A121707

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Contents

1 Introduction

From the OEIS[1]

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%I A121707
%S 35,55,77,95,115,119,143,155,161,187,203,
   209,215,221,235,247,253,275,287,295,299,
%N Numbers n > 1 such that n^3 divides
   Sum_{k=1..n-1} k^n = A121706(n).
%A Alexander Adamchuk, Aug 16 2006
%E Sequence corrected by
   Robert G. Wilson v, Apr 04 2011
```

Additional comments from Thomas Ordowski and Robert Israel:

- Note that n^2 divides $\sum_{k=1}^{n-1} k^n$ for every odd number $n > 1$.
- Conjecture 1: these are the odd numbers $n > 1$ such that n divides $\sum_{k=1}^{n-1} k^{n-1}$. (proven by Andrzej Schinzel)
- Conjecture 2: these are the "anti-Carmichael numbers": $n > 1$ such that for every prime p dividing n, $p-1$ does not divide $n-1$.

So let

$$\begin{aligned} S_n &= \sum_{k=1}^{n-1} k^n \\ T_n &= \sum_{k=1}^{n-1} k^{n-1} \end{aligned}$$

2 S_n mod n^3 for odd n

Since $n-1$ is even, pair up the terms of S_n:

$$\begin{aligned} S_n &= \sum_{k=1}^{n-1} k^n \\ &= \sum_{k=1}^{(n-1)/2} k^n + (n-k)^n \\ &= \sum_{k=1}^{(n-1)/2} \left[\sum_{j=1}^{n} \binom{n}{j} n^j (-k)^{n-j} \right] \end{aligned}$$

The inner sum starts at $j = 1$ because k^n cancels the $j = 0$ term.

The $j = 2$ term is $[n(n-1)/2]n^2(-k)^{n-2}$, and since n is odd, the 2 divides $n-1$: the term is a multiple of n^3. The subsequent terms are obviously multiples of n^3, and so (modulo n^3)

$$\begin{aligned} S_n &\equiv \sum_{k=1}^{(n-1)/2} \binom{n}{1} n^1 (-k)^{n-1} \\ &\equiv n^2 \sum_{k=1}^{(n-1)/2} k^{n-1} \end{aligned}$$

As Ordowski notes, n^2 divides S_n for odd n.

That last sum is tantalizingly close to T_n of the conjecture 1. When it is a multiple of n, n^3 divides S_n, and n is in A121707.

Let $U_n = S_n/n^2$, and consider U_n mod n:

$$\begin{aligned} 2U_n &\equiv \sum_{k=1}^{(n-1)/2} k^{n-1} + \sum_{k=1}^{(n-1)/2} k^{n-1} \\ &\equiv \sum_{k=1}^{(n-1)/2} k^{n-1} + \sum_{k=1}^{(n-1)/2} (n-k)^{n-1} \\ &\equiv \sum_{k=1}^{(n-1)/2} k^{n-1} + \sum_{k=(n+1)/2}^{n-1} k^{n-1} \equiv T_n \end{aligned}$$

For the second line, $k \to -k$ because the exponent is even; $-k \to n-k$ because it's modulo n.

The modulus is odd, so $T_n \equiv 2U_n \equiv 0 \bmod n$ just when $U_n \equiv 0$: conjecture 1 defines the odd terms of A121707. We have only to show that there are no even terms.

3 S_n for even n

First, a lemma. Let $z = 2^a$ be a power of two, and k be odd. Then $k^z \equiv 1 \bmod z$. Induce on a:

Basis: it's plainly true when $a = 1, z = 2$.

Step: let $k^z = zx + 1$. Then $k^{2z} = (k^z)^2 = z^2x^2 + 2zx + 1 = 2z((z/2)x^2 + x) + 1 \equiv 1 \bmod 2z$.

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Let 2^a be the highest power of two which divides n: $n = m \cdot 2^a$. So $n > a$, and the even terms of the S_n sum are divisible by 2^n and by 2^a.

There are $n/2$ odd terms in the S_n sum. When $a = 1$, they sum to an odd number: S_n is not divisible by 2, and not by n.

When $a > 1$, the odd terms can be paired up much as before:

$$\begin{aligned} O_n &= \sum_{k=1,3,5,\ldots}^{n-1} k^n \\ &= \sum_{k \text{ odd}}^{n/2-1} k^n + (n-k)^n \\ &= \sum_{k \text{ odd}}^{n/2-1} \left[k^n + \sum_{j=0}^{n} \binom{n}{j} n^j (-k)^{n-j} \right] \end{aligned}$$

Working modulo 2^a, the $j > 0$ terms of the inner sum vanish, and

$$\begin{aligned} O_n &\equiv \sum_{k \text{ odd}}^{n/2-1} k^n + (-k)^n \\ &\equiv \sum_{k \text{ odd}}^{n/2-1} (k^m)^{2^a} + ((-k)^m)^{2^a} \\ &\equiv \sum_{k \text{ odd}}^{n/2-1} 1 + 1 \\ &\equiv (n/4) \cdot 2 \equiv n/2 \equiv 2^{a-1} \end{aligned}$$

