Notes on A341602 and A341603

Peter Bala, Nov 25 2022

Let $\mathrm{F}(n) = \mathrm{A000045(n)}$ denote the n-th Fibonacci number. The purpose of this note is to show that A) the sequence $\left\{\mathrm{F}\left(2^{2n}\right)\right\}$ converges in the ring of 2-adic integers $\mathbb{Z}_2$ to A341603, the expansion of the 2-adic integer sqrt($-3/5$) that is $\equiv 3 \pmod 4$ and B) the sequence $\left\{\mathrm{F}\left(2^{2n+1}\right)\right\}$ converges in $\mathbb{Z}_2$ to A341602, the expansion of the 2-adic integer sqrt($-3/5$) that is $\equiv 1 \pmod 4$.

- -

A) In $\mathbb{Z}_2$, lim_{$n \to \infty$} $\mathrm{F}(2^{2n}) = \mathrm{A341603}$.

In Proposition 2 below, we will establish the following congruence property for the Fibonacci numbers:

$$\mathrm{F}\left(2^{2n+2}\right) \equiv \mathrm{F}\left(2^{2n}\right) \left(\bmod\ 2^{2n+1}\right),\ n \geq 0. \tag{1}$$

Assuming this for the moment, it follows that the sequence $\left\{\mathrm{F}\left(2^{2n}\right)\right\}$ is a Cauchy sequence in $\mathbb{Z}_2$, which therefore converges to some 2-adic integer, call it α. We aim to prove that $5\alpha^2 + 3 = 0$ with $\alpha \equiv 3 \pmod 4$.

Now by Proposition 1, equation (4) below, $\mathrm{F}\left(2^{2n}\right) \equiv 3 \pmod 4$ for $n \geq 1$, and hence in the limit we also have $\alpha \equiv 3 \pmod 4$.

For notational convenience, let $\mathrm{A}(n) = \mathrm{F}(2^{2n})$. The recurrence equation

$$\mathrm{A}(n+1)^2 = \mathrm{A}(n)^2 \left(5\mathrm{A}(n)^2 + 2\right)^2 \left(5\mathrm{A}(n)^2 + 4\right) \tag{2}$$

holds with the initial condition $\mathrm{A}(1) = 3$.

Proof. Let $u(n) = \mathrm{F}(2^n)$. The recurrence $u(n)^2 = u(n-1)^2 \left(5u(n-1)^2 + 4\right)$ may be verified using the Binet formula for the Fibonacci numbers: $\mathrm{F}(n) = \dfrac{1}{\sqrt{5}}\left(\phi^n - (-1/\phi)^n\right)$, where $\phi = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio. Then it is straightforward to check that $u(2n) = \mathrm{F}\left(2^{2n}\right)$ satisfies (2). $\square$

Taking the 2-adic limit of (2) as $n \to \infty$ gives $\alpha^2 = \alpha^2\left(5\alpha^2+2\right)^2\left(5\alpha^2+4\right)$, so that α is a root of the polynomial equation $5\alpha^2\left(5\alpha^2+3\right)\left(5\alpha^4+5\alpha^2+1\right) = 0$. Since $\alpha \equiv 3 \pmod 4$, we find that $\alpha^2 \equiv 1 \pmod 4$ and $5\alpha^4 + 5\alpha^2 + 1 \equiv 3 \pmod 4$, so it must be the case that $5\alpha^2 + 3 = 0$ in $\mathbb{Z}_2$ (a ring without zero divisors). Therefore α is the 2-adic integer sqrt($-3/5$) with $\alpha \equiv 3 \pmod 4$. Thus $\alpha = \mathrm{A341603}$.

B) In $\mathbb{Z}_2$, lim_$\{n \to \infty\}$ $\mathrm{F}(2^{2n-1}) = $ A341602.

