Classical and Quantum Gravity

PAPER

The covariant formulation of $f(T)$ gravity

To cite this article: Martin Krššák and Emmanuel N Saridakis 2016 *Class. Quantum Grav.* **33** 115009

IOP Publishing Classical and Quantum Gravity
Class. Quantum Grav. **33** (2016) 115009 (15pp) doi:10.1088/0264-9381/33/11/115009

The covariant formulation of $f(T)$ gravity

Martin Krššák[1,4] **and Emmanuel N Saridakis**[2,3]

[1] Instituto de Física Teórica, Universidade Estadual Paulista Rua Dr. Bento Teobaldo Ferraz 271, 01140-070 São Paulo, SP, Brazil
[2] CASPER, Physics Department, Baylor University, Waco, TX 76798-7310, USA
[3] Instituto de Física, Pontificia Universidad de Católica de Valparaíso, Casilla 4950, Valparaíso, Chile

E-mail: krssak@ift.unesp.br and Emmanuel_Saridakis@baylor.edu

Received 23 December 2015, revised 9 March 2016
Accepted for publication 21 March 2016
Published 28 April 2016

Abstract

We show that the well-known problem of frame dependence and violation of local Lorentz invariance in the usual formulation of $f(T)$ gravity is a consequence of neglecting the role of spin connection. We re-formulate $f(T)$ gravity starting from, instead of the 'pure tetrad' teleparallel gravity, the covariant teleparallel gravity, using both the tetrad and the spin connection as dynamical variables, resulting in a fully covariant, consistent, and frame-independent version of $f(T)$ gravity, which does not suffer from the notorious problems of the usual, pure tetrad, $f(T)$ theory. We present the method to extract solutions for the most physically important cases, such as the Minkowski, the Friedmann–Robertson–Walker (FRW) and the spherically symmetric ones. We show that in covariant $f(T)$ gravity we are allowed to use an arbitrary tetrad in an arbitrary coordinate system along with the corresponding spin connection, resulting always in the same physically relevant field equations.

Keywords: modified gravity, $f(T)$ gravity, teleparallel gravity

1. Introduction

Modified gravity [1, 2] is one of the two main approaches that one can follow in order to describe the two accelerated phases of expansion at early and late times, respectively, (the other one is to introduce the concept of dark energy in the framework of general relativity [3, 4]), which moreover has the additional motivation of alleviating the difficulties in the quantization of gravity and improving its ultraviolet behavior [5, 6]. However, even if one decides to take the serious step of modifying gravity, there is still the question of what

[4] Author to whom any correspondence should be addressed.

formulation of gravity to modify. Most of the works in the literature start from the usual, curvature-based formulation, and modify/extend the Einstein–Hilbert action, with the simplest example being the $f(R)$ paradigm in which the Lagrangian is considered to be a non-linear function of the curvature scalar [7, 8].

However, one could start from the teleparallel equivalent of general relativity (TEGR) [9–19], in which gravity is described through torsion and the gravitational Lagrangian is the torsion scalar T, and construct various extensions. Along these lines, the simplest torsional modification is the $f(T)$ paradigm, in which the Lagrangian is taken to be a non-linear function of T [20–23]. The crucial issue is that although TEGR coincides completely with general relativity at the level of equations, $f(T)$ is different from $f(R)$ gravity, with novel features (amongst others note the significant advantage that the field equations of $f(T)$ gravity are of second order while those of $f(R)$ are of fourth order) and interesting cosmological implications, and that is why it has gained a lot of interest in the literature [24–53].

Unfortunately, in the standard formulation of $f(T)$ gravity local Lorentz invariance is either completely absent or strongly restricted [54, 55], due to the strong imposition made in [20–23] that the spin connection vanishes. Although this assumption has a good motivation, namely to make the theory simpler in order to be able to extract solutions, and although it is based on the fact that the spin connection is not a tensor and under a local Lorentz transformation it transforms non-covariantly and thus it is always possible to transform to a frame that it is zero, in general this assumption makes the theory frame-dependent since the solution of the field equations depends on the choice of the frame [54]. One can neglect this issue and investigate solutions in particular frames (this is in analogy with the investigation of electromagnetism in the particular class of inertial frames), however strictly speaking the problem is there and will become obvious when questions about frame transformations and Lorentz invariance are raised, which is usual in the case of spherically symmetric solutions.

The feature of local Lorentz invariance violation is clearly a deficit of the standard formulation of $f(T)$ gravity. While this problem could be avoided by including higher derivative terms [56, 57], the resulting theories lose the most attractive feature of $f(T)$ theories, which is that the field equations are of second order. In the present work we desire to reformulate $f(T)$ gravity in order to be fully covariant, and this will be achieved by relaxing the strong assumption of setting the spin connection to zero. Hence, we obtain a theory that has both attractive features: it preserves local Lorentz symmetry and the field equations are of second order.

The outline of this paper is as follows. In section 2 we briefly introduce the covariant formulation of teleparallel gravity, keeping both the tetrad and the spin connection as dynamical variables, and focusing on the role of the inertial effects in the theory, explaining why the field equations are not affected by them. Based on this, in section 3 we construct the covariant and consistent version of $f(T)$ theories, deriving the field equations which include the spin connection. In section 4 we present the method of extracting solutions in such theories, illustrating it in the most physically relevant cases such as the Minkowski, the Friedmann–Robertson–Walker (FRW) and the spherically symmetric ones. Lastly, in section 5 we summarize our results.

2. Covariant teleparallel gravities

In this section we discuss the Lorentz invariance and the various versions of teleparallel gravity. In section 2.1 we present the covariant formulation of teleparallel gravity, while section 2.2 we present the covariant formulation of the teleparallel equivalent of general relativity, in which one keeps both the tetrad and the spin connection as dynamical variables.

2.1. General covariance, tetrad and spin connection

Let us first start by discussing ordinary teleparallel gravity and its origins. In his first relevant papers, Einstein was motivated by the observation that a tetrad has 16 independent components, of which only 10 are needed to determine the metric tensor and hence describe gravity, and thus the additional six degrees of freedom could describe the electromagnetic field [9–11]. However, later on [11–16] it was realized that the six additional degrees of freedom in the tetrad were actually related to the possible ways of choosing the observer, and therefore related to the inertial effects instead of electromagnetism, and the corresponding theory was named teleparallel gravity (due to the way one 'parallelizes' the tetrads from a 'distance'). We mention that in this formulation of teleparallel gravity one still uses the original Einstein definition of teleparallelism, which in our modern language can be defined as a geometry where the spin connection vanishes identically. Hence, the only dynamical variable is the tetrad, and therefore we refer to it as pure tetrad teleparallel gravity. The disadvantage of such a construction is that the torsion tensor is effectively replaced by the coefficients of anholonomy, that are not tensors under local Lorentz transformations, which is the cause of violation of local Lorentz symmetry in this theory. However, it turns out that these violations do not affect the field equations, which are still invariant under local Lorentz transformations [58], which ensures that the metric tensor obtained from pure tetrad teleparallel gravity is the correct one (see [59] for a recent review on this subject).

