} + q^{145} - ...,$$
$$A_2 = -q^2 + q^5 + q^{26} - q^{35} - q^{77} + q^{92} + q^{155} - ...,$$
$$A = y(q) = A_0 + A_1 + A_2.$$

It is natural to look for algebraic relations between A_0, A_1, A_2, but we need some algorithmic tools to find them. A few years ago I wrote such tools in PARI-GP. Frank Garvan's q-series Maple package also has similar tools. We start with two or more power series and the relation tool can find algebraic relations given enough terms and computation. While we will usually be finding homogeneous relations, it is easy to find non-homogeneous relations by supplying 1 as an extra power series. With A_0, A_1, A_2, by using only the coefficients from q^0 up to q^9, and assuming there is a homogeneous cubic relation, the tool produces

$$0 = A_2A_0^2 + A_0A_1^2 + A_1A_2^2.$$

It may be surprising that such a simple and symmetric relation holds, and easily found, but after finding many hundreds of similar relations it becomes more of a familiar phenomenon.

The proof of this relation depends on the trisection of $y(q)^3$ and leads to a 2007 result by Bruce Berndt and William Hart. Consider the series $y(q)^3$ with power series expansion

$$A = 1 - 3q + 5q^3 - 7q^6 + 9q^{10} - 11q^{15} + 13q^{21} - ....$$

The striking pattern of alternating odd integer coefficients and the triangular number exponents was known to Jacobi and resembles the pattern for $y(q)$ itself. We trisect the series and find that

$$A_0 = 1 + 5q^3 - 7q^6 - 11q^{15} + 13q^{21} + 17q^{36} - 19q^{45} - ...,$$
$$A_1 = -3q + 9q^{10} - 15q^{28} + 21q^{55} - ... = -3qA(q^9),$$
$$A_2 = 0, A = y(q)^3 = A_0 + A_1 + A_2.$$

Notice that A_2 is zero and that A_1 is similar to the original power series. We will encounter situations like this later on. In this case the fact that A_2 is zero is precisely the cubic relation that we previously found for the trisection of $y(q)$ itself.

Let w be a non-trivial cube root of unity. Then we find that

$$A(wq) = A(q) + (w - 1)A_1(q), A(w^2q) = A(q) + (w^2 - 1)A_1(q)$$

follows immediately from the trisection since $A = A_0 + A_1$. Similarly

$$A(wp) = A(p) + (w - 1)A_1(p), A(w^2p) = A(p) + (w^2 - 1)A_1(p)$$

where p is another variable. We do the algebra and find that

$$A(wp)A(wq) + wA(w^2p)A(w^2q) + 3(w + 1)A_1(p)A_1(q) = (w + 1)A(p)A(q)$$

which is essentially the identity of Berndt and Hart. Robin Chapman in "An eta-function identity" did something similar using cube roots of unity instead of multisection. He also generalized the result.

3 The Rogers-Ramanujan continued fraction

The next step would be to try a quadrisection of $y(q)$, but the results are not so simple, so we will skip that. We quintisect $y(q)$ and find that

$$A_0 = 1 + q^5 - q^{15} - q^{35} - q^{40} - q^{70} + q^{100} + q^{145} + ...,$$
$$A_1 = -q + q^{26} + q^{51} - q^{126} - q^{176} + q^{301} + q^{376} + ...,$$
$$A_2 = -q^2 + q^7 - q^{12} + q^{22} + q^{57} - q^{77} + q^{92} - q^{117} + ...,$$
$$A = y(q) = A_0 + A_1 + A_2 + A_3 + A_4,$$
$$A_1 = -qy(q^{25}) = -qA(q^{25}),$$
$$A_3 = A_4 = 0.$$

Notice that A_3, A_4 are zero and A_1 is similar to the original power series and this is like the previous $y(q)^3$ case. Using the coefficients from q^0 to q^5, and assuming there is a homogeneous quadratic relation, the relation tool produces

$$0 = A_0A_2 + A_1^2.$$

This is simpler than the trisection of $y(q)$, and probably this is the only irreducible relation between the three power series. Now Ramanujan essentially proves this relation using quintisection in a fragment in the 1988 Narosa edition of the "Lost Notebook and Other Unpublished Papers" on page 238, equations (20.1) to (20.4) and he first gives Jacobi's identity on $y(q)^3$ as equation (20.11). The relation is an immediate consequence of a formula in Bruce Berndt, "Ramanujan's Notebooks", Part III, page 82

$$f(-q) + qf(-q^{25}) = f(-q^{25}) \left\{ \frac{f(-q^{15}, -q^{10})}{f(-q^{20}, -q^5)} - q^2 \frac{f(-q^5, -q^{20})}{f(-q^{15}, -q^{10})} \right\}.$$

