Riordan Array Proofs of Identities in Gould's Book

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Chapter 1

Generating functions and Riordan arrays

1.1 Properties of special numbers

The main properties of widely used special numbers.

(BS). Symmetry of binomial coefficients:

$$\boxed{\binom{n}{k} = \binom{n}{n-k}}$$

(BN). Negation for binomial coefficients;

$$\boxed{\binom{n}{k} = \binom{-n+k-1}{k}(-1)^k}$$

(BX). Cross product of binomial coefficients:

$$\boxed{\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}}$$

(BR). Recurrence for binomial coefficients:

$$\boxed{\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}}$$

(BD). Reduction of binomial coefficients:

$$\boxed{\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}}$$

(BZ). Binomial coefficients being 0:

$$\binom{n}{r} = 0 \quad \text{for} \quad r < 0, r > n > 0, r \notin \mathbb{N}$$

(FK). Binomial coefficient with fraction:

$$\binom{n-k}{k} \cdot \frac{n}{n-k} = \binom{n-k}{k} + \binom{n-k-1}{k-1}$$

$$\binom{x+k}{k} \frac{1}{x+k} = \frac{1}{x}\left(\binom{x+k}{k} - \binom{x+k-1}{k-1}\right)$$

(XK) Simplification of fraction:

$$\binom{n}{k} \frac{x}{x+k} = \binom{n+x}{n}^{-1} \binom{n+x}{n-k} \binom{x+k-1}{x-1}$$

$$\binom{n}{k} \frac{x}{x+k} = \binom{n}{k} \binom{x+k}{k} \binom{x+k}{k}^{-1} \frac{x}{x+k} =$$

$$= \frac{n!(x+k)!x!k!}{k!(n-k)!(x+k)!x!k!} \cdot \frac{x}{x+k} \cdot \frac{(n+x)!}{(n+x)!} = \binom{n+x}{n}^{-1} \binom{n+x}{n-k} \binom{x+k-1}{x-1}.$$

1.2 Coefficient extraction

The main rules for the "coefficient of" functionals:

(1KE) Linearity

$$[t^n](af(t) + bg(t)) = a[t^n]f(t) + b[t^n]g(t)$$

(2KE) Shifting

$$[t^n]tf(t) = [t^{n-1}]f(t)$$

(3KE) Differentiation

$$[t^n]f'(t) = (n+1)[t^{n+1}]f(t)$$

(4KE) **(conv)** Convolution

$$[t^n]f(t)g(t) = \sum_{k=0}^{n} [t^k]f(t) \cdot [y^{n-k}]g(y)$$

(5KE) Composition

$$[t^n]f(g(t)) = \sum_{k=0}^{\infty}([t^k]f(t)) \cdot [y^n]g(y)^k$$

where ord$g(t) \geq 1$.

(6KE) Inversion (Lagrange Inversion Formula - LIF)

$$[t^n]f^{[-1]}(t) = \frac{k}{n}[t^{n-k}]\left(\frac{t}{f(t)}\right)^n$$

where $f^{[-1]}(t)$ is the compositional inverse of $f(t)$, i.e., $f(f^{[-1]}(t)) = f^{[-1]}(f(t)) = t$.

(0TR) The transformation $t \mapsto at$:

$$[t^n]f(at) = a^n[t^n]f(t)$$

(1TR) The transformation $t^j \mapsto t$:

$$[t^{jn}]f(t^j) = [t^n]f(t)$$

(1LF) A useful formulation of the Lagrange Inversion Formula:

$$[t^n]\left[F(w) \;\middle|\; w = t\phi(w)\right] = \frac{1}{n}[t^{n-1}]F'(t)\phi(t)^n$$

(2LF) Another useful formulation of the Lagrange Inversion Formula:

$$[t^n]\left[F(w) \;\middle|\; w = t\phi(w)\right] = [t^n]F(t)\phi(t)^{n-1}(\phi(t) - t\phi'(t))$$

(0RF)

$$[t^n]\frac{1}{1-zt} = z^n$$

(1RF)

$$[t^n]\frac{1}{(1+rt)(1+st)} = \frac{r^{n+1} - s^{n+1}}{r-s}(-1)^n$$

(2RF) - The denominator is an irreducible polynomial of second degree

$$[t^n]\frac{1}{1+bt+ct^2} = \frac{2(\sqrt{c})^{n+1}\sin(n+1)\theta}{\sqrt{4c-b^2}}$$

$$\theta = \arctan\frac{\sqrt{4c-b^2}}{-b} + \pi[b>0]$$

(1ZV)

$$[t^k]\frac{1}{(1-t)^{m+1}}\ln\frac{1}{1-t} = (H_{m+k}-H_m)\binom{m+k}{m}$$

1.3 Generating Functions

The rules of the *generating function operator*:

(GF1) Linearity

$$\mathcal{G}\left(af_k+bg_k\right) = a\mathcal{G}\left(f_k\right)+b\mathcal{G}\left(g_k\right)$$

(GF2) Shifting

$$\mathcal{G}\left(f_{k+1}\right) = \frac{\mathcal{G}\left(f_k\right)-f_0}{t}$$

(GF3) Differentiation

$$\mathcal{G}\left(kf_k\right) = tD\mathcal{G}\left(f_k\right)$$

(GF4) Convolution

$$\mathcal{G}\left(\sum_{k=0}^{n} f_k g_{n-k}\right) = \mathcal{G}\left(f_k\right)\cdot\mathcal{G}\left(g_k\right)$$

(GF5) Composition

$$\sum_{n=0}^{\infty} f_n(\mathcal{G}\left(g_k\right))^n = \mathcal{G}\left(f_k\right)\circ\mathcal{G}\left(g_k\right)$$

(GF6) Diagonalization

$$\mathcal{G}\left([t^n]F(t)\phi(t)^n\right) = \left[\frac{F(w)}{1-t\phi'(w)}\;\middle|\; w=t\phi(w)\right]$$

The most frequently used generating functions are listed here. The notation is as follows:

k	summation variable
m, n, p	integer constants
x, y, z	real constants
a, b, c	parameters (int. or real acc. to context)
t, u, v, w	indeterminates

(PW0)

$$\mathcal{G}\left(z^k\right) = \frac{1}{1 - zt}$$

(PW1)

$$\mathcal{G}\left(k^p\right) = \sum_{k=0}^{p} \left\{ {p \atop k} \right\} \frac{k!t^k}{(1-t)^{k+1}}$$

(BC1). Simple binomial coefficients:

$$\mathcal{G}\left(\binom{x}{k} a^k\right) = (1 + at)^x$$

(BC2). Weighted binomial coefficients.

$$\mathcal{G}\left(k\binom{n}{k}\right) = nt(1+t)^{n-1}$$

Proof:

$$\mathcal{G}\left(k\binom{n}{k}\right) = tD(1+t)^n = nt(1+t)^{n-1}$$

(BC3). Doubly weighted binomial coefficients.

$$\mathcal{G}\left(k^2\binom{n}{k}\right) = nt(1+nt)(1+t)^{n-2}$$

Proof:

$$\mathcal{G}\left(k^2\binom{n}{k}\right) = tDnt(1+t)^{n-1} = nt(1+nt)(1+t)^{n-2}$$

(BC4). Progressive binomial coefficients:

$$\mathcal{G}\left(\binom{x+k}{k} a^k\right) = \frac{1}{(1-at)^{x+1}}$$

Proof:

$$\mathcal{G}\left(\binom{x+k}{k}a^k\right) = \mathcal{G}\left(\binom{-x-k+k-1}{k}(-a)^k\right) = \mathcal{G}\left(\binom{-x-1}{k}(-a)^k\right) = \frac{1}{(1-at)^{x+1}}.$$