Later on it was realized that if the teleparallel geometry is defined more generally as a geometry with zero curvature, the most general spin connection that satisfies this requirement is the purely inertial spin connection, analogous to a pure gauge connection in gauge theories. Such a connection vanishes only in a very special class of frames called proper frames, which are the frames in which inertial effects are absent. This is a very subtle, but important difference in the pure tetrad formulation, where the spin connection is considered to vanish in all frames. The main advantage of this approach is that if the purely inertial connection is used, teleparallel gravity has manifest local Lorentz invariance. We refer to this theory as the covariant teleparallel gravity. This approach was pioneered in [60] in the framework of metric-affine theories [61], and then investigated further in [58, 62] and fully adopted in [19].

The resulting, covariant, teleparallel gravity seems to be more complicated than the pure tetrad one, due to the appearance of the spin connection. However, as it turns out, one can find a self-consistent method to solve the field equations and determine the spin connection [63]. In addition to having the theory that respects the crucial local Lorentz symmetry, we are able to separate the gravitational from inertial effects and remove the infrared divergences from the action [63, 64].

2.2. Teleparallel equivalent of general relativity

Having discussed above the general idea of covariantizing teleparallel gravity, in this subsection we present the covariant formulation of TEGR. In such a formulation the fundamental variables are the tetrad $h^a_{\ \mu}$ and the spin connection $\omega^a{}_{b\mu}$. The tetrad is a set of four orthonormal vectors that represents a frame of reference for the physical observer, and is related to the metric tensor through

$$g_{\mu\nu} = \eta_{ab} h^{a}_{\ \mu} h^{b}_{\ \nu}, \tag{1}$$

where $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$ (we use the notation where the Latin letters denote the tangent space indices, while Greek letters denote the spacetime indices).

The spin connection determines the torsion and curvature tensors, which in return completely characterize the spin connection, through

$$T^{a}_{\ \mu\nu}(h^{a}_{\ \mu}, \omega^{a}_{\ b\mu}) = \partial_\mu h^{a}_{\ \nu} - \partial_\nu h^{a}_{\ \mu} + \omega^{a}_{\ b\mu} h^{b}_{\ \nu} - \omega^{a}_{\ b\nu} h^{b}_{\ \mu}, \tag{2}$$

and

$$R^{a}_{\ b\mu\nu}(\omega^{a}_{\ b\mu}) = \partial_\mu \omega^{a}_{\ b\nu} - \partial_\nu \omega^{a}_{\ b\mu} + \omega^{a}_{\ c\mu}\omega^{c}_{\ b\nu} - \omega^{a}_{\ c\nu}\omega^{c}_{\ b\mu}. \tag{3}$$

In general relativity one uses the Levi–Civita connection $\overset{\circ}{\omega}{}^{a}_{\ b\mu}$, which by construction gives vanishing torsion, and thus all the information of the gravitational field is embedded in the curvature (Riemann) tensor, while the gravitational Lagrangian is the curvature (Ricci) scalar. On the other hand, in TEGR one uses the teleparallel (Weitzenböck) spin connection, which by construction gives vanishing curvature, and thus all the information on the gravitational field is embedded in the torsion tensor, while the gravitational Lagrangian is the torsion scalar. In particular, in TEGR the dynamics is derived from the Lagrangian density[5]

$$\mathcal{L} = \frac{h}{4\kappa} T, \tag{4}$$

with $h = \det h^{a}_{\ \mu}$ and $\kappa = 8\pi G$ the gravitational constant, where

$$T = T^{a}_{\ \mu\nu} S_{a}^{\ \mu\nu} \tag{5}$$

is the torsion scalar constructed by contractions of the torsion tensor, with

$$S_{a}^{\ \mu\nu} = K^{\mu\nu}_{\ \ \ a} - h_{a}^{\ \nu} T^{\alpha\mu}_{\ \ \ \alpha} + h_{a}^{\ \mu} T^{\alpha\nu}_{\ \ \ \alpha} \tag{6}$$

the superpotential and $K^{\mu\nu}_{\ \ \ a}$ the contortion tensor defined as

$$K^{\mu\nu}_{\ \ \ a} = \frac{1}{2}(T_{a}^{\ \mu\nu} + T^{\nu\mu}_{\ \ \ a} - T^{\mu\nu}_{\ \ \ a}). \tag{7}$$

The field equations are obtained by varying the gravitational Lagrangian (4) and the matter Lagrangian $\mathcal{L}_M$ with respect to the tetrad, namely

$$h^{-1}\partial_\sigma(h S_{a}^{\ \rho\sigma}) - h_{a}^{\ \mu} S_{b}^{\ \nu\rho} T^{b}_{\ \nu\mu} + \frac{h_{a}^{\ \rho}}{4} T - \omega^{b}_{\ a\sigma} S_{b}^{\ \rho\sigma} = \kappa \Theta_{a}^{\ \rho}, \tag{8}$$

where, as usual, we have defined the energy-momentum tensor of matter through

$$\Theta_{a}^{\ \rho} = \frac{1}{h}\frac{\delta \mathcal{L}_M}{\delta h^{a}_{\ \mu}}. \tag{9}$$

Let us now discuss the Lorentz transformation issues. A local Lorentz transformation is represented by the Lorentz matrix $\Lambda^{a}_{\ b} = \Lambda^{a}_{\ b}(x)$ that obeys

$$\eta_{ab} = \eta_{cd}\, \Lambda^{c}_{\ a} \Lambda^{d}_{\ b}, \tag{10}$$

[5] For convenience, in this manuscript we follow the conventions used in TEGR literature that slightly differ from those in *f*(*T*) gravity works by a factor of 2. Therefore, the superpotential in *f*(*T*) gravity is usually defined as one half of our definition (6).

under which the tetrad and spin connection transform simultaneously as

$$h'^{a}{}_{\mu} = \Lambda^{a}{}_{b} h^{b}{}_{\mu}, \qquad \text{and} \qquad \omega'^{a}{}_{b\mu} = \Lambda^{a}{}_{c}\omega^{c}{}_{d\mu}\Lambda_{b}{}^{d} + \Lambda^{a}{}_{c}\partial_{\mu}\Lambda_{b}{}^{c}, \tag{11}$$

where $\Lambda_{a}{}^{b} = (\Lambda^{-1})^{b}{}_{a}$ is the inverse Lorentz matrix. Hence, if we keep both the tetrad and the spin connection in the formulation of TEGR, the theory preserves local Lorentz invariance.