So if n is even, 2^a does not divide S_n, nor does $m \cdot 2^a = n$. Conjecture 1 is correct.

4 The anti-Carmichael conjecture

Again, an anti-Carmichael number is a value n such that for all primes p dividing n, $p-1$ does not divide $n-1$. (Ordowski says $n > 1$; I'm happy to include $n = 1$.)

$2-1$ divides $n-1$, so anti-Carmichael numbers are odd.

If n is odd, $3-1$ divides $n-1$; so anti-Carmichael numbers are not divisible by 3.

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Let n be an odd number, and let p^a be a component of the factorization of n: $n = mp^a, m \perp p$.[1]

$$\begin{aligned} T_n &= \sum_{k=1}^{n-1} k^{n-1} \\ &\equiv \sum_{k=1, k\perp p}^{mp^a-1} k^{n-1} \bmod p^a \\ &\equiv m \sum_{k=1, k\perp p}^{p^a-1} k^{n-1} \bmod p^a \end{aligned}$$

because the deleted terms are multiples of p^{n-1} and of p^a.

4.1 Groups modulo p^a

The modulo-p^a multiplicative group, G, has those values from 1 to $p^a - 1$ which are coprime to p: there are $p^a - p^{a-1} = |G|$ of them.

Since p is odd, the group is cyclic. Let g be a generator of the group.

[1] Theorem 119 of Hardy&Wright[2] suffices when n is squarefree.

That last expression for $T_n \bmod p^a$ can be written

$$T_n \equiv m \sum_{k \in G} k^{n-1} \bmod p^a$$

Let $h = g^{p-1}$, and let H be the subgroup of G generated by h.

First, $h \equiv 1 \bmod p$ (Fermat's little theorem), and each H element ($h^x \equiv (g^x)^{p-1}$) is $1 \bmod p$. There are $p^{a-1} = |H|$ of them: all of the 1-mod-p elements of G. So other powers of g yield other values, not $\equiv 1 \bmod p$, and

$g^k \equiv 1 \bmod p$ just if $p-1$ divides k.

4.2 $p-1$ divides $n-1$

If $p-1$ divides $n-1$, then $g^{n-1} = h^{(n-1)/(p-1)}$, an element of H. Those exponents are coprime to p and to $|H|$, and so the latter exponentiation just permutes the H elements. Modulo-p^a,

$$\begin{aligned}
\sum_{k \in H} k^{(n-1)/(p-1)} &\equiv \sum_{k \in H} k \\
&\equiv \sum_{j=0}^{p^{a-1}-1} 1 + jp \\
&\equiv p^{a-1} + p \sum_{j=0}^{p^{a-1}-1} j \\
&\equiv p^{a-1} + p \cdot \frac{(p^{a-1}-1)p^{a-1}}{2} \\
&\equiv p^{a-1}
\end{aligned}$$

In the following sum, the elements of G map to H: each H receives $(p-1)$ of the Gs.

$$\begin{aligned}
T_n &\equiv m \sum_{k \in G} k^{n-1} \\
&\equiv m(p-1) \sum_{k \in H} k^{(n-1)/(p-1)} \\
&\equiv m(p-1)p^{a-1} \\
&\equiv -mp^{a-1} \not\equiv 0
\end{aligned}$$

So p^a does not divide T_n.

4.3 $p-1$ does not divide $n-1$

The elements of G are $g^0, g^1, ..., g^{|G|-1}$, and so

$$\begin{aligned}
T_n &\equiv m \sum_{k \in G} k^{n-1} \bmod p^a \\
&\equiv m \sum_{j=0}^{|G|-1} (g^j)^{n-1} \\
&\equiv m \sum_{j=0}^{|G|-1} (g^{n-1})^j \\
(g^{n-1}-1)T_n &\equiv m((g^{n-1})^{|G|} - 1) \equiv m \cdot 0
\end{aligned}$$

So p^a divides $(g^{n-1}-1)T_n$. But $p-1$ does not divide $n-1$, and p does not divide $g^{n-1}-1$: p^a divides T_n.

4.4 Conclusion

For odd n, component p^a divides T_n just if $p-1$ does not divide $n-1$. That applies to each component of the factorization of n.

If a number $n > 1$ is anti-Carmichael, then n is odd, and for each component p^a dividing n: $p-1$ does not divide $n-1$, and so p^a divides T_n. Therefore n divides T_n, and $n \in$ A121707.
Other numbers greater than 1 have some $p-1$ which divides $n-1$, p^a and n do not divide T_n, and $n \notin$ A121707.

A121707 is the anti-Carmichael numbers except for $n = 1$.

References

[1] Neil Sloane, *The Online Encyclopedia of Integer Sequences*, `http://oeis.org`

[2] G.H.Hardy, E.M.Wright, *An Introduction to the Theory of Numbers*, fifth edition, Oxford University Press, 1983.