A note on the sequence of numerators of a rational function

Peter Bala, Feb 2019

Given integer polynomials $P(x)$ and $Q(x)$, we investigate the sequence of numerators of the rational numbers $P(n)/Q(n), n = 0, 1, 2, ...$, and give conditions for the sequence to be a quasi-polynomial in n.

1. Recall the *greatest common divisor* (gcd) of two integers a and b, not both zero, is the largest positive integer that divides both a and b. The gcd function has the following property:

$$d \mid a \text{ and } d \mid b \Longleftrightarrow d \mid \gcd(a, b). \tag{1}$$

Proposition 1. *Let $P(x), Q(x) \in \mathbb{Z}[x]$ be coprime integer polynomials. Then the arithmetical function $g(n) = gcd\,(P(n), Q(n))$ is a purely periodic function, that is, there exists a positive integer N such that*

$$g(n + N) = g(n), \quad n \in \mathbb{Z}. \tag{2}$$

Proof. Viewing $P(x)$ and $Q(x)$ as elements of $\mathbb{Q}[x]$ we can apply Bézout's identity [5] to find rational polynomials $A(x), B(x) \in \mathbb{Q}[x]$ such that

$$A(x)P(x) + B(x)Q(x) \quad = \quad 1.$$

Clearing denominators we obtain integer polynomials $\overline{A}$(x) and $\overline{B}$(x) and a positive integer N (equal to the lcm of the denominators of $A(x)$ and $B(x)$) such that

$$\overline{A}(x)P(x) + \overline{B}(x)Q(x) \quad = \quad N.$$

In particular,

$$\overline{A}(n)P(n) + \overline{B}(n)Q(n) \quad = \quad N, \quad n \in \mathbb{Z}. \tag{3}$$

From the definition of the greatest common divisor we have that $g(n) = \gcd(P(n), Q(n))$ divides both $P(n)$ and $Q(n)$, and hence from (3) we see that

$$g(n) \text{ divides } N \text{ for all } n. \tag{4}$$

We shall prove $g(n) = g(n + N)$ for all $n \in \mathbb{Z}$ by showing that $g(n)$ divides $g(n+N)$ and also that $g(n+N)$ divides $g(n)$ (recall $g(n)$ and $g(n+N)$ are both positive integers by definition). Using the binomial expansion we easily see that

$$\begin{aligned} P(n + N) &= P(n) + \text{an integer multiple of } N \\ Q(n + N) &= Q(n) + \text{an integer multiple of } N. \end{aligned} \tag{5}$$

For integer n, $g(n)$ divides $P(n)$ and $Q(n)$. By (4), $g(n)$ also divides N. Hence by (5), $g(n)$ divides both $P(n+N)$ and $Q(n+N)$. Then by (1), $g(n)$ divides $g(n+N) = \gcd(P(n+N), Q(n+N))$.

The argument in the other direction is exactly similar: $g(n+N)$ divides $P(n+N)$ and $Q(n+N)$ by the definition of the gcd function. Again by (4), $g(n+N)$ divides N. Hence by (5), $g(n+N)$ divides both $P(n)$ and $Q(n)$. Therefore by (1), $g(n+N)$ divides $g(n) = \gcd(P(n), Q(n))$. ■

Renark. The period N of the function $g(n)$ obtained in the Proposition may not be the least period.

We will be interested in sequences formed from the numerators of the rational numbers $P(n)/Q(n)$, where $P(x)$ and $Q(x)$ are integral polynomials. One minor problem is that rational numbers have many representations and the numerator of a rational number is not unambiguously defined. For example,

$$\frac{12}{20} = \frac{-12}{-20} = \frac{6}{10} = \frac{3}{5} = \frac{-3}{-5}.$$

We need to make an unambiguous choice for the numerator of a rational number a/b. We restrict our attention to the case where b is a positive integer. Then we have

$$a = a'\gcd(a,b), \quad b = b'\gcd(a,b),$$

where a' and b' are coprime integers and b' is necessarily positive since the gcd is a positive function. Clearly, $a/b = a'/b'$. We define the *numerator of the rational number* a/b with $b > 0$ by

$$\text{numerator}\left(\frac{a}{b}\right) = a',$$

or equivalently,

$$\text{numerator}\left(\frac{a}{b}\right) = \frac{a}{\gcd(a,b)}. \tag{6}$$

Recall [6] an arithmetical function $f : \mathbb{N} \to \mathbb{N}$ is a *quasi-polynomial* if there exist polynomials $p_0, ..., p_{m-1}$ such that $f(n) = p_i(n)$ when $n \equiv i \pmod{m}$.