As one can see, the most general spin connection with vanishing curvature is the purely inertial spin connection [19]

$$\omega^{a}{}_{b\mu} = \Lambda^{a}{}_{c}\partial_{\mu}\Lambda_{b}{}^{c}, \tag{12}$$

which represents only the inertial effects, since it depends only on the choice of the frame. However, since it depends only on the choice of the observer represented by a Lorentz matrix, it is not determined uniquely. In particular, various spin connections represent different inertial effects for the observer, which, however, do not affect the field equations [63]. This follows from the fact that the teleparallel Lagrangian can be written as [64]

$$\mathcal{L}(h^{a}{}_{\mu}, \omega^{a}{}_{b\mu}) = \mathcal{L}(h^{a}{}_{\mu}, 0) + \frac{1}{\kappa}\partial_{\mu}(h\omega^{\mu}), \tag{13}$$

where $\omega^{\mu} = \omega^{ab}{}_{\nu} h_{a}{}^{\nu} h_{b}{}^{\mu}$. Therefore, the spin connection enters the teleparallel action only as a surface term. A variation of the surface term vanishes, and hence the choice of the spin connection does not effect the field equations. This property allows us to solve the field equations using an arbitrary spin connection. In practice, we usually choose the zero spin connection, which is trivially of the correct form (12).

On the other hand, the torsion tensor (2) is a function of both the tetrad and spin connection. As a consequence, torsion can represent the field strength of both gravity and inertia [64]. However, there exists a preferred choice of the spin connection, which results in torsion that is the field strength of gravity only. This spin connection can be obtained by considering the reference tetrad defined by setting the gravitational constant to zero

$$h^{a}_{(\mathrm{r})^{\mu}} \equiv h^{a}{}_{\mu}|_{G\to 0}, \tag{14}$$

and requiring torsion to vanish there, namely $T^{a}{}_{\mu\nu}(h^{a}_{(\mathrm{r})^{\mu}}, \omega^{a}{}_{b\mu}) \equiv 0$. We find that this defines the spin connection as

$$\omega^{a}{}_{b\mu} = \overset{\circ}{\omega}{}^{a}{}_{b\mu}(h_{(\mathrm{r})}). \tag{15}$$

If the spin connection is chosen in this way the torsion tensor (2) is indeed the field strength of gravity only. Consequently, the action (4) represents the gravitational effects only, without the spurious inertial contributions (see [63, 64] for further details).

We close this section by mentioning the crucial fact that in TEGR the spin connection does not effect the field equations, which allows us to use an arbitrary spin connection to solve the field equations first, and then calculate the appropriate spin connection from the solution of the field equations. This feature lies behind the success of the pure tetrad formulation of teleparallel gravity, where the spin connection is considered to vanish identically. The solution of the field equations will represent, in addition to gravity, the inertial effects, but these do not affect the metric tensor. Therefore, as far as we are strictly interested in the metric tensor only, the role of the spin connection can be neglected. However, this is not the case when one modifies the teleparallel Lagrangian and abandons its linearity in the torsion scalar, and indeed in this case the covariant formulation is necessary.

Having developed the techniques for covariantizing the theory, in the following section we construct the covariant and fully consistent formulation of *f*(*T*) gravity, which is the main goal of this work.

3. Covariant *f*(*T*) gravity

As was mentioned above, in the usual formulation of *f*(*T*) gravity one generalizes the pure tetrad teleparallel gravity [20–23], and thus the violation of local Lorentz invariance is inherited. However, the crucial novel feature is that although in pure tetrad TEGR this violation does not affect the field equations that are still invariant under local Lorentz transformations, which allows us to use an arbitrary, for instance zero, spin connection to solve the field equations, in the case of pure tetrad *f*(*T*) gravity this is no longer true and the solutions of the field equations do depend on the frame choice. Hence, in the usual formulation of *f*(*T*) gravity both the action and the field equations are not invariant under local Lorentz transformations [54].

In order to clearly see the above problem we recall that according to (13) we can write the torsion scalar as

$$T(h^a{}_\mu,\, \omega^a{}_{b\mu}) = T(h^a{}_\mu,\, 0) + \frac{4}{h}\partial_\mu(h\omega^\mu). \tag{16}$$

Hence, as long as the Lagrangian remains a function linear in T, then the spin connection appears in the action only as a surface term, and its presence does not affect the field equations. However, if the function *f*(*T*) is anything other than a linear function then the total divergence in (16) is not a total divergence in the whole *f*(*T*) Lagrangian (17). In such a case the variation of the second term will be, in general, non-vanishing, and terms proportional to the spin connection will enter the field equations. This implies that the solution of the field equations will depend on the choice of the spin connection, which can be understood as that the field equations are potentially the field equations for both gravity and inertia.

In order to elaborate this problem further, let us recall that the spin connection represents just a choice of the observer, i.e. the inertial effects associated with the observer. On the other hand, the solution of the field equations determines the metric tensor, which describes the spacetime geometry. Therefore, we deduce that we are running into a severe problem, since the metric tensor of spacetime depends on the inertial effects associated with the observer, i.e. the geometry itself becomes frame-dependent.

The cause of the problem is that in the usual, pure tetrad, formulation of *f*(*T*) gravity the action was constructed using only the first term on the right-hand side of (16) and the surface term was neglected. However, in the *f*(*T*) case, as we have just discussed, this neglected term is crucial and the solution of the field equations depends on it. Thus, the pure tetrad *f*(*T*) theory leads to physically sensible results only in the case where the total divergence in (16) is actually zero, which is the case only for the proper tetrad in which the spin connection vanishes. This is why people were forced to discuss and construct 'good' and 'bad' tetrads [65–67].

