Enumeration of Threshold Functions of Eight Variables

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Abstract—The number of threshold functions of eight variables is counted by ILLIAC II, the computer of the University of Illinois. Sets of optimum weights of majority elements realizing these functions also are investigated. Actually, canonical positive self-dual threshold functions of nine variables are investigated instead of directly investigating threshold functions of eight variables because it is easier to deal with them. The number and optimum weights of threshold functions of eight variables are easily obtained from these functions of nine variables and their realization.

First a linear program to minimize the total input weight is considered. Canonical positive self-dual functions of nine variables are generated by modifying Winder's method and tested by the simplex method to find whether or not they are threshold functions. The number of canonical positive self-dual threshold functions of exactly nine variables and the number of all threshold functions of exactly eight variables are 172 958 and 17 494 930 604 032, respectively. It had been an open question whether there exists any completely monotonic function of exactly eight variables which is not a threshold function. But the computational result verified that there was no such function. Unlike the case of threshold functions of seven or fewer variables, there are some threshold functions of exactly eight variables whose extreme optimum weights include fractional numbers. Other properties of these functions are also explored.

Next a linear program to minimize the threshold is considered. The linear program is considerably different from the previous linear program to minimize the total input weight with respect to uniqueness of an optimum solution. The number of threshold functions which have multiple optimum solutions is counted.

Index Terms—Completely monotonic functions, enumeration of threshold functions, linear programming, optimum structure, switching theory, threshold logic.

I. Introduction

It is an interesting problem to find how many Boolean functions of n variables are actually threshold functions. Threshold functions have been enumerated by a a few authors for n equal to seven or fewer [1], [2], [12]. In this paper we enumerate all threshold functions of eight variables and investigate algebraic properties of these functions and sets of optimum structures which realize them. The digital computer of the University of Illinois, ILLIAC II, was used for this study.

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First a linear program to minimize the total input weight is considered. Properties of optimum weights of threshold functions of eight variables which will be discussed in this paper are quite different from those of seven or fewer variables. For example, functions of seven or fewer variables have only integral and unique optimum weights. But some of the functions of eight variables have fractional optimum weights and some others multiple sets of optimum weights. Whether there are completely monotonic functions which are not threshold functions for eight variables has been an interesting open question, because there is none for seven or fewer variables but there is at least one known for nine variables discovered by Gabelman [9]; but no such function was found in our enumeration.

Next a linear program to minimize the threshold is considered. This linear program is somewhat different from the one to minimize the total input weight.

These two linear programs and their solutions are discussed in this paper.

For the definition of terminology, see [2]–[4], [14], [16], and [20].

II. Linear Program to Minimize Total Input Weight

First let us formulate the following linear programming problem [3], [4] for a positive threshold function of n variables, $f(x_1, \cdots, x_n)$. Minimize the *objective function*

$$W = \sum_{i=0}^{n} w_i \tag{1}$$

under the constraints

$$w_0 \geq 0 \text{ and } w_i \geq 0 \qquad (i = 1, \cdots, n), \tag{2}$$

$$\sum_{i=1}^{n} w_i \xi_i^{(j)} + w_0 \xi_0 \geq 1 \qquad \text{for } f(x_1^{(j)}, \cdots, x_n^{(j)}) = 1, \tag{3}$$

and

$$\sum_{i=1}^{n} w_i \xi_i^{(j)} + w_0 \xi_0 \leq -1 \qquad \text{for } f(x_1^{(j)}, \cdots, x_n^{(j)}) = 0, \tag{4}$$

where

$$\xi_i^{(j)} = 2x_i^{(j)} - 1 \qquad (i = 1, 2, \cdots, n).$$

Then a solution $w_1, \cdots, w_n$ and $w_0 \xi_0$ for which the value of the objective function (1) is minimum and finite is called an *optimum solution*, *optimum weights* of f, or an *optimum structure* of f.

The sum of the absolute values of all weights, i.e., $\sum_{i=0}^{n} |w_i|$, is called the *total input weight*. Since $w_i \geq 0$ for $i = 0, 1, 2, \cdots, n$, (1) is the total input weight.

The minimization of (1) has engineering significance. When it is minimized, the input tolerance and reliability of a majority element are maximized [10].