Proposition 2. *Let $P(x), Q(x) \in \mathbb{Z}[x]$ be coprime integer polynomials such that $Q(n) > 0$ for $n \in \mathbb{N}$. Then the sequence $\left(\text{numerator}\left(\frac{P(n)}{Q(n)}\right)\right)_{n\geq 0}$ of numerators of the rational numbers $\frac{P(n)}{Q(n)}$ is a quasi-poynomial in n.*

Proof. From the definiton of the numerator function in (6) we have

$$\text{numerator}\left(\frac{P(n)}{Q(n)}\right) = \frac{P(n)}{g(n)}, \quad n \in \mathbb{N},$$

where $g(n) = \gcd(P(n), Q(n))$. By Proposition 1, $g(n)$ is a purely periodic arithmetical function. Denote the least period of $g(n)$ by m. Then $\text{numerator}\left(\frac{P(n)}{Q(n)}\right) = p_i(n)$ when $n \equiv i \pmod{m}$, where $p_i(n) = \frac{P(n)}{g(i)}$ is polynomial in n. Thus the function $\text{numerator}\left(\frac{P(n)}{Q(n)}\right)$ is a quasi-polynomial in n. ■

2. Example. The simplest case of Proposition 2 is when $P(n)$ and $Q(n)$ are both linear polynomials in n. Let k be a positive integer and consider the family of sequences $a_k(n)$ defined by

$$a_k(n) = \text{numerator}\left(\frac{n}{n+k}\right), \quad n \in \mathbb{N},$$

which by (6) becomes

$$\begin{aligned} a_k(n) &= \frac{n}{\gcd(n, n+k)} \\ &= \frac{n}{\gcd(n,k)}. \end{aligned} \tag{7}$$

Many examples of these sequences are listed in the OEIS: see A026741 ($k = 2$), A051176 ($k = 3$), A060819 ($k = 4$), A060791 ($k = 5$), A060789 ($k = 6$), A106608 through A106612 ($k = 7, \ldots, 11$), A051724 ($k = 12$) and A106614 through A106621 ($k = 13, \ldots, 20$). Proposition 2 tells us that each of these sequences is a quasi-polynomial in n.

We mention two other general properties of the sequence $a_k(n)$.

A) For a fixed positive integer k, the sequence $a_k(n)$, $n = 1, 2, 3, \ldots$, is a strong divisibility sequence [2], that is,

$$\gcd(a_k(n), a_k(m)) \quad = \quad a_k(\gcd(n, m)) \quad \text{for all } n, m \geq 1. \tag{8}$$

In particular, each sequence $a_k(n)$ is a divisibility sequence, that is, if n divides m then $a_k(n)$ divides $a_k(m)$.

Proof. We require the following three properties of the greatest common divisor function $\gcd(a, b)$ and the least common multiple function $\text{lcm}(a, b)$ [3]:

(i) If m is a nonnegative integer then

$$m \gcd(a, b) \quad = \quad \gcd(ma, mb). \tag{9}$$

(ii) The greatest common divisor function and the least common multiple function are related by

$$\gcd(a, b)\text{lcm}(a, b) \quad = \quad |\, ab \,| \,. \tag{10}$$

(iii) The distributivity law

$$\operatorname{lcm}(a, \gcd(b,c)) \quad = \quad \gcd(\operatorname{lcm}(a,b), \operatorname{lcm}(a,c)). \tag{11}$$

Suppose a, b, c are positive integers. We can use (10) to remove the lcm functions in (11) to give

$$\frac{a \gcd(b,c)}{gcd(a, \gcd(b,c)} \quad = \quad \gcd\left(\frac{ab}{\gcd(a,b)}, \frac{ac}{\gcd(a,c)}\right).$$

Using (9) we can remove the common factor of a from the right and left-hand sides to arrive at the identity

$$\frac{\gcd(b,c)}{gcd(a, \gcd(b,c))} \quad = \quad \gcd\left(\frac{b}{\gcd(a,b)}, \frac{c}{\gcd(a,c)}\right).$$

Now set $a = k$, $b = n$ and $c = m$ to give

$$\frac{\gcd(n,m)}{\gcd(\gcd(n,m),k)} \quad = \quad \gcd\left(\frac{n}{\gcd(n,k)}, \frac{m}{\gcd(m,k)}\right).$$

By (7), this is equivalent to

$$a_k\left(\gcd(n,m)\right) \quad = \quad \gcd(a_k(n), a_k(m)).$$

Thus the sequence $a_k(n)$ is a strong divisibility sequence as claimed. ■

B) We show that the ordinary generating function of the sequence $a_k(n)$ is the rational function

$$\sum_{n=1}^{\infty} a_k(n) x^n = \sum_{d|k} \psi(d) \frac{x^d}{(1-x^d)^2}, \tag{12}$$

where $\psi(n)$ is the multiplicative arithmetical function defined on prime powers by $\psi(p^k) = 1 - p$. Thus $\psi(n) = \prod_{\text{prime } p|n} (1-p)$. The particular case of the generating function (12) when $k = p$, a prime, was noted by Hanna in A106608. Indeed, it was Hanna's observation that suggested the general form of the generating function given above. The first few values of the function $\psi(n)$ are tabled below.

n	1	2	3	4	5	6	7	8	9	10	11	12
$\psi(n)$	1	-1	-2	-1	-4	2	-6	-1	-2	4	-10	2

See OEIS sequence A023900.