In this section we resolve the above severe problem, by re-formulating *f*(*T*) gravity starting from, instead of the pure tetrad teleparallel gravity, the covariant teleparallel gravity presented in the previous section. Hence, we can indeed formulate the fully covariant, consistent, and frame-independent, version of *f*(*T*) gravity, which does not suffer from the notorious problems of the usual, pure tetrad, *f*(*T*) theory.

In order to construct the covariant *f*(*T*) gravity, we generalize the covariant TEGR. In particular, the action of the theory will be

$$\mathcal{L}_f = \frac{h}{4\kappa} f(T), \tag{17}$$

where $f(T)$ is an arbitrary function of the torsion scalar (5). The field equations are derived through a variation with respect to the tetrad. As shown in detail in the appendix, they are found to be:

$$E_a{}^{\mu} \equiv h^{-1} f_T\, \partial_\nu (h S_a{}^{\mu\nu}) + f_{TT}\, S_a{}^{\mu\nu} \partial_\nu T - f_T\, T^b{}_{\nu a} S_b{}^{\nu\mu} + f_T\, \omega^b{}_{a\nu} S_b{}^{\nu\mu} + \frac{1}{4} f(T) h_a{}^{\mu} = \kappa \Theta_a{}^{\mu}, \tag{18}$$

where f_T and f_{TT} denote first and second order derivatives of $f(T)$ with respect to the torsion scalar T.

We mention that the field equations (18) coincide with the field equations of the usual $f(T)$ gravity in the case $\omega^b{}_{a\nu} = 0$ [20–23].

In summary, the theory (17) and the corresponding field equations (18) are indeed the covariant version of the theory we were looking for.

4. Solutions

In the previous section we formulated the covariant $f(T)$ gravity, resulting in the covariant field equations (18). Nevertheless, we now face the problem of how to extract solutions. In particular, although in TEGR the spin connection enters the action only as a surface term, and thus we can first solve the field equations to determine the tetrad and then calculate the spin connection from the solution, this no longer holds in the $f(T)$ case where the solution of the field equation does depend on the choice of the spin connection, but in order to solve the equations we need to have the spin connection. This feature implies that in principle we face a loop difficulty.

In order to avoid this problem we need a method of determining the spin connection that does not rely on the solution of the field equations. We can recall that in the TEGR case the spin connection (15) is calculated from the knowledge of the reference tetrad only. Although in the $f(T)$ case it is not possible to determine the reference tetrad from the solution of the field equations, the correct reference tetrad can be guessed by making some reasonable assumptions based on the symmetries of the geometry. As we will show, this is usually possible to achieve using knowledge of the coordinate system in which the tetrad is written.

In particular, in practice the starting point for any calculation is the ansatz tetrad, which is given by the symmetry of the problem being investigated. Assuming knowledge of the reference tetrad corresponding to the ansatz tetrad we can find the spin connection using (15), and then solve the field equations. If our guess of the reference tetrad is right then we will be able to remove the inertial effects from the action and hence solve the purely gravitational field equations that are not contaminated by the spurious inertial contributions. In the following subsections we demonstrate this method for various examples.

4.1. Minkowski spacetime

An illustrative example of the relevance of the spin connection in $f(T)$ theories is the simple Minkowski spacetime. We consider two different tetrads representing the Minkowski spacetime, each in a different coordinate system. Let us start with a diagonal tetrad in the Cartesian coordinate system:

$$h^a{}_{\mu} = \mathrm{diag}(1,\ 1,\ 1,\ 1). \tag{19}$$

It is easy to check that this tetrad is a proper tetrad, i.e. the associated inertial spin connection vanishes. This can be seen from the fact that the torsion tensor vanishes for this tetrad, namely

$$T^{a}_{\ \mu\nu}(h^{a}_{\ \mu}, 0) = 0. \tag{20}$$

The field equations in both TEGR and *f*(*T*) cases are then trivially satisfied. This is an expected result, since the field equations should be equations for gravity, which is absent in Minkowski spacetime.

On the other hand, if we consider a Minkowski diagonal tetrad in the spherical coordinate system

$$h^{a}_{\ \mu} = \text{diag}(1,\, 1,\, r,\, r\sin\theta), \tag{21}$$

we find that this tetrad is not a proper tetrad, since for zero spin connection the corresponding torsion tensor for this tetrad is non-vanishing, namely

$$T^{a}_{\ \mu\nu}(h^{a}_{\ \mu}, 0) \neq 0. \tag{22}$$

One can clearly see the crucial difference between the TEGR and *f*(*T*) cases. In particular, in the TEGR case, despite the fact that (22) represents the field strength of the inertial effects, and consequently the associated Lagrangian is non-vanishing

$$\mathcal{L}(h^{a}_{\ \mu}, 0) = \frac{1}{\kappa}\sin\theta, \tag{23}$$

the field equations are still satisfied. This is because the holographic relation (13) allows the Lagrangian (23) to be written as a surface term and hence the field equations to be trivially satisfied. By contrast, this does not happen in the *f*(*T*) case.

Following the essence of the covariant formulation described in the previous sections, the above problem can be solved by calculating an appropriate spin connection, which along this tetrad will remove these spurious inertial contributions from the tetrad. In particular, for the tetrad (21), the non-vanishing components of the inertial spin connection are

$$\omega^{\hat{1}}_{\ \hat{2}\theta} = -1, \quad \omega^{\hat{1}}_{\ \hat{3}\phi} = -\sin\theta, \quad \omega^{\hat{2}}_{\ \hat{3}\phi} = -\cos\theta. \tag{24}$$

If we use this spin connection, the torsion will vanish identically

$$T^{a}_{\ \mu\nu}(h^{a}_{\ \mu}, \omega^{a}_{\ b\mu}) = 0, \tag{25}$$

and hence the field equations are trivially satisfied in both the TEGR and *f*(*T*) cases.

4.2. FRW universe

Let us now proceed to the physically more interesting case of a FRW geometry, where one obtains non-trivial gravitational field equations.

We start with the Cartesian coordinate system and the diagonal tetrad that represent the FRW metric:

$$h^{a}_{\ \mu} = \text{diag}(1,\, a(t),\, a(t),\, a(t)), \tag{26}$$

where $a(t)$ is the scale factor (for simplicity we restrict to the flat case, however the non-flat case can be studied straightforwardly). The above tetrad leads to the torsion scalar[6] $T = -12\,H^2$, and hence the field equations (18) give rise to the Friedmann equations

$$\kappa\rho_M = 6H^2 f_T + \frac{1}{4}f, \tag{27}$$

[6] We recall that the superpotential (6) is defined without the factor $1/2$ usually used in *f*(*T*) gravity literature, and therefore the torsion scalar here differs from the usual value by a factor of 2.

$$\kappa(p_M + \rho_M) = 2\dot{H}(24f_{TT}H^2 - f_T), \tag{28}$$

where the subscript T denoting the derivative with respect to T, $H = \dot{a}/a$ is the Hubble parameter, and ρ_M and p_M are the energy density and pressure of the matter fluid, respectively. These are the correct $f(T)$ modified Friedmann equations capable of explaining the accelerated expansion of the Universe, as shown in [24–53].