Winder [6] showed an important property that if the self-dualized function of a given non-self-dual function $f(x_1, \cdots, x_n)$ is $[(n+1)/3]$-monotonic, where $[x]$ is the greatest integer which is not greater than x, f is completely monotonic. This can be easily strengthened into the necessary and sufficient condition, which Winder apparently knew but did not state. This condition is useful for finding whether a given threshold function of n variables is completely monotonic, by examining whether the self-dualized function is $[(n+1)/3]$-monotonic. We make use of this property.

When we solve the linear program (1)–(4), it is desirable to eliminate unnecessary extremal inequalities before solving by the simplex method. Two elimination methods based on extrinsic extremal vectors of a function were explored. When a function is individually given, some of the extremal inequalities in (3) and (4) can be eliminated by these two elimination methods and then the linear program (1)–(4) can be solved with fewer inequalities.

When all functions of a certain specified number of variables are to be generated and the linear programs for them are to be solved, different approaches might be appropriate. Winder [2] devised an efficient method of generating extremal inequalities for canonical positive self-dual functions. In our enumeration of threshold functions of eight variables, or equivalently canonical positive self-dual functions of nine variables, we adopted Winder's generation method to generate the extremal inequalities for two-monotonic functions which are candidates for threshold functions. His generation method, however, eliminates some of the inequalities which can be eliminated by the above two elimination methods but not all of them (this incomplete elimination was due to Winder's treating x_1 differently from other variables in his generation method), although a majority of inequalities which can be eliminated by two-monotonicity are eliminated in Winder's generation method. (Any inequalities which can be eliminated by three-or-more-monotonicity are not eliminated.) In our computer program, we eliminate further inequalities by one of the above two elimination methods after using Winder's generation method since it is easy to incorporate.

The variables of each canonical two-monotonic positive self-dual function generated are ordered as

$$x_1 \gtrsim x_2 \gtrsim \cdots \gtrsim x_n. \tag{5}$$

The inequalities

$$w_1 \geq w_2 \geq w_3 \cdots w_{n-1} \geq w_n \geq 0 \tag{6}$$

are added. Since the generated functions are self-dual, only the inequalities of (3) need to be considered since $w_0 = 0$ and (4) may be ignored. When an extremal inequality

$$\sum_{i=1}^{n} w_i \xi_i^{(j)} \geq 1 \tag{7}$$

is redundant in terms of (6) and other extremal inequalities, in other words, when we can prove by using (6) and other extremal inequalities that the left side of (7) is not smaller than 1, the extremal inequality (7) may be eliminated. The extremal n-tuples generated by Winder's method do not include the extremal n-tuples which are redundant in terms of the inequalities (6) of $w_2, \cdots, w_n$, but do include redundant ones in terms of the inequalities with w_1. The elimination of these redundant ones takes little computer time but reduction of computer time for the linear programming due to this elimination is significant.

Then we considered the following linear program for a canonical positive self-dual function, by converting the unknown w_i of the linear program (1)–(4) into their differences. Minimize

$$\sum_{i=1}^{n-1} i(w_i - w_{i+1}) + nw_n \tag{8}$$

under the constraints

$$(w_1 - w_2) \geq 0, (w_2 - w_3) \geq 0, \cdots, \\ (w_{n-1} - w_n) \geq 0, w_n \geq 0 \tag{9}$$

and the extremal inequalities left after the above elimination method,

$$\sum_{i=1}^{n-1} (w_i - w_{i+1}) \sum_{k=1}^{i} \xi_k^{(j)} \\ + w_n \sum_{i=1}^{n} \xi_i^{(j)} \geq 1 \qquad (j = 1, 2, \cdots, r). \tag{10}$$

As pointed out by Winder [2], taking the differences $(w_i - w_{i+1})$ as the unknowns will make the number of iterations fewer than that by taking w_i themselves as the unknowns, in solving the following dual linear program.

Then the dual linear program of the above linear program was formulated as follows. Maximize

$$\sum_{j=1}^{r} v_j \tag{11}$$

under the constraints

$$v_j \geq 0 \qquad \text{for } j = 1, 2, \cdots, r \tag{12}$$

and

$$\sum_{j=1}^{r} v_j(\xi_1^{(j)}) \leq 1 \\ \sum_{j=1}^{r} v_j\left(\sum_{k=1}^{2} \xi_k^{(j)}\right) \leq 2 \\ \cdots \\ \sum_{j=1}^{r} v_j\left(\sum_{k=1}^{n} \xi_k^{(j)}\right) \leq n. \tag{13}$$

At this stage some of the inequalities of (13) can be eliminated, reducing the size of the linear program further. (According to the literature this elimination scheme has not been used before.)