(BC5). Binomial coefficients with constant denominator:

$$\boxed{\mathcal{G}\left(\binom{p+k}{m}\right) = \frac{t^{m-p}}{(1-t)^{m+1}}}$$

Proof:

$$\binom{p+k}{m} = \binom{p+k}{p+k-m} = [t^k]t^{m-p}(1+t)^{p+k} \overset{GF6}{=}$$

$$= [t^k]\left[w^{m-p}(1+w)^{p+1} \;\middle|\; w = t(1+w)\right] = [t^k]\frac{t^{m-p}}{(1-t)^{m-p}} \cdot \frac{1}{(1-t)^{p+1}} = [t^k]\frac{t^{m-p}}{(1-t)^{m+1}}.$$

(CB1). Central binomial coefficients:

$$\boxed{\mathcal{G}\left(\binom{2k}{k}x^k\right) = \frac{1}{\sqrt{1-4xt}}}$$

Proof:

$$[t^n](1+t)^{2n} \overset{GF6}{=} [t^n]\left[\frac{1}{1-2t(1+w)} \;\middle|\; w = t(1+w)^2\right] =$$

$$= [t^n]\left[\frac{1+w}{1-w} \;\middle|\; w = \frac{1-t-\sqrt{1-4t}}{2t}\right] = [t^n]\frac{1}{\sqrt{1-4t}};$$

then we apply rule (0TR).

(CB2). Modified central binomial coefficients:

$$\boxed{\mathcal{G}\left(\frac{1}{2k-1}\binom{2k}{k}\right) = -\sqrt{1-4t}}$$

(CB3). Weighted central binomial coefficients.

$$\boxed{\mathcal{G}\left(k\binom{2k}{k}\right) = \frac{2t}{(1-4t)^{3/2}}}$$

Proof:

$$\mathcal{G}\left(k\binom{2k}{k}\right) = tD\frac{1}{\sqrt{1-4t}} = \frac{2t}{(1-4t)^{3/2}}.$$

(CB4). Oddly weighted central binomial coefficients.

$$\boxed{\mathcal{G}\left(\binom{2k}{k}(2k+1)\right) = \frac{1}{(1-4t)^{3/2}}}$$

Proof:

$$\mathcal{G}\left(\binom{2k}{k}(2k+1)\right) = \frac{4t}{(1-4t)^{3/2}} + \frac{1}{(1-4t)^{1/2}} = \frac{1}{(1-4t)^{3/2}}.$$

(CB5). A variant of oddly weighted central binomial coefficients.

$$\boxed{\mathcal{G}\left(\binom{2k}{k}\frac{(-1)^k(2k+1)}{4^k}\right) = \frac{1}{(1+t)^{3/2}}}$$

(IC0)

$$\boxed{\mathcal{G}\left(4^n\binom{2n}{n}^{-1}\right) = \sqrt{\frac{t}{(1-t)^3}}\arctan\sqrt{\frac{t}{1-t}} + \frac{1}{1-t}}$$

$$4^{n+1}\binom{2n+2}{n+1}^{-1} = \frac{2n+2}{2n+1}4^n\binom{2n}{n}^{-1} \to (2n+1)f_{n+1} = 2(n+1)f_n;$$

$$2t(1-t)f'(t) - (1+2t)f(t) + 1 = 0; \qquad \int\frac{1+2t}{2t(1-t)}\mathrm{d}t = \frac{1}{2}\int\frac{\mathrm{d}t}{t} + \frac{3}{2}\int\frac{\mathrm{d}t}{1-t} = \ln\sqrt{\frac{t}{(1-t)^3}};$$

$$f(t) = \sqrt{\frac{t}{(1-t)^3}}\left(-\int\sqrt{\frac{(1-t)^3}{t}}\cdot\frac{\mathrm{d}t}{2t(1-t)} + C\right).$$

This integral can be solved by setting:

$$z = \sqrt{\frac{t}{1-t}} \qquad \text{or} \qquad \frac{\mathrm{d}t}{\mathrm{d}z} = \frac{2z}{(1+z^2)^2}.$$

$$\frac{1}{2}\int\sqrt{\frac{1-t}{t}}\cdot\frac{\mathrm{d}t}{t} = \frac{1}{2}\int\frac{1}{z}\cdot\frac{1+z^2}{z^2}\cdot\frac{2z\mathrm{d}z}{(1+z^2)^2} = \int\frac{\mathrm{d}z}{z^2(1+z^2)} = -\frac{1}{z} - \arctan z.$$

$$f(t) = \sqrt{\frac{t}{(1-t)^3}}\arctan\sqrt{\frac{t}{1-t}} + \frac{1}{1-t} + C\sqrt{\frac{t}{(1-t)^3}}; \qquad C = 0 \text{ being } f(0) = 1.$$

(IC6) Doubly weighted inverse central binomial coefficients.

$$\boxed{\mathcal{G}\left(\frac{16^k}{2k^2}\binom{2k}{k}^{-1}\right) = \left(\arctan\sqrt{\frac{4t}{1-4t}}\right)^2}$$

(CN1) Catalan numbers:

$$\boxed{\mathcal{G}\left(\frac{1}{k+1}\binom{2k}{k}\right) = \frac{1-\sqrt{1-4t}}{2t}}$$

(GP1) Geometric progression: the g. f. of the sequence $(1, 1, 1, \ldots, 1, 0, 0, \ldots)$, with n starting 1's:

$$\boxed{\mathcal{G}(1, 1, \ldots, 1, 0, 0, \ldots) = \frac{1-t^{n+1}}{1-t}}$$

Proof: The generating function is $1 + t + t^2 + \cdots + t^{n-1} + t^n$. By the rule for the sum of a geometric progression we immediately obtain the formula.

(GP2) Weighted geometric progression: the g. f. of the sequence $(0, 1, 2, \ldots, n, 0, 0, \ldots)$:

$$\mathcal{G}(0, 1, 2, \ldots, n, 0, 0, \ldots) = \frac{t - t^{n+2}}{(1-t)^2} - (n+1)\frac{t^{n+1}}{1-t}$$

Proof: The generating function is $0 + t + 2t^2 + \cdots + (n-1)t^{n-1} + nt^n$, that is t times the derivative of the previous g.f..

(EX1) Exponential function:

$$\mathcal{G}\left(\frac{a^k}{k!}\right) = e^{at} = \exp(at)$$

(FB1) Fibonacci numbers:

$$\mathcal{G}(F_k) = \frac{t}{1-t-t^2}$$

(FB2) Even Fibonacci numbers:

$$\mathcal{G}(F_{2n}) = \frac{t}{1-3t+t^2}$$

Proof:

$$\mathcal{G}(F_{2n}) = \frac{1}{2}\left(\frac{\sqrt{t}}{1-\sqrt{t}-t} - \frac{\sqrt{t}}{1+\sqrt{t}-t}\right) = \frac{\sqrt{t}}{2} \cdot \frac{1+\sqrt{t}-t-1+\sqrt{t}+t}{(1-t)^2-t} = \frac{t}{1-3t+t^2}.$$

(FB3). Odd Fibonacci numbers:

$$\mathcal{G}(F_{2n+1}) = \frac{1-t}{1-3t+t^2}$$

Proof:

$$\mathcal{G}(F_{2n+1}) = \frac{1}{2\sqrt{t}}\left(\frac{\sqrt{t}}{1-\sqrt{t}-t} + \frac{\sqrt{t}}{1+\sqrt{t}-t}\right) = \frac{1}{2} \cdot \frac{1+\sqrt{t}-t+1-\sqrt{t}-t}{(1-t)^2-t} = \frac{1-t}{1-3t+t^2}.$$