On the other hand, in the spherical coordinate system a natural choice for the FRW tetrad is the diagonal one

$$h^a_{\ \mu} = \mathrm{diag}(1, (1 - kr^2)^{-\frac{1}{2}}a, ra, ra\sin\theta), \tag{29}$$

which, similar to the Minkowski case, is not a proper tetrad. It is easy to check that the field equations for this tetrad and vanishing spin connection are satisfied only in the TEGR case.

Similarly to the Minkowski case, this can be easily solved by finding the purely inertial spin connection corresponding to (29). We start with defining the reference tetrad by $a(t) = 1$, and using (15) we calculate the non-vanishing components of the spin connection as

$$\omega^{\hat{1}}_{\ \hat{2}\theta} = -(1 - kr^2)^{\frac{1}{2}}, \qquad \omega^{\hat{1}}_{\ \hat{3}\phi} = -(1 - kr^2)^{\frac{1}{2}}\sin\theta, \qquad \omega^{\hat{2}}_{\ \hat{3}\phi} = -\cos\theta. \tag{30}$$

Using this spin connection it is straightforward to check that the tetrad (29) leads to the same field equations (27)–(28) as the ones obtained from the tetrad (26), however we now have the advantage of general covariance. This verifies that if we take the role of inertial spin connection into consideration, then indeed both tetrads are equally good in $f(T)$ gravity, and the theory is no longer frame-dependent.

4.3. Spherically symmetric geometry

The spherically symmetric spacetime is an important issue that must be addressed properly in any theory of gravity. In the framework of $f(T)$ theories it has attracted lot of attention recently [65–87]. In most of these works it was argued that only very specific forms of the tetrad, with off-diagonal components, can lead to the physical outcome [65–81], but the physical motivation for this was not understood. Having constructed the covariant formulation of $f(T)$ gravity in the previous sections, we can now show that every tetrad corresponding to the desired metric is equally good and leads to solutions as long as the correct spin connection is used. Additionally, we will also provide an explanation for the necessity to use the complicated off-diagonal tetrad in the previous non-covariant formulation of $f(T)$ gravity.

As shown in [88], the spherically symmetric spacetime is necessarily static, and hence its metric can be written as

$$ds^2 = A(r)^2dt^2 - B(r)^2dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{31}$$

where $A(r)$ and $B(r)$ are arbitrary functions of the coordinate r. The most natural choice of the tetrad that corresponds to this metric has the simple diagonal form

$$h^a_{\ \mu} = \mathrm{diag}(A(r), B(r), r, r\sin\theta). \tag{32}$$

It is straightforward to check that if we assume the trivial spin connection $\omega^a_{\ b\mu} = 0$, then the field equation $E_{\hat{2}}^{\ \theta} = -\frac{8f_{TT}}{r^5}\cot\theta = 0$ gives us necessarily the condition $f_{TT} = 0$, which restricts the theory to TEGR. In the literature this feature is wrongly interpreted as 'the diagonal tetrad is not a good tetrad for spherically symmetric solutions in $f(T)$ gravity' [65–81].

Let us now show that the above issue is an artifact of the non-covariant formulation of $f(T)$ gravity. In particular, using the covariant formulation presented in the previous sections we will calculate the appropriate spin connection, which will allow us to use any tetrad giving the metric (31) without restricting the functional dependence of the Lagrangian. Similar to the previous examples, due to the fact that the solution of the field equations is unknown, we have to start with a guess for the reference tetrad corresponding to the tetrad (32). It is natural to expect that in the absence of gravity the diagonal tetrad should reduce to the tetrad (21) representing the Minkowski spacetime in spherical coordinates. Therefore, the corresponding spin connection is again given by

$$\omega^{\hat{1}}{}_{\hat{2}\theta} = -1, \quad \omega^{\hat{1}}{}_{\hat{3}\phi} = -\sin\theta, \quad \omega^{\hat{2}}{}_{\hat{3}\phi} = -\cos\theta. \tag{33}$$

Using this spin connection we can now remove the spurious inertial contributions and obtain the gravitational torsion tensor. The torsion scalar constructed from it is given by

$$T(h^a{}_\mu, \omega^a{}_{b\mu}) = -\frac{4(-1+B)(A-AB+2rA')}{r^2AB^2}, \tag{34}$$

where primes denote derivatives with respect to the radial coordinate r. Thus, using the tetrad (32) and the non-zero spin connection (33), the field equations (18) become

$$\begin{aligned} E_{\hat{0}}{}^t \equiv\ & (4r^4AB^5)^{-1}[8(24f_{TT} - f_T r^2)B^3 + 8(f_T r^2 - 8f_{TT})B^4 + fr^4B^5 \\ & + 64f_{TT}(rB' - 2rBB' + B)] + 2(r^4AB^3)^{-1}[(8f_{TT}r + f_T r^3)B' - 24f_{TT}] \\ & \times (r^3A^3B^5)^{-1}\{16f_{TT}r(B-1)^2BA'^2 + 2A(B-1)[f_T r^2B^3A' + 8f_{TT}B^2(A' - rA'') \\ & - 16f_{TT}rA'B' + 8f_{TT}B(A'(rB'-1) + rA'')]\} = 0, \end{aligned} \tag{35}$$

$$E_{\hat{1}}{}^r \equiv \frac{A(8f_T B - 8f_T + fr^2B^2) + 8f_T r(B-2)A'}{4r^2AB^3} = 0, \tag{36}$$

$$\begin{aligned} E_{\hat{2}}{}^\theta \equiv\ & (4r^5A^3B^5)^{-1}\{32f_{TT}r^2AA'\{B^3A' + 2rA'B' + B^2(rA'' - 3A') \\ & - B[A'(rB'-2) + rA'']\} - 32f_{TT}r^3(B-1)BA'^3\} \\ & + (4r^5B^5)^{-1}[(96f_{TT} - 4f_T r^2)B^3 + 8(f_T r^2 - 4f_{TT})B^4 + (fr^4 - 4f_T r^2)B^5 \\ & + 32f_{TT}rB' + 32f_{TT}B(1 - 2rB')] + \frac{(8f_{TT}r + f_T r^3)B' - 24f_{TT}}{r^5B^3} \\ & + (r^4AB^5)^{-1}\{2f_T r^2B^4A' + 24f_{TT}rA'B' - 8f_{TT}B[A'(-2+4rB') + rA'']\} \\ & + (r^4AB^3)^{-1}\{16f_{TT}rA'' - 32f_{TT}A'^2 + (8f_{TT}r + f_T r^3)B'A' \\ & - 1B[(3f_T r^2 - 16f_{TT})A' + r(8f_{TT} + f_T r^2)A'']\} = 0, \end{aligned} \tag{37}$$

$$E_{\hat{3}}{}^\phi \equiv \frac{1}{\sin\theta}E_{\hat{2}}{}^\theta = 0. \tag{38}$$