Then the dual linear program of (11)–(13) is solved by the simplex method. The advantage of dealing with the dual linear program is the immediate availability of the initial feasible solution without introduction of artificial variables.

We exhausted all extreme solutions in order to get all optimum solutions. The last tableau is examined to determine whether there is a symptom for multiple extreme optimum solutions, according to the method suggested by Gomory as reported in [10].

III. Computational Results

First, 319 124 canonical two-monotonic positive self-dual functions were generated. We then solved the same number of linear programs. 175 428 of them turned out to be threshold functions and the remaining 143 696 to be nonthreshold functions. By our modification of Winder's generation, we had a total of 1 898 947 extremal inequalities for threshold functions (3 477 178 extremal inequalities for all generated functions). This means that each threshold function on the average had 10.8 extremal inequalities. (Each of all generated functions had an average of 10.9 extremal inequalities.) The maximum number of extremal inequalities for a threshold function was 23.

Since a large number of linear programs had to be solved, the computer program for the simplex method had to be fast. Special features of ILLIAC II were used [8].

The statistics of the number of threshold functions and other types of functions are shown in Table I. In this table NPN type means the class of functions which are equivalent by negation and/or permutation of variables and/or negation of a function. NP type means the class of functions which are equivalent by negation and/or permutation of variables (no operation on a function). Some figures were available already and our results were checked against them. (One obvious mistake in Winder's table [2], i.e., number of nondegenerate threshold functions of seven variables, was corrected.)

Let R_n be the number of threshold functions of up to n variables. The bounds on R_n are known [15]–[19] as

$$\frac{1}{2} \leq \lim_{n\to\infty} \frac{1}{n^2} \log_2 R_n \leq 1.$$

The values of $(1/n^2) \log_2 R_n$ from the figures in Table I are computed in Table II. The value reaches a minimum at $n=5$ and increases again for larger n.

We enumerated also the symmetry types of canonical positive self-dual threshold functions, i.e., how many sets of symmetrical variables each of the canonical positive self-dual threshold functions has, and then their numbers.

A set of optimum weights which includes the maximum w_1^0 was 42, 22, 18, 15, 13, 10, 8, 4, 3.

A set of optimum weights which includes the maximum w_9^0 was 28, 22, 20, 19, 18, 15, 14, 12, 11.

The maximum of the optimum total input weight, i.e., $\sum_{i=1}^{9} w_i^0$, was 209. The set of optimum weights (unique) for this function was 34, 32, 28, 27, 24, 20, 18, 15, 11 ($T=105$). The minimum was of course 9.

Each of the optimum weights for the 175 428 threshold functions were added up and were shown in Table III with the average size for each weight. For functions which had multiple extreme optimum solutions, average optimum weights were used (i.e., since all these functions had only two extreme optimum solutions, the optimum weights at the midpoint were added).

Twelve canonical positive self-dual threshold functions of exactly nine variables were discovered to have multiple extreme optimum solutions, though none of those of eight or fewer variables had multiple solutions. These are listed in Table IV. Each function has only a single pair of symmetrical variables whose weights are encircled. If encircled weights are interchanged, the other extreme optimum solution is obtained.

Two functions were discovered to have an extreme optimum solution which included fractional weights. Both solutions were unique extreme optimum ones. These are shown in Table V. When a function has multiple extreme optimum solutions of integral weights, we may have an extreme optimum solution having fractional weights in the last tableau which were not originally extreme solutions but were brought in by the constraints (6). Of course, such solutions were not counted.

Each time we discovered a set of optimum weights for a generated canonical positive self-dual threshold function of up to nine variables, we counted how many of the extremal inequalities used for the primal linear program are satisfied by equality by the solution and how many by strict inequality.

The maximum number of inequalities which are satisfied by strict inequality by an optimum solution was ten. Six functions were found to have this number of ten strict inequalities. But some of these strict inequalities can be eliminated as the inequalities corresponding to extrinsic prime implicants. In this sense these are interesting functions. Let us discuss one of these six functions as an example.