In Proposition 3 below, we establish the following congruence property for the Fibonacci numbers:

$$\mathrm{F}\left(2^{2n+1}\right) \equiv \mathrm{F}\left(2^{2n-1}\right) \pmod{2^{2n}} \quad \text{for } n \geq 1.$$

It follows that the sequence $\left\{\mathrm{F}\left(2^{2n-1}\right)\right\}$ is a Cauchy sequence in $\mathbb{Z}_2$, which therefore converges to some 2-adic integer, call it β. From Proposition 4 below, we have

$$\text{lim\_}\{n \to \infty\} \left(\mathrm{F}\left(2^{2n-2}\right) + F\left(2^{2n-1}\right)\right) = \alpha + \beta = 0.$$

Thus $\beta = -\alpha$ is the other root in $\mathbb{Z}_2$ of $5x^2 + 3 = 0$ and $\beta \equiv 1 \pmod 4$. Therefore, $\beta =$ lim_$\{n \to \infty\}$ $\mathrm{F}(2^{2n-1}) = $ A341602. □

Remark. Just as in (2), one can show that $\mathrm{B}(n) := \mathrm{F}(2^{2n-1})$ satisfies the recurrence equation

$$\mathrm{B}(n+1)^2 = \mathrm{B}(n)^2 \left(5\mathrm{B}(n)^2 + 2\right)^2 \left(5\mathrm{B}(n)^2 + 4\right), \tag{3}$$

the same as for $\mathrm{A}(n)$, but with the initial condition $\mathrm{B}(1) = 1$.

It remains to prove the four Propositions concerning Fibonacci numbers used in the above proofs.

Proposition 1.

$$\mathrm{F}\left(2^{2n}\right) \equiv 3 \pmod 4 \quad \text{for } n \geq 1 \tag{4}$$

$$\mathrm{F}\left(2^{2n+1}\right) \equiv 1 \pmod 4 \quad \text{for } n \geq 0. \tag{5}$$

Proof. Recall the Binet formulas for the Fibonacci numbers and Lucas numbers $\mathrm{L}(n) = $ A000032(n):

$$\mathrm{F}(n) = \frac{1}{\sqrt{5}}\left(\phi^n - (-1/\phi)^n\right) \quad \text{and} \quad \mathrm{L}(n) = \phi^n + (-1/\phi)^n,$$

where $\phi = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio. A consequence of Binet's formula for the Lucas numbers is the recurrence equation

$$\mathrm{L}\left(2^n\right) = \mathrm{L}\left(2^{n-1}\right)^2 - 2. \tag{6}$$

An induction argument then shows that

$$\mathrm{L}\left(2^n\right) \equiv 3 \pmod 4 \quad \text{for } n \geq 1. \tag{7}$$

A well-known identity connecting the Fibonacci and Lucas numbers, which follows immediately from the Binet formulas, is

$$\mathrm{F}(2n) = \mathrm{F}(n)\mathrm{L}(n).$$

Hence

$$F(2^n) = F(2^{n-1})L(2^{n-1}). \tag{8}$$

Using (7) and (8), a straightforward induction argument with base cases $F(2) = 1$ and $F(4) = 3$ completes the proof of (4) and (5). □

Proposition 2. The congruence

$$F\left(2^{2n+2}\right) \equiv F\left(2^{2n}\right) \pmod{2^{2n+1}}$$

holds for $n \geq 0$.

Proof. The case $n = 0$ is easily checked. Assume now that $n \geq 1$. The Lucas numbers $L(n)$ are known to satisfy the Gauss congruences

$$L(mp^r) \equiv L\left(mp^{r-1}\right) \pmod{p^r} \tag{9}$$

for all primes p and all positive integers m and r.