As a brief inspection shows, the field equations (35)–(38) do not restrict the form of the $f(T)$ function. Hence, due to the covariant formulation used above, the field equations for every $f(T)$ form can be satisfied by all tetrads related through the Lorentz transformation and corresponding to the spherically symmetric metric (31), and not only by specifically constructed ones.

We stress that the above field equations (35)–(38), generated from the diagonal tetrad (32) and the non-zero spin connection (33), coincide with those obtained in the usual, non-covariant, formulation of $f(T)$ gravity, for zero spin connection, but for the specific and peculiar non-diagonal tetrad [65–67]

$$\tilde{h}^{a}_{\ \mu} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B\cos\phi\sin\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ 0 & -B\cos\theta & r\sin\theta & 0 \\ 0 & B\sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \end{pmatrix}. \tag{39}$$

This coincidence of the field equations can be easily explained. The off-diagonal tetrad (39) is related to the diagonal tetrad (32) by a local Lorentz transformation of the form

$$\tilde{h}^{a}_{\ \mu} = \Lambda^{a}_{\ b} h^{b}_{\ \mu}, \tag{40}$$

where the Lorentz matrix is given explicitly by

$$\Lambda^{a}_{\ b} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi\sin\theta & \cos\phi\cos\theta & -\sin\phi \\ 0 & -\cos\theta & \sin\theta & 0 \\ 0 & \sin\phi\sin\theta & \sin\phi\cos\theta & \cos\phi \end{pmatrix}. \tag{41}$$

We should now recall that a local Lorentz transformation simultaneously transforms both the tetrad and spin connection through (11), and thus the spin connection (33) also gets transformed. Interestingly enough, the transformed spin connection through (41) is identically zero, namely

$$\tilde{\omega}^{a}_{\ b\mu} = 0. \tag{42}$$

Hence, we can see that the off-diagonal tetrad (39) is a proper tetrad, i.e. a tetrad in which the inertial spin connection vanishes, and that is why the obtained field equations coincide with the ones of the covariant formulation. In other words, in the usual, non-covariant, formulation of $f(T)$ gravity, one considers specific peculiar non-diagonal tetrads, thus making the theory frame-dependent, as a naive way to be consistent with a vanishing spin connection. However, as we show, the correct and general way to acquire consistency is to use the covariant formulation of $f(T)$ gravity, in which case frame dependence is absent. In particular, one is allowed to use any form of the tetrad provided that the corresponding spin connection is calculated. The off-diagonal tetrad (39) no longer has a privileged position; it is just a specific tetrad in which the corresponding spin connection happens to be zero.

5. Conclusions

Taking the serious decision to modify gravity, one still faces the question of which formulation of gravity to modify. The usual approach is to start from the curvature-based formulation, i.e. general relativity, and modify its action. However, one could start from the torsion-based formulation of gravity, i.e from TEGR. The crucial issue is that although TEGR coincides with general relativity at the level of equations, their modifications correspond to different gravitational theories.

$f(T)$ gravity is the simplest modification of TEGR, as $f(R)$ is the simplest modification of GR. However, it is well known that it does not satisfy local Lorentz invariance [54]. The reason for this is the following: in the usual formulation of $f(T)$ gravity [20–23] one starts from the pure tetrad teleparallel gravity, i.e it is assumed that the spin connection vanishes identically, and hence, although the theory becomes simpler, the torsion tensor is effectively replaced by the coefficients of anholonomy, which are not tensors under local Lorentz transformations. Although in un-modified gravity, i.e in TEGR, the resulting violation of local Lorentz symmetry is often neglected, since it does not affect the field equations, this becomes manifest and a severe problem in $f(T)$ gravity.

In this work we solved the above problem by constructing a consistent, covariant formulation of $f(T)$ gravity. In particular, starting from, instead of the pure tetrad teleparallel gravity, the covariant teleparallel gravity, we were able to re-formulate $f(T)$ gravity in a frame-independent way, which does not suffer from the notorious problems of the usual, pure tetrad $f(T)$ theory. In such a theory one uses both the tetrad and the spin connection in a way that for every tetrad choice a suitably constructed connection makes the whole theory covariant.

Covariant $f(T)$ gravity is a little bit more involved than the usual, non-covariant one, due to the necessity of finding the appropriate spin connection to the tetrad. While in covariant TEGR the spin connection enters the action only as a surface term, and thus we can first solve the field equations to determine the tetrad and then calculate the spin connection from the solution, this no longer holds in the $f(T)$ case where the solution of the field equation does depend on the choice of the spin connection, which naively leads to a loop difficulty. In order to avoid this problem we need a method of determining the spin connection that does not rely on the solution of the field equations. In particular, the correct reference tetrad can be guessed by making some reasonable assumptions based on the symmetries of the geometry. Assuming the knowledge of the reference tetrad corresponding to the ansatz tetrad, we can find the spin connection first, and then solve the field equations.

As examples we presented a method of extracting solutions in covariant $f(T)$ gravity in the most physically relevant cases such as the Minkowski, the FRW and the spherically symmetric geometries. As we have shown, the field equations for every $f(T)$ form can be satisfied by all tetrads related through a Lorentz transformation, and not only by specifically constructed ones. Hence, there are no 'good' or 'bad' tetrads in $f(T)$ gravity, there is no-frame dependence, as long as one abandons the strong imposition of zero spin connection (the peculiar non-diagonal, 'good' tetrads were just a naive way of being consistent with a vanishing spin connection).

