Missouri State University's Advanced Problem of February 2010

Vincent PANTALONI (Orléans, France)

Problem : Evaluate the following integral :

$$\int_0^1 \left(\left(\frac{1}{x}\right)\right) \, \mathrm{d}x$$

where $((x))$ denotes the fractional part of a number x.

Solution : ....

$\forall x \in \mathbb{R},\ ((x)) = x - [x]$. Where $[x]$ is the integer part of x. The above integral is undefined in 0 so we have to evaluate :

$$\lim_{n\to+\infty} \int_{\frac{1}{n}}^1 \left(\left(\frac{1}{x}\right)\right) \, \mathrm{d}x$$

Let n be an integer greater than 2 and consider : $I_n = \int_{\frac{1}{n}}^1 \left(\left(\frac{1}{x}\right)\right) \, \mathrm{d}x$. Then :

$$\begin{aligned}
I_n &= \int_{\frac{1}{n}}^1 \frac{1}{x} - \left[\frac{1}{x}\right] \, \mathrm{d}x \\
&= \int_{\frac{1}{n}}^1 \frac{1}{x} \, \mathrm{d}x - \int_{\frac{1}{n}}^1 \left[\frac{1}{x}\right] \, \mathrm{d}x \\
&= \ln(n) - \int_{\frac{1}{n}}^1 \left[\frac{1}{x}\right] \, \mathrm{d}x \\
&= \ln(n) - \sum_{k=2}^{n} \int_{\frac{1}{k}}^{\frac{1}{k-1}} \left[\frac{1}{x}\right] \, \mathrm{d}x \\
&= \ln(n) - \sum_{k=2}^{n} \int_{\frac{1}{k}}^{\frac{1}{k-1}} k - 1 \, \mathrm{d}x \\
&= \ln(n) - \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) \times (k-1) \\
&= \ln(n) - \sum_{k=2}^{n} \left(1 - \frac{k-1}{k}\right) \\
&= \ln(n) - \sum_{k=2}^{n} \frac{1}{k} \\
&= 1 + \ln(n) - \sum_{k=1}^{n} \frac{1}{k}
\end{aligned}$$

We know that $\lim\limits_{n\to+\infty} \left(\left(\sum_{k=1}^n \frac{1}{k}\right) - \ln(n)\right) = \gamma$. Where $\gamma \approx 0.5772156649$ is the EULER constant. Hence :

$$\boxed{\int_0^1 \left(\left(\frac{1}{x}\right)\right) \, \mathrm{d}x = 1 - \gamma}$$