(GO1). Gould first generating function:

$$\mathcal{G}\left(\frac{x}{x+kz}\binom{x+kz}{k}\right) = \left[(1+w)^x \;\middle|\; w = t(1+w)^z\right]$$

Proof:

$$\frac{x}{x+kz}\binom{x+kz}{k} = \binom{x+kz}{k} - z\binom{x+kz-1}{k-1} = [t^k](1+t)^{x+kz} - z[t^{k-1}](1+t)^{x+kz-1} =$$

$$= [t^k](1-(z-1)t)(1+t)^{x+kz-1} = [t^k]\left[\frac{(1-(z-1)w)(1+w)^{x-1}}{1-wz(1+w)^{z-1}/(1+w)^z} \;\middle|\; w = t(1+w)^z\right] =$$

$$= [t^n]\left[(1+w)^x \;\middle|\; w = t(1+w)^z\right].$$

(GO2). Gould second generating function:

$$\mathcal{G}\left(\binom{x+kz}{k}\right) = \left[\frac{(1+w)^{x+1}}{1-(z-1)w} \mid w = t(1+w)^z\right]$$

Proof:

$$\binom{x+kz}{k} = [t^k](1+t)^{x+kz} = [t^k]\left[\frac{(1+w)^x}{1-wz(1+w)^{z-1}/(1+w)^z} \mid w = t(1+w)^z\right] =$$

$$= [t^n]\left[\frac{(1+w)^{x+1}}{1-(z-1)w} \mid w = t(1+w)^z\right].$$

1.4 Riordan Array transformations

Here are the most important transformations related to Riordan Arrays:

(G). The general summation rule of Riordan Arrays $(d(t), h(t))$:

$$\sum_k d_{n,k} f_k = [t^n] d(t) f(th(t))$$

Proof:

$$\sum_k d_{n,k} f_k = \sum_k [t^n] d(t)(th(t))^k f_k = [t^n] d(t) \sum_k f_k (th(t))^k \stackrel{comp}{=} [t^n] d(t) f(th(t))$$

(A). This rule can also be applied when $b = a$, provided the generating function $f(t)$ is a polynomial; in that case, the rule will be denoted by **(A*)**:

$$\sum_k \binom{n+ak}{m+bk} z^{n-m+(a-b)k} f_k = [t^n] \frac{t^m}{(1-zt)^{m+1}} \cdot f\left(\frac{t^{b-a}}{(1-zt)^b}\right) \qquad b > a$$

Because of the complex condition on the power of z, most times rule (A) is applied with $z = 1$:

$$\sum_k \binom{n+ak}{m+bk} f_k = [t^n] \frac{t^m}{(1-t)^{m+1}} \cdot f\left(\frac{t^{b-a}}{(1-t)^b}\right) \qquad b > a$$

Proof:

$$\binom{n+ak}{m+bk} z^{n-m+(a-b)k} = \binom{n+ak}{n+ak-b-bk} z^{n-m+(a-b)k} =$$

$$= \binom{-n-ak+n+ak-m-bk-1}{n+ak-m-bk} (-z)^{n-m+(a-b)k} = \binom{-m-bk-1}{(n-m)+(a-b)k} (-z)^{(n-m)+(a-b)k} =$$

$$= [t^{(n-m)+(a-b)k}] \frac{1}{(1-zt)^{m+1+bk}} = [t^n] \frac{t^m}{(1-zt)^{m+1}} \left(\frac{t^{b-a}}{(1-zt)^b}\right)^k.$$

Therefore we have a Riordan Array and apply the general rule (G).

(B). This rule can also be applied when $b = 0$, provided the generating function $f(t)$ is a polynomial; in that case, the rule will be denoted by **(B*)**:

$$\sum_k \binom{n+ak}{m+bk} z^{m+bk} f_k = [t^m](1+zt)^n f(t^{-b}(1+zt)^a) \qquad b < 0$$

Proof:

$$\binom{n+ak}{m+bk} z^{m+bk} = [t^{m+bk}](1+zt)^{n+ak} = [t^m](1+zt)^n \left(t^{-b}(1+zt)^a\right)^k .$$

Therefore we have a Riordan Array and apply the general rule (G).

(P). Formula for partial sums:

$$\boxed{\sum_{k=0}^{n} f_k = [t^n]\frac{f(t)}{1-t}}$$

(E). Euler transformation; a specialization of (A):

$$\boxed{\sum_k \binom{n}{k} z^{n-k} f_k = [t^n]\frac{1}{1-zt} f\left(\frac{t}{1-zt}\right)}$$

(Pf). Pfaff reflection rule:

$$\boxed{\frac{1}{(1-t)^a} F\left(\begin{matrix} a, \ b \\ c \end{matrix} \,\middle|\, \frac{-t}{1-t}\right) = F\left(\begin{matrix} a, \ c-b \\ c \end{matrix} \,\middle|\, t\right)}$$

The hypergeometric function satisfies the recurrence relation:

$$\frac{f_{k+1}}{f_k} = -\frac{(a+k)(b+k)}{(c+k)(k+1)}$$

$$(c+k)(k+1)f_{k+1} = (a+k)(b+k)f_k$$

$$c(k+1)f_{k+1} + k(k+1)f_{k+1} = -abf_k - (a+b)tkf_k - k^2 f_k$$

$$cf'(t) + tf''(t) = -abf(t) - (a+b)tf'(t) - tf''(t) - t^2 f''(t)$$

$$abf(t) + (at+bt+t+c)f'(t) + t(1+t)f''(t) = 0. \tag{1.4.1}$$

Let us apply the Riordan array transformation to this differential equation. We set:

$$g(t) = \frac{1}{(1-t)^a} f\left(\frac{t}{1-t}\right) \quad \text{or} \quad f\left(\frac{t}{1-t}\right) = (1-t)^a g(t)$$

$$f'\left(\frac{t}{1-t}\right) = -a(1-t)^{a+1}g(t) + (1-t)^{a+2}g'(t)$$

$$f''\left(\frac{t}{1-t}\right) = a(a+1)(1-t)^{a+2}g(t) - 2(a+1)(1-t)^{a+3}g'(t) + (1-t)^{a+4}h''(t).$$

By substituting in equation (1.4.1) we get:

$$abf\left(\frac{t}{1-t}\right) + \frac{at+bt+t-ct+c}{1-t} f'\left(\frac{t}{1-t}\right) + \frac{t}{(1-t)^2} f''\left(\frac{t}{1-t}\right)$$

$$ab(1-t)^a g(t) + \frac{at+bt-ct+t+c}{1-t}\left(-a(1-t)^{a+1}g(t) + (1-t)^{a+2}g'(t)\right) +$$

$$+\frac{t}{(1-t)^2}\left(a(a+1)(1-t)^{a+2}g(t) - 2(a+1)(1-t)^{a+3}g'(t) + (1-t)^{a+4}h''(t)\right) = 0.$$