Covariant $f(T)$ gravity is the correct and consistent way to modify gravity starting from its torsion-based formulation. There is an additional difficulty in finding the spin connection, however we have developed a consistent method to solve this issue. In summary, we are able to present a consistent torsional alternative to curvature modifications such as $f(R)$ gravity. Hence, covariant $f(T)$ gravity and its applications to cosmology and spherically symmetric geometries must be investigated in detail.

Acknowledgments

We would like to thank to José G Pereira for fruitful discussions. This work was supported by FAPESP.

Appendix. Derivation of the field equations

In this appendix we derive the field equations for the Lagrangian (17), keeping both the tetrad and the spin connection in the definition of the torsion tensor (2). The left-hand side of the field equations is given as the Euler–Lagrange expression for the Lagrangian (17), namely

$$E_a{}^{\mu} \equiv \frac{\partial \mathcal{L}_f}{\partial h^a{}_\mu} - \partial_\nu \frac{\partial \mathcal{L}_f}{\partial(\partial_\nu h^a{}_\mu)}. \tag{A1}$$

Up to an overall $1/(4\kappa)$ factor, the first term is given by

$$\frac{\partial h f(T)}{\partial h^a{}_\mu} = h f_T \frac{\partial T}{\partial h^a{}_\mu} + f(T) h h_a{}^\mu, \tag{A2}$$

while the second term reads

$$\partial_\nu \frac{\partial h f(T)}{\partial(\partial_\nu h^a{}_\mu)} = \partial_\nu \left[h f_T \frac{\partial T}{\partial(\partial_\nu h^a{}_\mu)} \right] = f_T\, \partial_\nu \left[h \frac{\partial T}{\partial(\partial_\nu h^a{}_\mu)} \right] + h(\partial_\nu T) f_{TT} \frac{\partial T}{\partial(\partial_\nu h^a{}_\mu)}. \tag{A3}$$

The derivatives of the torsion scalar are well known in the ordinary TEGR case (for instance see the appendix C in [19]), namely

$$\frac{\partial T}{\partial(\partial_\nu h^a{}_\mu)} = -4 S_a{}^{\mu\nu} \tag{A4}$$

$$\frac{\partial T}{\partial(h^a{}_\mu)} = -4 T^b{}_{\nu a} S_b{}^{\nu\mu} + 4 \omega^b{}_{a\nu} S_b{}^{\nu\mu}. \tag{A5}$$

Assembling the above we finally obtain

$$E_a{}^\mu = h^{-1} f_T\, \partial_\nu (h S_a{}^{\mu\nu}) + f_{TT}\, S_a{}^{\mu\nu} \partial_\nu T - f_T\, T^b{}_{\nu a} S_b{}^{\nu\mu} + f_T\, \omega^b{}_{a\nu} S_b{}^{\nu\mu} + \frac{1}{4} f(T) h_a{}^\mu. \tag{A6}$$

Hence, the field equations can be written as

$$E_a{}^\mu = \kappa \Theta_a{}^\mu, \tag{A7}$$

where as usual we have defined the energy-momentum tensor of matter through

$$\Theta_a{}^\rho = \frac{1}{h} \frac{\delta \mathcal{L}_M}{\delta h^a{}_\mu}. \tag{A8}$$

References

[1] Nojiri S and Odintsov S D 2006 *eConf* **C0602061** 06
Nojiri S and Odintsov S D 2007 *Int. J. Geom. Meth. Mod. Phys.* **4** 115
[2] Capozziello S and Laurentis M De 2011 *Phys. Rept.* **509** 167
[3] Copeland E J, Sami M and Tsujikawa S 2006 *Int. J. Mod. Phys.* D **15** 1753
[4] Cai Y F, Saridakis E N, Setare M R and Xia J Q 2010 Quintom cosmology: Theoretical implications and observations *Phys. Rept.* **493** 1
[5] Stelle K S 1977 *Phys. Rev.* D **16** 953
[6] Biswas T, Gerwick E, Koivisto T and Mazumdar A 2012 *Phys. Rev. Lett.* **108** 031101
[7] De Felice A and Tsujikawa S 2010 *Living Rev. Rel.* **13** 3
[8] Nojiri S and Odintsov S D 2011 *Phys. Rept.* **505** 59
[9] Einstein A 1928 *Sitz. Preuss. Akad. Wiss.* **17** 217
Einstein A 1928 *Sitz. Preuss. Akad. Wiss.* **17** 224
[10] Unzicker A and Case T arXiv:physics/0503046
[11] Sauer T 2006 *Historia Math.* **33** 399–439
[12] Møller C and Dan K 1961 *Vidensk. Selsk. Mat. Fys. Skr.* **1** 10
[13] Hayashi K and Nakano T 1967 *Prog. Theor. Phys.* **38** 491
[14] Cho Y M 1976 *Phys. Rev.* D **14** 2521
[15] Hayashi K 1977 *Phys. Lett.* B **69** 441