The self-dual threshold function expressed by the following optimum weights (unique optimum solution) 23, 17, 15, 13, 11, 9, 7, 5, 3 ($T=52$) has 47 prime implicants, 35 of which are extrinsic and 12 of which are intrinsic. By elimination of inequalities, 20 extremal inequalities are left in the primal linear program and the corresponding dual linear program is solved by the simplex method. By the optimum solution obtained, ten of these inequalities are satisfied by equality and the remaining ten by strict inequality. However, eight of these ten strict inequalities can be eliminated as the inequalities corresponding to extrinsic prime implicants but the remaining two strict inequalities correspond to intrinsic prime implicants.

There was no three-monotonic self-dual function of exactly nine variables which was not a threshold function. This means that *there is no completely monotonic self-dual*

TABLE I
NUMBER OF FUNCTIONS BY CLASSIFICATION

n	0	1	2	3	4	5	6	7	8
Switching functions of up to n variables	2	4	16	256	65 536	about 4.3×10^9	about 1.8×10^{19}	about 3.4×10^{38}	about 1.16×10^{77}
Switching functions of exactly n variables	2	2	10	218	64 594	about 4.3×10^9	about 1.8×10^{19}	about 3.4×10^{38}	about 1.16×10^{77}
Threshold functions of up to n variables	2	4	14	104	1882	94 572	15 028 134	8 378 070 864	17 561 539 552 946
Threshold functions of exactly n variables	2	2	8	72	1536	86 080	14 487 040	8 274 797 440	17 494 930 604 032
N-type threshold functions of up to n variables	2	3	6	20	150	3287	244 158	66 291 591	68 863 243 522
N-type threshold functions of exactly n variables	2	1	2	9	96	2690	226 360	64 646 855	68 339 572 672
N-type self-dual threshold functions of up to $n+1$ variables	1	2	4	12	81	1684	123 565	33 207 256	34 448 225 389
N-type self-dual threshold functions of exactly $n+1$ variables	1	0	1	4	46	1322	112 519	32 267 168	34 153 652 752
NP types of up to n variables	2	3	6	22	402	1 228 158	400 507 806 843 728	—	—
NP types of exactly n variables	2	1	3	16	380	1 227 756	400 507 805 615 570	—	—
NP threshold functions of up to n variables	2	3	5	10	27	119	1113	29 375	2 730 166
NP threshold functions of exactly n variables	2	1	2	5	17	92	994	28 262	2 700 791
NPN types of up to n variables	1	2	4	14	222	616 126	200 253 952 527 184	—	—
NPN types of exactly n variables	1	1	2	10	208	615 904	200 253 951 911 058	—	—
NPN threshold functions of up to n variables	1	2	3	6	15	63	567	14 755	1 366 318
NPN threshold functions of exactly n variables	1	1	1	3	9	48	504	14 188	1 351 563
Self-duality types of up to $n+1$ variables	1	1	3	7	83	109 958	—	—	—
Self-duality types of exactly $n+1$ variables	1	0	2	4	76	109 875	—	—	—
Self-duality type threshold functions of up to $n+1$ variables	1	1	2	3	7	21	135	2470	175 428
Self-duality type threshold functions of exactly $n+1$ variables	1	0	1	1	4	14	114	2335	172 958.

Note that the 7th, 8th, and the last four rows are for $n+1$ variables instead of for n variables.

TABLE II

n Number of Variables	$(1/n^2) \log_2 R_n$
1	2
2	0.95184
3	0.74449
4	0.67988
5	0.66117
6	0.66226
7	0.67273
8	0.68740

TABLE III
AVERAGE SIZE OF WEIGHTS

	Total Sum	Average Size
w_1	3 367 459	19.2
w_2	2 643 212	15.1
w_3	2 150 516.5	12.3
w_4	1 762 341	10.0
w_5	1 442 171	8.2
w_6	1 145 993.5	6.5
w_7	882 375.5	5.0
w_8	628 050.5	3.6
w_9	379 946	2.2

function of exactly nine variables which is not a threshold function and also that there is no completely monotonic function of eight or fewer variables which is not a threshold function. Therefore, as Gabelman had shown [9], for nine variables there appears for the first time a completely monotonic function which is not a threshold function.

The computer program which we wrote in NICAP for ILLIAC II contains various error-checking procedures. All strongly asymmetrical threshold functions of exactly nine variables are printed out and compared with the self-dual threshold functions of exactly eight variables. This partly served to insure that the generation program generated all nine variable functions, because strongly asymmetrical canonical positive threshold functions of exactly n variables have a one-to-one correspondence with canonical positive threshold functions of exactly $(n-1)$ variables [7].

