APPROXIMATIONS TO $\sum_{n\ge 1}(-1)^{\lfloor n\sqrt{2}\rfloor}/n$

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ABSTRACT. The constant in sequence A228639 is estimated by replacing the square root of two by its standard successive rational approximants.

1. Theme

The constant

$$c \equiv \sum_{n\ge 1} \frac{(-1)^{\lfloor n\sqrt{2}\rfloor}}{n} \approx -0.5154 \tag{1}$$

is defined in sequence [6, A228639]. The floor function in the exponent is the Beatty sequence of $\sqrt{2}$ [6, A001951], and the constant is some sort of aperiodic binary Dirichlet L-series based on the parity of this Beatty sequence [5].

2. Rational Approximations

By replacing $\sqrt{2}$ in the exponent by one of its rational approximations, the binary sequence in the numerator becomes periodic, and approximations of the constant are obtained by summation of this periodic binary sequence. In the sequel, the rational approximations are the convergents of the continued fraction of $\sqrt{2}$, q_j as in Table 1.

If we write the rational number with coprime numberator and denominator as $q_j = q_{j,n}/q_{j,d}$, then

$$c \approx c_j = \sum_{n\ge 1} \frac{(-1)^{\lfloor n q_j\rfloor}}{n} \tag{2}$$

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TABLE 1. Rational approximations of $\sqrt{2}$ [6, A001333,A000129].

j	q_j	q_j
3	17/12	1.41666666666...
4	41/29	1.4137931034482...
5	99/70	1.4142857142857...
6	239/169	1.4142011834319 ...
7	577/408	1.4142156862745 ...
8	1393/985	1.4142131979695 ...
9	3363/2378	1.4142136248948 ...

where

$$(-1)^{\lfloor (n+q_{j,d})q_j \rfloor} = (-1)^{\lfloor nq_j + q_{j,n} \rfloor} = (-1)^{\lfloor nq_j \rfloor}(-1)^{q_{j,n}}. \tag{3}$$

In the terminology of discrete Fourier transforms, the binary sequence has even or odd symmetry depending on whether the numerator $q_{j,n}$ is even or odd. By the recurrence of sequence [6, A001333] we conclude that the parity is actually always odd,

$$(-1)^{q_{j,n}} = -1. \tag{4}$$

This ensures that these series are converging, because the binary sequence sums to zero over the full period of length $2q_{j,d}$:

$$\sum_{1\le k\le 2q_{j,d}} (-1)^{\lfloor kq_j \rfloor} = 0. \tag{5}$$

[This is a general criterion for convergence of non-principal Dirichlet L-series at $s = 1$ [3].]

Considering the sequence of ± 1 with half period length $q_{j,d}$, the approximation becomes [2][1, 6.3][4, 0.267]

$$\begin{aligned} c_j &= -\sum_{1\le k\le 2q_{j,d}} (-1)^{\lfloor kq_j \rfloor} \left[\frac{1}{k} + \frac{1}{k+2q_{j,d}} + \frac{1}{k+4q_{j,d}} + \dots\right] \\ &= -\sum_{1\le k\le 2q_{j,d}} (-1)^{\lfloor kq_j \rfloor} \frac{\psi(\frac{k}{2q_{j,d}})}{2q_{j,d}} \\ &= -\sum_{1\le k\le q_{j,d}} (-1)^{\lfloor kq_j \rfloor} \left[\frac{1}{k} - \frac{1}{k+q_{j,d}} + \frac{1}{k+2q_{j,d}} - \frac{1}{k+3q_{j,d}} + \frac{1}{k+4q_{j,d}} + \dots\right] \\ &= \sum_{1\le k\le q_{j,d}} (-1)^{\lfloor kq_j \rfloor} \left[-\frac{\psi(\frac{k}{2q_{j,d}})}{2q_{j,d}} + \frac{\psi(\frac{k+q_{j,d}}{2q_{j,d}})}{2q_{j,d}}\right] \\ &= \frac{1}{2q_{j,d}} \sum_{1\le k\le q_{j,d}} (-1)^{\lfloor kq_j \rfloor} \left[\psi(\frac{k}{2q_{j,d}} + \frac{1}{2}) - \psi(\frac{k}{2q_{j,d}})\right] \end{aligned} \tag{6}$$

where ψ is the digamma function. These results are assembled in Table 2.

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Table 2. Approximations c_j

j	c_j
6	-.516347135840390
7	-.518431563885121
8	-.515577792256100
9	-.515935423023041
10	-.515445793635595
11	-.515507153371839
12	-.515423146252091
13	-.515433673918488
14	-.515419260575476
15	-.515421066837501
16	-.515418593898806
17	-.515418903802575
18	-.515418479516339

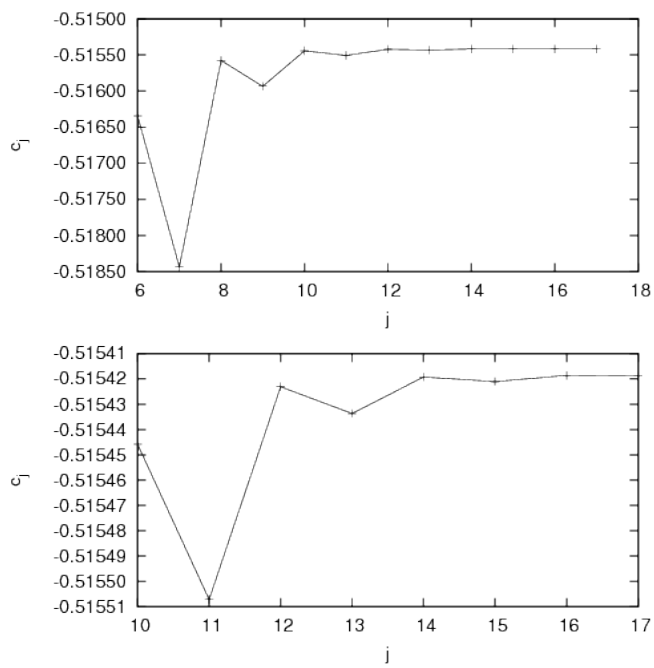

Figure 1. The approximations of Table 2 in graphical form.