We can now simplify and consider the coefficients of $h(t), h'(t), h''(t)$:

$$\begin{array}{ll} \text{coeff}(g(t)) & a(b-c)(1-t) \\ \text{coeff}(g'(t)) & (bt-at-ct-t+c)(1-t) \\ \text{coeff}(g''(t)) & t(1-t)^2 \end{array}$$

and this corresponds to the differential equation satisfied by $h(t)$, the transformed function:

$$a(b-c)g(t) - ((a-b+c+1)t-c)g'(t) + t(1-t)g''(t) = 0.$$

We extract the coefficients of t^n:

$$\begin{aligned} [t^n]a(b-c)g(t) &= a(b-c)g_n \\ [t^n]((a-b+c-1)t-c)g'(t) &= (a-b+c-1)ng_n + c(n+1)g_{n+1} \\ [t^n](t-t^2)g''(t) &= (n+1)ng_{n+1} - n(n-1)g_n. \end{aligned}$$

By separating g_n and g_{n+1}, an easy factorization yields to:

$$(a+n)(n+c-b)g_n = (n+1)(c+n)h_{n+1}$$

$$\frac{g_{n+1}}{g_n} = \frac{(a+n)(c-b+n)}{(n+1)(c+n)} \rightsquigarrow F\left(\begin{matrix} a,\ c-b \\ c \end{matrix} \,\middle|\, t\right).$$

($\mathbf{T}_+$) Bizley (6.43).

$$\boxed{\sum_k \binom{n}{a+k}\binom{m}{p-k}\binom{q+k}{n+m} = \binom{q-a}{m+n-p-a}\binom{q+p-m}{p+a}}$$

Let us call r the denominator of the third binomial coefficient:

$$S = \sum_k \binom{n}{a+k}\binom{m}{p-k}\binom{q+k}{n+m} = \sum_k [t^{a+k}](1+t)^n[v^{p-k}](1+v)^m[w^r](1+w)^{q+k} =$$

$$= [v^p](1+v)^m[w^r](1+w)^q\sum_k [t^k]\frac{(1+t)^n}{t^a}(v(1+w))^k = [v^p](1+v)^m[w^r](1+w)^q\frac{(1+v+vw)^n}{v^a(1+w)^a} =$$

$$= [v^{p+a}](1+v)^{m+n}[w^r](1+w)^{q-a}\left(1+\frac{v}{1+v}w\right) = [v^{p+a}](1+v)^{m+n}\sum_{k=0}^{r}\binom{q-a}{k}\binom{n}{r-k}\frac{v^{r-k}}{(1+v)^{r-k}} =$$

$$= \sum_{k=0}^{r}\binom{q-a}{k}\binom{n}{r-k}[v^{p+a-r+k}](1+v)^{m+n-r+k} = \sum_k \binom{q-a}{k}\binom{n}{r-k}\binom{m+n-r+k}{m+n-p-a}.$$

We try to apply the cross product for binomial coefficients. This can be done if:

1. $k = m+n-r+k$, that is $r = m+n$; or
2. $n-r+k = m+n-r+k$, that is $m = 0$.

Consequently, we should suppose $r = m+n$, and in that case we have:

$$S = \sum_k \binom{n}{m+n-k}\binom{q-a}{k}\binom{k}{m+n-p-a} = \binom{q-a}{m+n-p-a}\sum_k \binom{n}{m+n-k}\binom{q-m-n+p}{k-m-n+p+a} \overset{B}{=}$$

$$= \binom{q-a}{m+n-p-a}[t^{m+n}](1+t)^n\left[u^{m+n-p-a}(1+u)^{q-m-n+p} \,\middle|\, u=t\right] = \binom{q-a}{m+n-p-a}\binom{q-m+p}{p+a}.$$

($\mathbf{T}_-$)

$$\boxed{\sum_k \binom{n}{a+k}\binom{m}{p-k}\binom{q-k}{n+m} = \binom{q-p}{m+n-p-a}\binom{q+a-n}{p+a}}$$

Let us call r the denominator of the third binomial coefficient:

$$S = \sum_k \binom{n}{a+k}\binom{m}{p-k}\binom{q-k}{n+m} = \sum_k [t^{a+k}](1+t)^n[v^{p-k}](1+v)^m[w^r](1+w)^{q-k} =$$

$$= [v^p](1+v)^m[w^r](1+w)^q \sum_k [t^k]\frac{(1+t)^n}{t^a}\left(\frac{v}{1+w}\right)^k = [v^{p+a}](1+v)^m[w^r](1+w)^{q-n+a}(1+v)^n\left(1+\frac{w}{1+v}\right)^n =$$

$$= [v^{p+a}](1+v)^{m+n}[w^r](1+w)^{q-n+a}\left(1+\frac{w}{1+v}\right)^n = [v^{p+a}](1+v)^{m+n}\sum_{k=0}^{r}\binom{q-n+a}{k}\binom{n}{r-k}\frac{1}{(1+v)^{r-k}} =$$

$$= \sum_{k=0}^{r}\binom{q-n+a}{k}\binom{n}{r-k}[v^{p+a}](1+v)^{m+n-r+k} = \sum_k \binom{q-n+a}{k}\binom{n}{r-k}\binom{m+n-r+k}{p+a}.$$

We try to apply the cross product for binomial coefficients. This can be done if:

1. $k = m+n-r+k$, that is $r = m+n$; or
2. $n-r+k = m+n-r+k$, that is $m = 0$.

Consequently, we should suppose $r = m+n$, and in that case we have:

$$S = \sum_k \binom{n}{m+n-k}\binom{q-n+a}{k}\binom{k}{p+a} = \binom{q-n+a}{p+a}\sum_k \binom{n}{m+n-k} q-n-pchoosek-p-a \stackrel{B}{=}$$

$$= \binom{q-n+a}{p+a}[t^{m+n}](1+t)^n\left[u^{p+a}(1+u)^{q-n-p} \;\middle|\; u=t\right] = \binom{q-n+a}{p+a}\binom{q-p}{m+n-p-a}.$$

Chapter 2

Gould's combinatorial identities

Proofs of Gould's combinatorial identities. Gould's comments are written in roman; my comments are written in italics.

2.1 Table 1: summations of the form S:1/0

This table contains 135 identities.

(1.1) - Binomial theorem, valid for arbitrary complex x, and complex $|z| < 1$, where the principal value of $(1+z)^x$ is taken. Convergence is irrelevant when we think of this as a generating function.

$$\boxed{\sum_{k=0}^{\infty} \binom{x}{k} z^k = (1+z)^x}$$

$$(I) \quad \sum_{k=0}^{\infty} \binom{x}{k} z^k = \left[(1+zt)^x \;\middle|\; t=1 \right] = (1+z)^x.$$

$$(II) \quad \sum_{k=0}^{n} \binom{n}{k} z^k = n! \sum_{k=0}^{n} \frac{z^k}{k!} \frac{1}{(n-k)!} = n! e^{zt} e^t = n! \frac{(z+1)^n}{n!} = (z+1)^n.$$

(1.2)

$$\boxed{\sum_{k=0}^{\infty} (-1)^k \binom{x}{k} = 0, \qquad \Re(x) > 0}$$

$$\sum_{k=0}^{\infty} \binom{x}{k} (-1)^k = \left[(1-t)^x \;\middle|\; t=1 \right] = 0.$$

(1.3)

$$\boxed{\sum_{k=0}^{\infty} \binom{n+k}{k} x^k = \frac{1}{(1-x)^{n+1}}, \qquad |x| < 1}$$

$$\sum_{k=0}^{\infty} \binom{n+k}{k} x^k \overset{BC2}{=} \left[\frac{1}{(1-xt)^{n+1}} \;\middle|\; t=1 \right] = \frac{1}{(1-x)^{n+1}}.$$