Computation time of our computer program was approximately ten hours on ILLIAC II. It includes time for auxiliary error-checking programs and intermediate printings.

IV. LINEAR PROGRAM TO MINIMIZE A THRESHOLD

In this section, the linear program to minimize a threshold is discussed.

Let us define a linear program corresponding to the threshold expression as follows.

$$\text{Minimize } T \tag{14}$$

under the constraints

$$w_i \geq 0 \qquad (i = 1, 2, \cdots, n), \tag{15}$$

$$\sum_{i=1}^{n} w_i x_i^{(j)} \geq T \qquad \text{for } f(x_1^{(j)}, \cdots, x_n^{(j)}) = 1, \tag{16}$$

and

TABLE IV
CANONICAL POSITIVE SELF-DUAL THRESHOLD FUNCTIONS WHICH HAVE MULTIPLE EXTREME OPTIMUM SOLUTIONS

Chow's Parameters	One Set of Extreme Optimum Weights*
83, 33, 31, 31, 19, 19, 19, 13, 13	13, 7, 6, 6, 4, 4, 4, ③, ②
87, 33, 31, 25, 25, 21, 9, 7, 7	17, 9, 8, ⑦, ⑥, 5, 3, 2, 2
72, 44, 32, 32, 30, 18, 18, 12, 12	13, 9, 7, 7, 6, 4, 4, ③, ②
79, 41, 31, 31, 23, 23, 11, 9, 9	14, 9, ⑦, ⑥, 5, 5, 3, 2, 2
77, 43, 31, 31, 25, 25, 9, 7, 7	17, 12, 8, 8, ⑦, ⑥, 3, 2, 2
68, 52, 32, 32, 22, 22, 22, 8, 8	11, 9, 6, 6, 4, 4, 4, ②, ①
66, 54, 32, 32, 24, 24, 20, 6, 6	13, 11, 7, 7, 5, 5, 4, ②, ①
67, 53, 37, 27, 27, 19, 19, 7, 7	13, 11, 8, 6, 6, 4, 4, ②, ①
66, 54, 36, 28, 28, 20, 18, 6, 6	15, 13, 9, 7, 7, 5, 4, ②, ①
59, 47, 45, 33, 25, 25, 19, 19, 7	13, 11, 10, 8, 6, 6, ⑤, ④, 2
65, 55, 41, 31, 23, 15, 15, 5, 5	16, 14, 11, 9, 6, 4, 4, ②, ①
64, 56, 40, 32, 24, 16, 14, 4, 4	18, 16, 12, 10, 7, 5, 4, ②, ①

* The other set of extreme optimum weights is obtained by interchanging encircled weights.

TABLE V
CANONICAL POSITIVE SELF-DUAL THRESHOLD FUNCTIONS WHOSE EXTREME OPTIMUM SOLUTIONS INCLUDE FRACTIONAL WEIGHTS

Chow's Parameters	Unique Optimum Weights*
66, 54, 40, 30, 24. 16, 16, 6, 6	14.5, 12.5, 9.5, 7.5, 6, 4, 4, 1.5, 1.5, $T=31$
65, 55, 39, 31, 25, 17, 15, 5, 5	16.5, 14.5, 10.5, 8.5, 7, 5, 4, 1.5, 1.5, $T=35$

* T is a threshold.

$$\sum_{i=1}^{n} w_i x_i^{(j)} \leq T - 1 \qquad \text{for } f(x_1^{(j)}, \cdots, x_n^{(j)}) = 0 \tag{17}$$

for a positive function f. Minimization of T has some engineering motivation [10]. It gives maximization of input tolerance (revision of the proof of this in [10] will be given elsewhere), yielding more reliable operation of threshold elements. The inequalities (16) and (17) are called a *normalized system of inequalities in threshold expression.*