where we have the identifications

$$A_0 = f(-q^{25}) \frac{f(-q^{15}, -q^{10})}{f(-q^{20}, -q^5)}$$

and

$$A_2 = -q^2 f(-q^{25}) \frac{f(-q^5, -q^{20})}{f(-q^{15}, -q^{10})}.$$

One immediate consequence of the quadratic relation we found is that the two ratios

$$A_2/A_1 = -A_1/A_0 = R(q^5)$$

are equal and the common value is

$$R(q^5) = q - q^6 + q^{11} - q^{21} + q^{26} - q^{31} + q^{36} - ....$$

We look up the coefficients of this power series in Neil Sloane's OEIS and discover it is sequence $A007325$ which is the Rogers-Ramanujan continued fraction. More precisely this is given by

Definition 3. The Rogers-Ramanujan continued fraction is defined by

$$R(q) = q^{1/5}/(1 + q/(1 + q^2/(1 + q^3/(1 + q^4/...)))).$$

It would be difficult to discover this property without being able to search the OEIS. Combining previous equations we have

$$\frac{y(q)}{qy(q^{25})} = 1/R(q^5) - 1 - R(q^5).$$

where the left side is the generating function of sequence $A096562$. Replacing q^5 by q we get immediately

$$\frac{y(q^{1/5})}{q^{1/5}y(q^5)} = 1/R(q) - 1 - R(q).$$

which is a known theorem about $R(q)$. It is essentially equation (20.2) in the Lost Notebook fragment mentioned earlier but written in terms of $f(-q)$. Another appearance is in Bruce Berndt, "Ramanujan's Notebooks", Part III, page 267. More notably, on page 50 of the "Lost Notebook", Ramanujan explictly exhibits both the 2-sections and 5-sections of the Rogers-Ramanujan continued fraction as well as of its reciprocal all without any comment as usual. However, he uses the dissection form and not the multisection form. This is proof that he sometimes used the dissection form of multisection.

4 The Ramanujan cubic continued fraction

Maybe we were just lucky before, so naturally we wonder if we can find something else equally interesting. One approach would be to try a slightly more complicated product of q-series and see what happens. Accordingly, let us try

$$A = y(q)y(q^2) = 1 - q - 2q^2 + q^3 + 2q^5 + q^6 - 2q^9 + q^{10} + ...,$$

and see what we get using trisection. We find that

$$\begin{aligned} A_0 &= 1 + q^3 + q^6 - 2q^9 - 2q^{12} - q^{15} + q^{21} - 2q^{24} - 2q^{30} + ..., \\ A_1 &= -q + q^{10} + 2q^{19} - q^{28} - 2q^{46} - q^{55} + 2q^{82} - q^{91} + ..., \\ A_2 &= -2q^2 + 2q^5 - 2q^{11} + 2q^{14} + 2q^{17} - 2q^{29} + 2q^{32} + ..., \\ A &= y(q)y(q^2) = A_0 + A_1 + A_2, \\ A_1 &= -qy(q^9)y(q^{18}) = -qA(q^9). \end{aligned}$$

Again, using the coefficients from q^0 to q^5, and assuming there is a homogeneous quadratic relation, the relation tool produces

$$0 = A_0 A_2 + 2A_1^2.$$

Again, an immediate consequence of the relation is that the two ratios

$$A_2/(2A_1) = -A_1/A_0 = G(q^3)$$

are equal and the common value is

$$G(q^3) = q - q^4 + 2q^{10} - 2q^{13} - q^{16} + 4q^{19} - 4q^{22} + ...$$

We look up the coefficients of this power series in Sloane's OEIS and discover it is sequence $A092848$ which is Ramanujan's cubic continued fraction. More precisely this is given by

Definition 4. The Ramanujan cubic continued fraction is defined by

$$G(q) = q^{1/3}/(1 + (q + q^2)/(1 + (q^2 + q^4)/(1 + (q^3 + q^6)/...))).$$

More properties are given in the "Lost Notebook" on page 366. Again, it would be hard to discover this series had been studied before without being able to search the OEIS. Combining previous equations we have

$$1 + \frac{y(q)y(q^2)}{qy(q^9)y(q^{18})} = 1/G(q^3) - 2G(q^3).$$

where the left side is the generating function of sequence $A058531$ which is the normalized McKay-Thompson series of class 18A for the Monster group and the right side is a symmetrization formula for it.