(1.4) - *(**NN**) This sum can be reduced to the successive sum (1.5)*:

$$\boxed{\sum_{k=a}^{n}(-1)^k\binom{x}{k} = (-1)^a\binom{x-1}{a-1} + (-1)^n\binom{x-1}{n}}$$

$$\sum_{k=a}^{n}(-1)^k\binom{x}{k} = \sum_{k=0}^{n}(-1)^k\binom{x}{k} - \sum_{k=0}^{a-1}(-1)^k\binom{x}{k}.$$

(1.5) - *(**NN**) This sum is a partial sum and the formula for partial sum can be applied*:

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x}{k} = (-1)^n\binom{x-1}{n} = \prod_{k=1}^{n}\left(1-\frac{x}{k}\right)}$$

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k} \stackrel{P}{=} [t^n]\frac{1}{1-t}(1-t)^x = [t^n](1-t)^{x-1} = (-1)^n\binom{x-1}{n}.$$

(1.6)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x}{k}k^r = \sum_{k=0}^{r}(-1)^k\binom{x}{k}\binom{n-x}{n-k}\left\{ r \atop k \right\}k!}$$

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}k^r \stackrel{PW1}{=} \sum_{k=0}^{n}(-1)^k\binom{x}{k}[t^k]\sum_{j=0}^{r}\left\{ r \atop j \right\}\frac{j!t^j}{(1-t)^{j+1}} =$$

$$= \sum_{k=0}^{n}(-1)^k\binom{x}{k}\sum_{j=0}^{r}\left\{ r \atop j \right\}j![t^{k-j}]\frac{1}{(1-t)^{j+1}} =$$

$$= \sum_{j=0}^{r}\left\{ r \atop j \right\}j!\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{k}{j} = \sum_{j=0}^{r}\left\{ r \atop j \right\}j!\binom{x}{j}\sum_{k=0}^{n}(-1)^k\binom{x-j}{k-j} \stackrel{P}{=}$$

$$= \sum_{j=0}^{r}\left\{ r \atop j \right\}j!\binom{x}{j}[t^n]\frac{(-t)^j(1-t)^{x-j}}{1-t} = \sum_{j=0}^{r}\left\{ r \atop j \right\}j!\binom{x}{j}(-1)^j[t^{n-j}](1-t)^{x-j-1} =$$

$$= \sum_{j=0}^{r}\left\{ r \atop j \right\}j!\binom{x}{j}(-1)^j\binom{x-j-1}{n-j}(-1)^{n-j} = \sum_{j=0}^{r}\left\{ r \atop j \right\}j!(-1)^j\binom{x}{j}\binom{n-x}{n-j}.$$

***(1.7)** - Problem 4551, American Mathematical Monthly, 1953, p. 482:

$$\boxed{\lim_{n\to\infty} n^{x-r}\sum_{k=0}^{n-1}(-1)^k\binom{x}{k}k^r = \frac{1}{(r-x)\Gamma(-x)}}$$

(1.8) - *(**NN**) The left hand side is a partial sum*:

$$\boxed{\sum_{k=0}^{n}\binom{x+1}{k}z^k = \sum_{k=0}^{n}\binom{x-n+k}{k}z^k(1+z)^{n-k}}$$

$$(1)\quad \sum_{k=0}^{n}\binom{x+1}{k}z^k \overset{P}{=} [t^n]\frac{(1+zt)^{x+1}}{1-t}.$$

$$(2)\quad \sum_{k=0}^{n}\binom{x-k}{n-k}z^{n-k}(1+z)^k \overset{B}{=} [t^n](1+zt)^x\left[\frac{1}{1-(1+z)u}\;\middle|\; u=\frac{t}{1+zt}\right] = [t^n]\frac{(1+zt)^{x+1}}{1-t}.$$

(1.9) - *(NN)* *The left hand side is a partial sum:*

$$\boxed{\sum_{k=0}^{n}\binom{x}{k}y^k = \sum_{k=0}^{n}\binom{n-x}{k}(-y)^k(1+y)^{n-k}}$$

$$(1)\quad \sum_{k=0}^{n}\binom{x}{k}y^k \overset{P}{=} [t^n]\frac{(1+yt)^x}{1-t}.$$

$$(2)\quad \sum_{k=0}^{n}\binom{n-x}{n-k}(-y)^{n-k}(1+y)^k \overset{B}{=} [t^n](1-yt)^{n-x}\left[\frac{1}{1-(1+y)u}\;\middle|\; u=t\right] \overset{GF6}{=}$$

$$= [t^n]\left[\frac{1}{(1-yw)^x}\cdot\frac{1}{1-(1+y)w}\cdot(1-yw)\;\middle|\; w=t(1-yw)\right] =$$

$$= [t^n](1+yt)^{x-1}\cdot\frac{1+yt}{1+yt-t-yt} = [t^n]\frac{(1+yt)^x}{1-t}.$$

(1.10)

$$\boxed{\sum_{k=0}^{n-1}\binom{z}{k}x^{n-k-1} = \sum_{k=1}^{n}\binom{z-k}{n-k}(x+1)^{k-1}}$$

$$(1)\quad \sum_{k=0}^{n-1}\binom{z}{k}x^{n-k-1} \overset{conv}{=} [t^{n-1}]\frac{(1+t)^z}{1-xt}.$$

$$(2)\quad \sum_{k=0}^{n-1}\binom{z-1-k}{n-1-k}(x+1)^k \overset{B}{=} [t^{n-1}](1+t)^{z-1}\left[\frac{1}{1-(x+1)u}\;\middle|\; u=\frac{t}{1+t}\right] = [t^{n-1}]\frac{(1+t)^z}{1-xt}.$$

(1.11) - *The left hand side is a convolution:*

$$\boxed{\sum_{k=0}^{n-1}\binom{z}{k}\frac{x^{n-k}}{n-k} = \sum_{k=1}^{n}\binom{z-k}{n-k}\frac{(x+1)^k-1}{k}}$$

$$(1)\quad \sum_{k=0}^{n}\binom{z}{k}\frac{x^{n-k}}{n-k} \overset{conv}{=} [t^n](1+t)^z\ln\frac{1}{1-xt}.$$

$$(2)\quad \sum_{k=0}^{n}\binom{z-k}{n-k}\frac{(x+1)^k}{k} \overset{B}{=} [t^n](1+t)^z\left[\ln\frac{1}{1-(x+1)u}\;\middle|\; u=\frac{t}{1+t}\right] = [t^n](1+t)^z\ln\frac{1+t}{1-xt}.$$

$$\sum_{k=0}^{n}\binom{z-k}{n-k} = [t^n](1+t)^z\left[\ln\frac{1}{1-u}\;\middle|\; u=\frac{t}{1+t}\right] = [t^n](1+t)^z\ln(1+t).$$

$$[t^n](1+t)^z\ln\frac{1+t}{1-xt} - [t^n](1+t)^z\ln(1+t) = [t^n](1+t)^z\ln\frac{1}{1-xt}.$$

***(1.12)**

$$\sum_{k=0}^{n}\binom{n}{k}\frac{x^{r+k}}{r+k}=\sum_{k=1}^{r}(-1)^{r-k}\binom{r-1}{r-k}\frac{(x+1)^{n+k}-1}{n+k}\qquad r\geq 1$$