Note that the inequalities in majority expression (3) and (4) can be converted into (16) and (17), respectively, by the conversion formula

$$w_0 \xi_0 = \sum_{i=1}^{n} w_i + 1 - 2T \tag{18}$$

and vice versa. The objective function (14) is equivalent to (3) by (18), when $\xi_0 = -1$ or $w_o = 0$, but they are different when $\xi_0 = +1$. In other words, the linear program to minimize $\sum_{i=0}^{n} w_i$ of (1)–(4) is equivalent to the linear program to minimize T of (14)–(17), if a given function is a minor function or a self-dual function. If the function is a major function, these two linear programs are not equivalent [10]. Actually they have different solutions for some major functions. For example, optimum solutions of the linear program to minimize T for the major function

$$f(x_1, x_2, x_3) = x_1 \vee x_2 x_3$$

are

$$w_1^0 \geq 2, w_2^0 = w_3^0 = 1 \text{ and } T^0 = 2,$$

TABLE VI
NUMBER OF CANONICAL POSITIVE MAJOR THRESHOLD FUNCTIONS AND NUMBER OF THOSE WHICH HAVE SINGLE LITERAL PRIME IMPLICATIONS

n, Number of Variables	$M(n)$, Number of Canonical Positive Major Threshold Functions of Exactly n Variables	$S(n)$, How Many of $M(n)$ Have at Least One Single Literal Prime Implicant	$M(n)-S(n)$, How Many of $M(n)$ Do Not Have Single Literal Prime Implicants
1	0	0	0
2	1	1	0
3	2	2	0
4	8	5	3
5	44	17	27
6	490	92	398
7	14 074	994	13 080
8	1 349 228	28 262	1 320 966

TABLE VII
NUMBER OF CANONICAL POSITIVE MAJOR THRESHOLD FUNCTIONS WHICH HAVE DIFFERENT OPTIMUM STRUCTURES FOR THE TWO LINEAR PROGRAMS

n, Number of Variables	$M(n)$, Number of Canonical Positive Major Threshold Functions of Exactly n Variables	Number of Functions, Among $M(n)$ Functions, which have Nonequivalent Threshold and Majority Structures due to:			Number of Functions, Among $M(n)$ Functions, Which Have Unique and Equivalent Threshold and Majority Structures
		1) Single Literal Prime Implicants are Possessed	2) Multiple Threshold Structures and a Unique Majority Structure	3) Threshold and Majority Structures Both Unique but Nonequivalent	
1	0	0	0	0	0
2	1	1	0	0	0
3	2	2	0	0	0
4	8	5	0	0	3
5	44	17	0	0	27
6	490	92	6	0	392
7	14 074	994	314	0	12 766

while its optimum solution for the linear program to minimize $\sum_{i=0}^{n} w_i$ is $(2, 1, 1; w_0\xi_0=1)$ which is unique. In other words, the linear program for this f defined by (14)–(17) has multiple optimum solutions, while that defined by (1)–(4) has a unique solution.

It is easy to prove the following which was observed by Yen.

Theorem 1: The linear program to minimize T, (14)–(17), for a positive threshold function has optimum weights of unbounded magnitude if and only if the function contains a prime implicant consisting of a single literal.

Therefore, at least for those functions which have single literal prime implicants, the linear program to minimize T has optimum solutions different from those of the linear program to minimize W, (1)–(4), since the latter linear program always has optimum weights of finite magnitude for threshold functions.

Table VI shows the number of canonical positive major threshold functions of exactly n variables (the class of functions which are equivalent by complementation and/or permutation of variables and/or complementation of a function) and it also shows how many of them have at least one single literal prime implicant.

A computer program was written for investigating how many of $(M(n)-S(n))$ of Table VI have different optimum structures in threshold expression from those in majority expression. The number of the interesting functions of eight variables, 1 320 966, is unfortunately too big to handle. So our investigation was limited only to the case of the functions of seven or fewer variables.

The linear programs to minimize T, (14)–(17), were formulated for all canonical positive major threshold functions of seven or fewer variables except those with single literal prime implicants. After solving these linear programs by simplex method, it was checked whether the optimum solution $(w_1^0, \cdots, w_n^0; T^0)$ obtained in the last tableau was identical to the optimum solution $(w_1^0, \cdots, w_n^0, w_0^0\, \xi_0)$ of the linear program of (1)–(4), by the conversion formula (18).

As was seen in Table VI, 3 (4 variables), 27 (5 variables), 398 (6 variables), and 13 080 (7 variables) major functions were actually investigated to find whether they have different structures in threshold expression and majority expression. Table VII shows the result of this investigation.