[16] Hayashi K and Shirafuji T 1979 *Phys. Rev.* D **19** 3524
Hayashi K and Shirafuji T 1982 *Phys. Rev.* D **24** 3312 (addendum)
[17] Maluf J W 1994 *J. Math. Phys.* **35** 335
[18] Arcos H I and Pereira J G 2004 *Int. J. Mod. Phys.* D **13** 2193
[19] Aldrovandi R and Pereira J G 2013 *Teleparallel Gravity: An Introduction* (Dordrecht: Springer)
[20] Ferraro R and Fiorini F 2007 *Phys. Rev.* D **75** 084031
[21] Bengochea G R and Ferraro R 2009 *Phys. Rev.* D **79** 124019
[22] Ferraro R and Fiorini F 2008 *Phys. Rev.* D **78** 124019
[23] Linder E V 2010 *Phys. Rev.* D **81** 127301
[24] Chen S H, Dent J B, Dutta S and Saridakis E N 2011 *Phys. Rev.* D **83** 023508
[25] Wu P and Yu H W 2010 *Phys. Lett.* B **693** 415
[26] Bengochea G R 2011 *Phys. Lett.* B **695** 405
[27] Dent J B, Dutta S and Saridakis E N 2011 *J. Cosmol. Astropart. Phys.* JCAP01(2011)009
[28] Zheng R and Huang Q G 2011 *J. Cosmol. Astropart. Phys.* JCAP03(2011)002
[29] Bamba K, Geng C Q, Lee C C and Luo L W 2011 *J. Cosmol. Astropart. Phys.* JCAP01(2011)021
[30] Cai Y-F, Chen S-H, Dent J B, Dutta S and Saridakis E N 2011 *Class. Quant. Grav.* **28** 215011
[31] Sharif M and Rani S 2011 *Mod. Phys. Lett.* A **26** 1657
[32] Li M, Miao R X and Miao Y G 2011 *J. High Energy Phys.* **1107** JHEP07(2011)108
[33] Capozziello S, Cardone V F, Farajollahi H and Ravanpak A 2011 *Phys. Rev.* D **84** 043527
[34] Daouda M H, Rodrigues M E and Houndjo M J S 2012 *Eur. Phys. J.* C **72** 1890
[35] Wu Y P and Geng C Q 2012 *Phys. Rev.* D **86** 104058
[36] Wei H, Guo X J and Wang L F 2012 *Phys. Lett.* B **707** 298
[37] Atazadeh K and Darabi F 2012 *Eur.Phys. J.* C **72** 2016
[38] Farajollahi H, Ravanpak A and Wu P 2012 *Astrophys. Space Sci.* **338** 23
[39] Karami K and Abdolmaleki A 2012 *J. Cosmol. Astropart. Phys.* JCAP04(2012)007
[40] Cardone V F, Radicella N and Camera S 2012 *Phys. Rev.* D **85** 124007
[41] Jamil M, Momeni D and Myrzakulov R 2012 *Eur. Phys. J.* C **72** 2267
[42] Ong Y C, Izumi K, Nester J M and Chen P 2013 *Phys. Rev.* D **88** 024019
[43] Amoros J, de Haro J and Odintsov S D 2013 *Phys. Rev.* D **87** 104037
[44] Nesseris S, Basilakos S, Saridakis E N and Perivolaropoulos L 2013 *Phys. Rev.* D **88** 103010
[45] Bamba K, Capozziello S, De Laurentis M, Nojiri S 'i and Sáez-Gómez D 2013 *Phys. Lett.* B **727** 194
[46] Kofinas G and Saridakis E N 2014 *Phys. Rev.* D **90** 084044
[47] Harko T, Lobo F S N, Otalora G and Saridakis E N 2014 *Phys. Rev.* D **89** 124036
[48] Haro J and Amoros J 2014 *J. Cosmol. Astropart. Phys.* JCAP12(2014)031
[49] Geng C Q, Lai C, Luo L W and Tseng H H 2014 *Phys. Lett.* B **737** 248
[50] El Hanafy W and Nashed G G L 2015 *Eur. Phys. J.* C **75** 279
[51] Darabi F, Mousavi M and Atazadeh K 2015 *Phys. Rev.* D **91** 084023
[52] Capozziello S, Luongo O and Saridakis E N 2015 *Phys. Rev.* D **91** 124037
[53] Nashed G L 2015 *Gen. Rel. Grav.* **47** 75
[54] Li B, Sotiriou T P and Barrow J D 2011 *Phys. Rev.* D **83** 064035
[55] Ferraro R and Fiorini F 2015 *Phys. Rev.* D **91** 064019
[56] Bahamonde S, Boehmer C G and Wright M 2015 *Phys. Rev.* D **92** 104042
[57] Junior E L B and Rodrigues M E arXiv:1509.03267 [gr-qc]
[58] Obukhov Y N and Rubilar G F 2006 *Phys. Rev.* D **73** 124017
[59] Maluf J W 2013 *Ann. Phys.* **525** 339
[60] Obukhov Y N and Pereira J G 2003 *Phys. Rev.* D **67** 044016
[61] Hehl F W, McCrea J D, Mielke E W and Ne'eman Y 1995 *Phys. Rep.* **258** 1
[62] Lucas T G, Obukhov Y N and Pereira J G 2009 *Phys. Rev.* D **80** 064043
[63] Krššák M and Pereira J G 2015 *Eur. Phys. J.* C **75** 519
[64] Krššák M arXiv:1510.06676 [gr-qc]
[65] Ferraro R and Fiorini F 2011 *Phys. Lett.* B **702** 75
[66] Ferraro R and Fiorini F 2011 *Phys. Rev.* D **84** 083518
[67] Tamanini N and Boehmer C G 2012 *Phys. Rev.* D **86** 044009
[68] Wang T 2011 *Phys. Rev.* D **84** 024042
[69] Hamani Daouda M, Rodrigues M E and Houndjo M J S 2011 *Eur. Phys. J.* C **71** 1817
[70] Nashed G G L 2013 *Eur. Phys. J.* C **73** 2394
[71] Atazadeh K and Mousavi M 2013 *Eur. Phys. J.* C **73** 2272

[72] Nashed G G L 2014 *Europhys. Lett.* **105** 10001
[73] Aftergood J and DeBenedictis A 2014 *Phys. Rev.* D **90** 124006
[74] Bhadra J and Debnath U 2014 *Int. J. Theor. Phys.* **53** 645
[75] de la Cruz-Dombriz Á, Dunsby P K S and Saez-Gomez D 2014 *J. Cosmol. Astropart. Phys.* JCAP12(2014)049
[76] Abbas G, Qaisar S and Meraj M A 2015 *Astrophys. Space Sci.* **357** 156
[77] Junior E L B, Rodrigues M E and Houndjo M J S 2015 *J. Cosmol. Astropart. Phys.* JCAP06(2015)037
[78] Das A, Rahaman F, Guha B K and Ray S 2015 *Astrophys. Space Sci.* **358** 36
[79] Zubair M and Abbas G 2016 *Astrophys. Space Sci.* **361** 27
[80] Ruggiero M L and Radicella N 2015 *Phys. Rev.* D **91** 104014
[81] Nashed G G L 2015 *Eur. Phys. J. Plus* **130** 124
[82] Boehmer C G, Mussa A and Tamanini N 2011 *Class. Quant. Grav.* **28** 245020
[83] Gonzalez P A, Saridakis E N and Vasquez Y 2012 *J. High Energy Phys.* JHEP07(2012)053
[84] Iorio L and Saridakis E N 2012 *Mon. Not. Roy. Astron. Soc.* **427** 1555
[85] Capozziello S, Gonzalez P A, Saridakis E N and Vasquez Y 2013 *J. High Energy Phys.* JHEP02(2013)039
[86] Kofinas G, Papantonopoulos E and Saridakis E N 2015 *Phys. Rev.* D **91** 104034
[87] Kofinas G 2015 *Phys. Rev.* D **92** 084022
[88] Dong H, Wang Y b and Meng X h 2012 *Eur. Phys. J.* C **72** 2002