(1.13) - *Gould only considers $j \leq n$. The Riordan Array approach allows to extend this sum to all j's, also greater than n. Egorychev ascribes to Govindarajulu and Suzuki (p. 186 (5.6.12)) this identity. Egorychev (1.1.2) calls* Tepper identity *the sum* $\sum_k (-1)^k \binom{n}{k}(x-k)^p$ $(p \leq n)$. *With the method of the next two sums, it can been shown to be equivalent to the present one*:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^j=\left\{ {j \atop n} \right\}(-1)^n n!$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k k^j \overset{E,PW1}{=} [t^n]\frac{1}{1-t}\left[\sum_{k=0}^{j}\left\{ {j \atop k} \right\}\frac{k!(-u)^k}{(1+u)^{k+1}}\ \Big|\ u=\frac{t}{1-t}\right]=$$

$$=[t^n]\sum_{k=0}^{j}\left\{ {j \atop k} \right\}k!(-1)^k t^k=\left\{ {j \atop n} \right\}n!(-1)^n.$$

(1.14) - *We use formula (1.13)*:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)^{n+1}=\frac{2x-n}{2}(n+1)!$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)^{n+1}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{r=0}^{n+1}\binom{n+1}{r}x^{n+1-r}(-1)^r k^r=$$

$$=\sum_{r=0}^{n+1}\binom{n+1}{r}x^{n+1-r}(-1)^r\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^r=$$

$$=\binom{n+1}{n}x(-1)^n\left\{ {n \atop n} \right\}(-1)^n n!+\binom{n+1}{n+1}(-1)^{n+1}\left\{ {n+1 \atop n} \right\}(-1)^n n!=$$

$$=x(n+1)!-\frac{(n+1)n}{2}n!=\frac{2x-n}{2}(n+1)!.$$

(1.15) - *We use formula (1.13)*:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)^{n+2}=\frac{3n^2+n+12x^2-12nx}{24}(n+2)!$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)^{n+2}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{r=0}^{n+2}\binom{n+2}{r}x^{n+2-r}(-1)^r k^r=$$

$$=\sum_{r=0}^{n+2}\binom{n+2}{r}x^{n+2-r}(-1)^r\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^r=$$

$$= \binom{n+2}{n} x^2 \left\{ n \atop n \right\} n! - \binom{n+2}{n+1} x \left\{ n+1 \atop n \right\} n! + \binom{n+2}{n+2} \left\{ n+2 \atop n \right\} n! =$$

$$= \frac{x^2}{2}(n+2)! - \frac{xn}{2}(n+2)! + \frac{3n+1}{24} n(n+2)! = \frac{3n^2+n+12x^2-12nx}{24}(n+2)!.$$

(1.16) - (1.22) *According to formula (1.13) we have:*

$$S_j = \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^{n+j} = \left\{ n+j \atop n \right\} (-1)^n n!.$$

Therefore, S_j is essentially given by the elements in the diagonals (top-left to bottom-right) of the Stirling triangle of the second kind. Let $\delta^{[k]}(t)$ be the generating function of the k-th diagonal. Clearly, $\delta^{[0]}(t) = 1/(1-t)$ and in general we have: $\delta_n^{[k]} = \left\{ n \atop n-k \right\}$. The recurrence of the Stirling numbers of the second kind gives:

$$\left\{ n+1 \atop n+1-k \right\} = (n+1-k) \left\{ n \atop n+1-k \right\} + \left\{ n \atop n-k \right\}$$

$$\delta_{n+1}^{[k]} = n\delta_n^{[k-1]} - (k-1)\delta_n^{[k-1]} + \delta_n^{[k]}.$$

Passing to generating functions:

$$\frac{\delta^{[k]}(t) - \delta_0^{[k]}}{t} = \delta^{[k]}(t) + tD\delta^{[k-1]}(t) - (k-1)\delta^{[k-1]}(t).$$

For $k > 0$ we have $\delta_0^{[k]} = 0$ and consequently:

$$\delta^{[k]}(t) = t\delta^{[k]}(t) + t^2 D\delta^{[k-1]}(t) - t(k-1)\delta^{[k-1]}(t)$$

$$\delta^{[k]}(t) = \frac{t}{1-t}\left(tD\delta^{[k-1]}(t) - (k-1)\delta^{[k-1]}(t) \right).$$

For example:

$$\delta^{[1]}(t) = \frac{t}{1-t}\left(tD\frac{1}{1-t} - 0 \right) = \frac{t}{1-t} \cdot \frac{t}{(1-t)^2} = \frac{t^2}{(1-t)^3}$$

$$\left\{ n \atop n-1 \right\} = [t^n]\frac{t^2}{(1-t)^3} = \binom{-3}{n-2}(-1)^{n-2} = \binom{n}{n-2} = \binom{n}{2}$$

$$\delta^{[2]}(t) = \frac{t}{1-t}\left(tD\frac{t^2}{(1-t)^3} - \frac{t^2}{(1-t)^3} \right) = \frac{t^3+2t^4}{(1-t)^5}.$$

$$\delta^{[3]}(t) = \frac{t}{1-t}\left(tD\frac{t^3+2t^4}{(1-t)^5} - 2\frac{t^3+2t^4}{(1-t)^5} \right) = \frac{t^4+8t^5+6t^6}{(1-t)^7}.$$

$$\left\{ n \atop n-2 \right\} = \frac{3n-5}{24}\binom{n}{3} \qquad \left\{ n \atop n-3 \right\} = \frac{n(n-1)(n-2)^2(n-3)^2}{48}$$

(1.23)

$$\boxed{\sum_{k=0}^{\infty} \binom{n+k}{k} \frac{1}{2^k} = 2^{n+1}}$$

$$\sum_{k=0}^{\infty} \binom{n+k}{k} \frac{1}{2^k} \stackrel{BC4}{=} \left[\frac{1}{(1-u/2)^{n+1}} \;\middle|\; u = 1 \right] = 2^{n+1}.$$

(1.24)

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} = 2^n}$$

$$(I) \qquad \sum_{k=0}^{n} \binom{n}{k} = \left[(1+u)^n \;\Big|\; u=1\right] = 2^n.$$

$$(II) \qquad \sum_{k=0}^{n} \binom{n}{k} = n! \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{1}{(n-k)!} \stackrel{conv}{=} n![t^n]e^{2t} = n! \cdot \frac{2^n}{n!} = 2^n.$$

$$(III) \qquad \sum_{k=0}^{n} \binom{n}{k} \stackrel{P}{=} [t^n]\frac{1}{1-t}\left[\frac{1}{1-u} \;\Big|\; u = \frac{t}{1-t}\right] = [t^n]\frac{1}{1-2t} = 2^n.$$

(1.25)

$$\boxed{\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \delta_{n,0}}$$

$$(I) \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \left[(1-u)^n \;\Big|\; u=1\right] = \delta_{n,0}.$$

$$(II) \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \cdot \frac{1}{(n-k)!} \stackrel{conv}{=} n![t^n]e^0 = n! \cdot 1 = \delta_{n,0}.$$

$$(III) \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \stackrel{P}{=} [t^n]\frac{1}{1-t}\left[\frac{1}{1-u} \;\Big|\; u = \frac{t}{1-t}\right] = [t^n]1 = \delta_{n,0}.$$

(1.26) - *Many identities involving trigonometric functions are better proved by passing to the exponential function. They are more exercises in Trigonometry than in combinatorial identities.*