A total of 320 functions of seven or fewer variables were found which do not have single literal prime implicants and which have a different optimum structure in threshold expression from an optimum structure in majority expression. All of those structures have the following properties: 1) multiple and bounded; 2) one of the extreme optimum

TABLE VIII
MULTIPLE OPTIMUM STRUCTURES OF FUNCTIONS OF SIX VARIABLES

w_1	w_2	w_3	w_4	w_5	w_6	T
6~7	4	3	3	2	2	8
5~6	4	3	3	2	2	7
7~8	5	4	3	2	2	9
6~7	5	4	3	2	2	8
5~6	3	3	2	2	2	7
4~5	3	3	2	2	2	6

TABLE IX
TYPES OF THRESHOLD FUNCTIONS OF SEVEN VARIABLES WHICH HAVE MULTIPLE OPTIMUM SOLUTIONS FOR LINEAR PROGRAM TO MINIMIZE T

Type	Number of Functions	w_1	w_2	w_3	w_4	w_5	w_6	w_7	T
1	200	$a\sim a+1$						2	$a+2$
2	22	$a\sim a+2$						3	$a+3$
3	6	$a\sim a+1$	$a\sim a+1$					2	$a+2$
4	46	$a\sim a+1$					$b+2$	b	$a+b+2$
5	18	$a\sim a+1$				$b+2$	b		$a+b+2$
6	10	$a\sim a+1$			$b+2$	b			$a+b+2$
7	4	$a\sim a+2$					$b+3$	b	$a+b+3$
8	8	Miscellany (See Table X)							

TABLE X
EXAMPLES OF THRESHOLD FUNCTIONS SHOWN IN TABLE IX

Type	Optimum Weights							T
1	6~ 7,	5,	4,	4,	3,	3,	2	8
2	7~ 9,	6,	5,	5,	4,	3,	3	10
3*	5~ 6,	5~6,	4,	3,	3,	2,	2	7
	6~ 7,	6~7,	5,	4,	3,	2,	2	8
	4~ 5,	4~5,	3,	3,	2,	2,	2	6
	7~ 8,	7~8,	5,	4,	3,	2,	2	9
	5~ 6,	5~6,	3,	3,	2,	2,	2	7
	6~ 7,	6~7,	4,	3,	3,	2,	2	8
4	6~ 7,	4,	4,	3,	3,	3,	1	9
5	8~ 9,	7,	6,	5,	5,	3,	3	13
6	10~11,	9,	7,	6,	4,	4,	2	16
7*	13~15,	9,	7,	6,	4,	4,	1	17
	11~13,	9,	7,	6,	4,	4,	1	15
	11~13,	7,	6,	5,	4,	4,	1	15
	9~11,	7,	6,	5,	4,	4,	1	13
8*	10~11,	9,	8,	5,	3,	2,	2	17
	8~ 9,	6,	5,	5,	2,	2,	2	12
	7~ 8,	6,	5,	5,	2,	2,	2	11
	9~10,	7,	6,	5,	2,	2,	2	13
	8~ 9,	7,	6,	5,	2,	2,	2	12
	9,	7 ~ 8,	5,	5,	3,	2,	2	14
	9,	6 ~ 7,	5,	5,	3,	2,	2	13
	9,	8,	6~7,	5,	3,	2,	2	13

* All functions of types 3, 7, and 8 in Table IX are shown here.

solutions agrees with the structure derived from the optimal majority structure, by the conversion formula (18).

Six functions of six variables with multiple solutions are listed in Table VIII. In this table w_1 is any number in the range shown. For example, w_1 of the first row is any number between 6 and 7. 314 functions of seven variables which have multiple optimum solutions are classified into eight types in Table IX. These fairly crude classifications are chosen only to demonstrate a few of the general patterns observed in the list of seven variable functions. For type 1, w_1 may be any number between a and $a+1$ where a is the number determined for each function. $w_7=2$, $w_2<a$, $T=a+2$ for all functions in this type, and $x_1\ x_7$ is a prime implicant but $x_2\ x_7$ is not. For type 3 both w_1 and w_2 may be any numbers between a and $a+1$, but not necessarily $w_1=w_2$. Type 8 is a miscellany which includes all functions not belonging to the previous seven types and all functions of type 8 are shown in Table X. The first function of type 8 in Table X has w_1 which may be any number between 10 and 11 but the threshold is not related in a way as in type 1. Table X shows some examples of functions of the types of Table IX. For types 3, 7, and 8, all functions are shown.

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