Recurrence relation for the second partial sums of m-th powers

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We build with Excel the table that calculates, for successive additions, the partial sums of powers of natural numbers:

n	n^m	1th sums	2th sums
1	…	…	…
2	…	…	d
3	…	b	e
4	a	c	f
5	…	…	…
6	…	…	…

We want to obtain the recurrence relation for the second sums, that is, a formula for calculating the n-th term in the column "2th sums" as a function of the previous terms.

The formula that we seek is obtained by analyzing the data in the table as follows:

c = a + b
e = b + d
f = c + e = a + b + e = a + e − d + e = 2e − d + a

Indicating with a(n) the n-th term of the sequence, we therefore have:

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$$a(n) = 2{*}a(n\text{-}1) - a(n\text{-}2) + n^m$$
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This recurrence relation is valid, by *induction*, for each (n, m) and for each order of the partial sums.