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} \cos kx = 2^n \cos\frac{nx}{2}\left(\cos\frac{x}{2}\right)^n}$$

$$\sum_{k=0}^{n} \binom{n}{k} (\cos kx + i \sin kx) = \sum_{k=0}^{n} \binom{n}{k} (e^{ix})^k \stackrel{E}{=} [t^n]\frac{1}{1-t}\left[\frac{1}{1-e^{ix}u} \;\Big|\; u = \frac{t}{1-t}\right] =$$

$$= [t^n]\frac{1}{1-(1+e^{ix})t} = (1+e^{ix})^n = (1+\cos x + i\sin x)^n = \left(2\cos^2\frac{x}{2} + i2\sin\frac{x}{2}\cos\frac{x}{2}\right)^n =$$

$$= 2^n\left(\cos\frac{x}{2}\right)^n\left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right)^n = 2^n\left(\cos\frac{x}{2}\right)^n e^{inx/2} = 2^n\left(\cos\frac{x}{2}\right)^n\left(\cos\frac{nx}{2} + i\sin\frac{nx}{2}\right).$$

(1.27) - *We take the imaginary part in the proof of formula (1.26)*:

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} \sin kx = 2^n \sin\frac{nx}{2}\left(\cos\frac{x}{2}\right)^n}$$

(1.28)

$$\boxed{\sum_{k=0}^{n} (-1)^k \binom{n}{k} \cos kx = (-2)^n \cos\frac{n(x+\pi)}{2}\left(\sin\frac{x}{2}\right)^n}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (\cos kx + i\sin kx) = (-1)^n \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (e^{ix})^k \stackrel{E}{=}$$

$$= (-1)^n [t^n] \frac{1}{1+t} \left[\frac{1}{1-e^{ix}u} \;\middle|\; u = \frac{t}{1+t} \right] = (-1)^n [t^n] \frac{1}{1+(1-e^{ix})t} =$$

$$= (1-e^{ix})^n = (1-\cos x - i\sin x)^n =$$

$$= \left(2\sin^2\frac{x}{2} - i2\sin\frac{x}{2}\cos\frac{x}{2}\right)^n = 2^n \left(\sin\frac{x}{2}\right)^n \left(-\cos\frac{\pi+x}{2} - i\sin\frac{\pi+x}{2}\right)^n =$$

$$= (-2)^n \left(\sin\frac{x}{2}\right)^n e^{in(\pi+x)/2} = (-2)^n \left(\sin\frac{x}{2}\right)^n \left(\cos\frac{n(\pi+x)}{2} + i\sin\frac{n(\pi+x)}{2}\right).$$

(1.29) - *We take the imaginary part in the proof of formula (1.28)*:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \sin kx = (-2)^n \sin\frac{n(x+\pi)}{2} \left(\sin\frac{x}{2}\right)^n$$

(1.30) - *Egorychev (2.1.4) Ex. 12. The natural sum is from* $-n/2$ *to* $n/2$*. We first perform this sum*:

$$\sum_{k=0}^{n/2} \binom{n}{k} \cos(n-2k)x = 2^{n-1}\cos^n x + \frac{1}{2}\binom{n}{n/2}$$

$$\sum_{k=-n/2}^{n/2} \binom{n}{k} \cos(n-2k)x \to \sum_k \binom{n}{k} e^{inx-i2kx} = e^{inx} \sum_k \binom{n}{k} (e^{-i2x})^k \stackrel{E}{=}$$

$$= e^{inx} [t^n] \frac{1}{1-(1+e^{-i2x})t} = e^{inx}(1+e^{-i2x})^n = e^{inx}(1+\cos(-2x)+i\sin(-2x))^n =$$

$$= e^{inx}(1+\cos(2x)-i\sin(2x))^n = e^{inx}(2\cos^2 x - i2\sin x\cos x)^n = e^{inx}2^n\cos^n x(\cos x - i\sin x)^n =$$

$$= e^{inx}2^n\cos^n(x)e^{-inx} = 2^n\cos^n x.$$

Clearly, $k \mapsto n-k$ *does not change the value; therefore, the addend is symmetric with respect to* $n/2$*. The original sum is one half of our result, plus one half of the central element, if this exists, that is if* n *is even. The central element is:* $\binom{n}{n/2}\cos 0 = \binom{n}{n/2}$*. In conclusion:*

$$\sum_{k=0}^{n/2} \binom{n}{k} \cos(n-2k)x = 2^{n-1}\cos^n x + \frac{1}{2}\binom{n}{n/2}$$

with the usual convention that the binomial coefficient is 0 if n *is odd.*

(1.31)

$$\sum_{k=0}^{n/2} \binom{n}{2k} \cos kx = 2^{n-1}\left(\cos^n\frac{x}{4}\cdot\cos\frac{nx}{4} + (-1)^n\sin^n\frac{x}{4}\cdot\cos\frac{n(2\pi+x)}{4}\right)$$

$$\sum_{k=0}^{n/2} \binom{n}{2k} e^{ikx} = [t^n]\frac{1}{1-t}\left[\frac{1}{1-ue^{ix}} \;\middle|\; u = \frac{t^2}{(1-t)^2}\right] = [t^n]\frac{1-t}{1-2t+t^2(1-e^{ix})} =$$

$$= [t^n]\frac{1-t}{(1-(1-e^{ix/2})t)\cdot(1-(1+e^{ix/2})t)} = \frac{1}{2}[t^n]\left(\frac{1}{1-(1-e^{ix/2})t} + \frac{1}{1-(1+e^{ix/2})t}\right) =$$

$$= \frac{(1-e^{ix/2})^n + (1+e^{ix/2})^n}{2} = \frac{(1-\cos x/2 - i\sin x/2)^n + (1+\cos x/2 + i\sin x/2)^n}{2} =$$

$$= \frac{1}{2}\left(\left(2\sin^2\frac{x}{4} - i2\sin\frac{x}{4}\cos\frac{x}{4}\right)^n + \left(2\cos^2\frac{x}{4} + i2\sin\frac{x}{4}\cos\frac{x}{4}\right)^n\right) =$$

$$= 2^{n-1}(-1)^n \sin^n\frac{x}{4}\left(\cos\frac{2\pi+x}{4} + i\sin\frac{2\pi+x}{4}\right)^n + 2^{n-1}\cos^n\frac{x}{4}\left(\cos\frac{x}{4} + i\sin\frac{x}{4}\right)^n =$$

$$= 2^{n-1}(-1)^n \sin^n\frac{x}{4}\left(\cos\frac{n(2\pi+x)}{4} + i\sin\frac{n(2\pi+x)}{4}\right) + 2^{n-1}\cos^n\frac{x}{4}\left(\cos\frac{nx}{4} + i\sin\frac{nx}{4}\right).$$

(1.32) - *We take the imaginary part in the proof of formula (1.31)*:

$$\boxed{\sum_{k=0}^{n/2}\binom{n}{2k}\sin kx = 2^{n-1}\left(\cos^n\frac{x}{4}\cdot\sin\frac{nx}{4} + (-1)^n\sin^n\frac{x}{4}\cdot\sin\frac{n(2\pi+x)}{4}\right)}$$

(1.33)

$$\boxed{\sum_{k=0}^{(n+1)/2}\cos(2k+1)x = 2^{n-1}\left(\cos^n\frac{x}{2}\cos\frac{nx}{2} - (-1)^n\sin^n\frac{x}{2}\cos\frac{n(\pi+x)}{2}\right)}$$

$$\sum_{k=0}^{(n+1)/2} e^{ix(2k+1)} = [t^n]\left[\frac{(1+e^{ix}\sqrt{t})^n - (1-e^{ix}\sqrt{t})^n}{2\sqrt{t}} \;\Big|\; t=1\right] = \frac{1}{2}\left((1+e^{ix})^n - (1-e^{ix})n\right).$$

$$(1+e^{ix})^n = (1+\cos x + i\sin x)^n = \left(2\cos^2\frac{x}{2} + i2\sin\frac{x}{2}\cos\frac{x}{2}\right)^n =$$

$$= 2^n\cos^n\frac{x}{2}\left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right)^n = 2^n\cos^n\frac{x}{2}\left(\cos\frac{nx}{2} + i\sin\frac{nx}{2}\right).$$

$$(1-e^{ix})^n = (1-\cos x - i\sin x)^n = \left(2\sin^2\frac{x}{2} - i2\sin\frac{x}{2}\cos\frac{x}{2}\right)^n =$$

$$= 2^n\sin^n\frac{x}{2}\left(-\cos\frac{\pi+x}{2} - i\sin\frac{\pi+x}{2}\right) = (-2)^n\sin^n\frac{x}{2}\left(\cos\frac{n(\pi+x)}{2} + i\sin\frac{n(\pi+x)}{2}\right).$$