5 7-section of $y(q)$ and Klein's quartic curve

Another path is to try the 7-section of $y(q)$. We find that

$$A_0 = 1 + q^7 - q^{35} - q^{70} - q^{77} - q^{126} + q^{210} + \ldots,$$
$$A_1 = -q - q^{15} + q^{22} + q^{57} + q^{92} + q^{155} - q^{176} + \ldots,$$
$$A_2 = -q^2 + q^{51} + q^{100} - q^{247} - q^{345} + q^{590} + \ldots,$$
$$A_5 = q^5 - q^{12} + q^{26} - q^{40} - q^{117} + q^{145} + \ldots,$$
$$A = y(q) = A_0 + A_1 + A_2 + A_3 + A_4 + A_5 + A_6,$$
$$A_2 = -q^2 y(q^{49}) = -q^2 A(q^{49}),$$
$$A_3 = A_4 = A_6 = 0.$$

Notice that A_3, A_4, A_6 are zero so there are only four nonzero sections. Using the coefficients from q^0 to q^{19}, and assuming there is a homogeneous cubic relation, the relation tool produces four relations

$$0 = A_0 A_1 A_5 - A_2^3,$$
$$0 = A_0 A_2^2 + A_2 A_1^2 + A_1 A_5^2,$$
$$0 = A_1 A_2^2 + A_2 A_5^2 + A_5 A_0^2,$$
$$0 = A_5 A_2^2 + A_2 A_0^2 + A_0 A_1^2.$$

The first relation dehomogenized yields

$$1 = (A_0/A_2)(A_1/A_2)(A_5/A_2).$$

This relation appears in Ramanujan's second notebook on page 239, end of entry 18(i). Several other 7-sections also appear on the same page. Since the three factor product is 1 we can assume that

$$A_0/A_2 = X/Z, A_1/A_2 = Y/X, A_5/A_2 = Z/Y$$

for some X, Y, Z. Rewriting the last three relations in terms of X, Y, Z and simplifying we find the three are equivalent to the one relation

$$0 = X^3Y + Y^3Z + Z^3X$$

which is Klein's quartic curve. A clue could have come also from the series expansion of A_0/A_2 which is OEIS sequence $A108483$ and has a reference to an article by William Duke published in 2005 which refers to a chapter by Noam Elkies in a 1999 book on Klein's quartic curve.

6 A_2 Rogers-Ramanujan identities

The 7-section of $y(q)$ also turns up in another context. In an article by George E. Andrews, Anne Schilling, S. Ole Warnaar published in 1999 is a theorem 5.2 with three equations and a comment that they had not found a fourth such equation. I noticed that the three equations given correspond to A_0, A_5, A_2 of the 7-section of $y(q)$, while the remaining nonzero section A_1 yields a fourth equation which was discovered a few years later, a fact for which I am indebted to Ole Warnaar. However, the connection to the 7-section apparently was unknown until I found it in June 2006.

To illustrate this connection, find the power series expansion of the infinite product in the first equation which we will write as

$$b(q) = 1 + 2q + 3q^2 + 5q^3 + 8q^4 + 11q^5 + 17q^6 + 24q^7 + 34q^8 + ...$$

and multiply by $y(q)$ to yield the power series we will write as

$$B(q) := b(q)y(q) = 1 + q - q^5 + q^{10} - q^{11} - q^{18} + q^{30} + ...$$

Comparing this with the 7-section of $y(q)$ we see that $A_0 = B(q^7)$. Exactly parallel reasoning holds for all four nonzero 7-sections of $y(q)$. Thus, the missing equation power series we will write as

$$c(q) = 1 + q + 3q^2 + 3q^3 + 6q^4 + 8q^5 + 13q^6 + 17q^7 + 25q^8 + ...$$

and multiply by $y(q)$ to yield the power series we will write as

$$C(q) := c(q)y(q) = 1 + q^2 - q^3 - q^8 - q^{13} - q^{22} + q^{25} + ...$$

Comparing this with the 7-section of $y(q)$ we see that $A_1 = -qC(q^7)$.

7 Summary and Conclusion

The multisection of simple q-series examined by simple tools lead to the discovery of results some of which have been known for a long time, some only recently, as well as those which appear to be new. However, these results do not require the extensive background and advanced tools that the original discoveries required. This is a good thing because it opens up a new world of results which would not otherwise have been found except by those with specialized knowledge.

The multisection examples given are those encountered in simple searches. There is no doubt in my mind that many others could now be found, but there is no way to know until they are looked for. Ramanujan himself explicitly and implicitly used multisection, but didn't really emphasize or explain that he was doing so. I hope that I have shown how easy it can be to discover interesting sequences and results in elliptic function theory in the style of Ramanujan using only simple ideas in algebra and a few simple computer tools.

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