It is not difficult to show that the function $\psi(n)$ as defined above is the Dirichlet inverse of Euler's totient function $\phi(n)$ (sequence A000010). In order to prove the generating function (12) we need the following property of the function $\psi(n)$.

Proposition 3.

$$\sum_{d|n} \frac{\psi(d)}{d} \quad = \quad \frac{1}{n}.$$

Proof. The Dirichlet generating function for the Euler totient function $\phi(n)$ is $\zeta(s-1)/\zeta(s)$. Thus the Dirichlet generating function for the Dirichlet inverse function $\psi(n)$ is simply the reciprocal function $\zeta(s)/\zeta(s-1)$. Hence

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} \quad = \quad \frac{\displaystyle\sum_{N=1}^{\infty} \frac{1}{N^s}}{\displaystyle\sum_{m=1}^{\infty} \frac{m}{m^s}},$$

which leads to

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} \sum_{m=1}^{\infty} \frac{m}{m^s} \quad = \quad \sum_{N=1}^{\infty} \frac{1}{N^s}.$$

Comparing the coefficients of N^{-s} on both sides of this identity gives the result. ■

We are now in a position to prove the rational generating function for the sequence $a_k(n)$ stated in (12). We have

$$\sum_{d|k} \psi(d) \frac{x^d}{(1-x^d)^2} \quad = \quad \sum_{d|k} \psi(d) \sum_{n=1}^{\infty} n x^{dn}.$$

The coefficient of x^N in the series on the right-hand side is given by

$$\sum_{\substack{d \mid k, \\ d \mid N}} \psi(d) \frac{N}{d}.$$

By (1), this is equal to

$$\begin{aligned} N \sum_{d|\gcd(N,k)} \frac{\psi(d)}{d} \quad &= \quad \frac{N}{\gcd(N,k)}, \quad \text{by Proposition 3,} \\ &= \quad a_k(N), \quad \text{by (7),} \end{aligned}$$

and this is the coefficient of x^N in the series on the left-hand side of (12). This completes the proof of the identity (12). ■

The previous results can be easily extended to the sequences

$$a_{k^m}\left(n^m\right) = \text{numerator}\left(\frac{n^m}{n^m+k^m}\right), \quad m = 1, 2, 3, \ldots.$$

By (7), we have

$$\begin{aligned} a_{k^m}\left(n^m\right) &= \frac{n^m}{\gcd(n^m, k^m)} \\ &= \left(\frac{n}{\gcd(n,k)}\right)^m \\ &= a_k(n)^m. \end{aligned}$$

By a similar calculation to the above case when $m = 1$, the ordinary generating function can be shown to be the rational function

$$\sum_{n=1}^{\infty} \left(a_k(n)\right)^m x^n = \sum_{d|k} \psi_m(d) \frac{A_m\left(x^d\right)}{\left(1-x^d\right)^{m+1}}, \tag{13}$$

where $\psi_m(n)$ is the multiplicative function defined by

$$\psi_m(n) = \prod_{\text{prime } p|n} (1-p^m),$$

and where $A_m(x)$ is the m-th Eulerian polynomial (see A008292 and also A123125). $\psi_m(n)$ is the Dirichlet inverse of the Jordan totient function $J_m(n)$ [4]. It has the property

$$\sum_{d|n} \frac{\psi_m(d)}{d^m} = \frac{1}{n^m}. \tag{14}$$

For $\psi_2(n)$ and $\psi_3(n)$ see A046970 and A063453.

The result (13) can be extended to negative values of m. For example, when $m = -1$ we obtain the generating function for the reciprocals of $a_k(n)$:

$$\sum_{n=1}^{\infty} \frac{x^n}{a_k(n)} = \sum_{d|k} \frac{\phi(d)}{d} \log\left(\frac{1}{1-x^d}\right),$$

where $\phi(n)$ denotes the Euler totient function.

References

[1] Wikipedia, *Dirichlet convolution*

[2] Wikipedia, *Divisibilty sequence*

[3] Wikipedia, *Greatest common divisor*

[4] Wikipedia, *Jordan's totient function*

[5] Wikipedia, *Polynomial greatest common divisor*

[6] Wikipedia, *Quasi-polynomial*