Using the Binet formulas it is easy to show that the Fibonacci and Lucas numbers are related by

$$5F(k)^2 + 2(-1)^k = L(2k).$$

Hence

$$5F\left(2^{2n}\right)^2 + 2 = L\left(2^{2n+1}\right) \tag{10}$$

and

$$5F\left(2^{2n+2}\right)^2 + 2 = L\left(2^{2n+3}\right). \tag{11}$$

Subtracting (10) from (11) gives

$$\begin{aligned} 5F\left(2^{2n+2}\right)^2 - 5F\left(2^{2n}\right)^2 &= L\left(2^{2n+3}\right) - L\left(2^{2n+1}\right) \\ &= \left(L\left(2^{2n+3}\right) - L\left(2^{2n+2}\right)\right) + \left(L\left(2^{2n+2}\right) - L\left(2^{2n+1}\right)\right) \\ &\equiv 0 \pmod{2^{2n+2}} \end{aligned}$$

by (9). It follows that

$$\left(F\left(2^{2n+2}\right) - F\left(2^{2n}\right)\right)\left(F\left(2^{2n+2}\right) + F\left(2^{2n}\right)\right) \equiv 0 \pmod{2^{2n+2}}. \tag{12}$$

Now by Proposition 1, equation (4), $F\left(2^{2n+2}\right) + F\left(2^{2n}\right)$ has the form $2(2N+3)$ for $n \geq 1$. Hence from (12) we conclude that

$$F\left(2^{2n+2}\right) - F\left(2^{2n}\right) \equiv 0 \pmod{2^{2n+1}}$$

for all $n \geq 0$. □

Proposition 3. The congruence

$$\mathrm{F}\left(2^{2n+1}\right) \equiv \mathrm{F}\left(2^{2n-1}\right) \pmod{2^{2n}}$$

holds for $n \geq 1$.

Sketch proof. Following a similar argument to that used in Proposition 2, we arrive at the congruence

$$\left(\mathrm{F}\left(2^{2n+1}\right) - \mathrm{F}\left(2^{2n-1}\right)\right)\left(\mathrm{F}\left(2^{2n+1}\right) + \mathrm{F}\left(2^{2n-1}\right)\right) \equiv 0 \pmod{2^{2n+1}}. \tag{13}$$

By (5), $\mathrm{F}\left(2^{2n+1}\right) \equiv 1 \pmod 4$. Thus the second factor $\mathrm{F}\left(2^{2n+1}\right) + \mathrm{F}\left(2^{2n-1}\right)$ on the left side of (13) is $\equiv 2 \pmod 4$, that is, $\mathrm{F}\left(2^{2n+1}\right) + \mathrm{F}\left(2^{2n}\right)$ is twice an odd number. It now follows from (13) that

$$\mathrm{F}\left(2^{2n+1}\right) - F\left(2^{2n-1}\right) \equiv 0 \pmod{2^{2n}}. \square$$

Proposition 4. The congruence

$$\mathrm{F}\left(2^{2n+1}\right) + \mathrm{F}\left(2^{2n}\right) \equiv 0 \pmod{2^{2n+1}}$$

holds for $n \geq 1$.

Sketch proof. Following a similar argument to that used in Proposition 2, we arrive at the congruence

$$\left(\mathrm{F}\left(2^{2n+1}\right) - \mathrm{F}\left(2^{2n}\right)\right)\left(\mathrm{F}\left(2^{2n+1}\right) + \mathrm{F}\left(2^{2n}\right)\right) \equiv 0 \pmod{2^{2n+2}}. \tag{14}$$

By (4) and (5), $\mathrm{F}\left(2^{2n}\right) \equiv 3 \pmod 4$ and $\mathrm{F}\left(2^{2n+1}\right) \equiv 1 \pmod 4$. Thus the first factor $\mathrm{F}\left(2^{2n+1}\right) - \mathrm{F}\left(2^{2n}\right)$ on the left side of (14) is $\equiv 2 \pmod 4$, that is, $\mathrm{F}\left(2^{2n+1}\right) - \mathrm{F}\left(2^{2n}\right)$ is twice an odd number. It follows from (14) that

$$\mathrm{F}\left(2^{2n+1}\right) + \mathrm{F}\left(2^{2n}\right) \equiv 0 \pmod{2^{2n+1}}. \square$$