(1.34) - *We take the imaginary part in the proof of formula (1.33)*:

$$\boxed{\sum_{k=0}^{(n+1)/2}\sin(2k+1)x = 2^{n-1}\left(\cos^n\frac{x}{2}\sin\frac{nx}{2} - (-1)^n\sin^n\frac{x}{2}\sin\frac{n(\pi+x)}{2}\right)}$$

(1.35) - *A direct proof by means of generating functions is possible, but rather lengthy. The conclusion follows from (1.24); the central element is present only if n is even*:

$$\boxed{\sum_{k=0}^{n/2}\binom{n}{k} = 2^{n-1} + \frac{1}{2}\binom{n}{n/2}}$$

(1.36) - *A direct proof by means of generating functions is possible, but rather lengthy. The conclusion follows from (1.24); the central element is is never present; for n even, we should subtract one half of the central element*:

$$\boxed{\sum_{k=0}^{(n-1)/2}\binom{n}{k} = 2^{n-1} - \frac{1}{2}\binom{n}{n/2}}$$

(1.37)

$$\sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{k+1}=\frac{(x+1)^{n+1}-1}{(n+1)x}$$

$$\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}x^k\overset{A}{=}\frac{1}{n+1}[t^{n+1}]\frac{t}{(1-t)^2}\left[\frac{1}{1-xu}\;\middle|\;u=\frac{t}{1-t}\right]=$$

$$=\frac{1}{n+1}[t^n]\frac{1}{(1-t)^2}\frac{1-t}{1-(x+1)t}=[t^n]\frac{1}{(n+1)x}\left(\frac{x+1}{1-(x+1)t}-\frac{1}{1-t}\right)=\frac{(x+1)^{n+1}-1}{(n+1)x}$$

(1.38)

$$\sum_{k=0}^{n/2}\binom{n}{2k}\frac{x^{2k}}{2k+1}=\frac{(x+1)^{n+1}-(1-x)^{n+1}}{2(n+1)x}$$

$$\frac{1}{n+1}\sum_{k=0}^{n/2}\binom{n+1}{2k+1}x^{2k}\overset{A}{=}\frac{1}{n+1}[t^{n+1}]\frac{t}{(1-t)^2}\left[\frac{1}{1-x^2u}\;\middle|\;u=\frac{t^2}{(1-t)^2}\right]=$$

$$=\frac{1}{n+1}[t^n]\frac{1}{(1-t)^2}\cdot\frac{(1-t)^2}{1-2t+t^2-x^2t^2}=[t^n]\frac{1}{2(n+1)x}\left(\frac{x+1}{1-(x+1)t}+\frac{1-x}{1-(1-x)t}\right)=$$

$$=\frac{(x+1)^{n+1}-(1-x)^{n+1}}{2(n+1)x}$$

(1.39)

$$\sum_{k=0}^{(n-1)/2}\binom{n}{2k+1}\frac{x^{2k}}{k+1}=\frac{(x+1)^{n+1}+(1-x)^{n+1}-2}{(n+1)x^2}$$

$$\frac{1}{x^2}\sum_{k=0}^{(n-1)/2}\binom{n}{2k+1}\frac{(x^2)^{k+1}}{k+1}\overset{A}{=}\frac{1}{x^2}[t^n]\frac{t}{(1-t)^2}\left[\frac{1}{u}\ln\frac{1}{1-x^2u}\;\middle|\;u=\frac{t^2}{(1-t)^2}\right]=$$

$$=\frac{1}{x^2}[t^n]\frac{t}{(1-t)^2}\cdot\frac{(1-t)^2}{t^2}\ln\frac{(1-t)^2}{1-t+(1-x^2)t^2}=\frac{1}{x^2}[t^{n+1}]\ln\frac{(1-t)^2}{(1-(1+x)t)(1-(1-x)t)}=$$

$$=\frac{1}{x^2}[t^{n+1}]\left(\ln\frac{1}{1-(1+x)t}+\ln\frac{1}{1-(1-x)t}-2\ln\frac{1}{1-t}\right)=\frac{(x+1)^{n+1}+(1-x)^{n+1}-2}{(n+1)x^2}.$$

***(1.40)** - *A proof by means of operators is given in Sprugnoli: An Introduction to Mathematical Methods in Combinatorics, Section 6.7, this same site*:

$$\sum_{k=0}^{\infty}(-1)^k\binom{x}{k}\frac{1}{z+k}=\frac{\Gamma(z)\Gamma(x+1)}{\Gamma(x+z+1)}=\frac{1}{z}\binom{x+z}{x}^{-1}\qquad\Re(x)>-1$$

(1.41)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x}{x+k}=\binom{x+n}{n}^{-1}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}\binom{x+k}{k}^{-1}\frac{x}{x+k} = \frac{n!x!}{(x+n)!}\sum_{k=0}^{n}(-1)^k\binom{x+n}{n-k}\binom{x+k-1}{k} \stackrel{B}{=}$$

$$= \frac{n!x!}{(x+n)!}[t^n](1+t)^{x+n}\left[\frac{1}{(1+u)^x}\;\middle|\; u=t\right] = \frac{n!x!}{(x+n)!}[t^n](1+t)^n = \binom{n+x}{n}^{-1}.$$

Closely related to this is the following sum of the form S:1/1:

(1.42) - The proof is related to Theorem (Z.23).

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}^{-1} = \frac{x}{x+n}}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}^{-1} = \sum_{k=0}^{n}(-1)^k\frac{n!k!x!}{k!(n-k)!(x+k)!} = \frac{n!x!}{(n+x)!}\sum_{k=0}^{n}\binom{n+x}{n-k}(-1)^k \stackrel{B}{=}$$

$$= \frac{n!x!}{(n+x)!}[t^n](1+t)^{n+x}\left[\frac{1}{1+u}\;\middle|\; u=t\right] = \frac{n!x!}{(n+x)!}\binom{n+x-1}{n} =$$

$$= \frac{n!x!}{(n+x)!}\cdot\frac{(n+x-1)!}{n!(x-1)!} = \frac{x}{x+n}.$$

(1.43)

$$\boxed{\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{x-k} = \binom{x}{n}^{-1}\frac{(-1)^n}{x-n}}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}\binom{x}{k}^{-1}\frac{(-1)^k}{x-k} = \sum_{k=0}^{n}\frac{n!k!(x-k)!}{k!(n-k)!x!}\cdot\frac{1}{x-k}\binom{x}{k}(-1)^k =$$

$$= \frac{n!(x-n)!}{(x-n)\cdot x!}\sum_{k=0}^{n}\binom{x-k-1}{n-k}\binom{x}{k}(-1)^k = \frac{n!(x-n)!}{(x-n)\cdot x!}[t^n](1+t)^{x-1}\left[(1-u)^x\;\middle|\; u=\frac{t}{1+t}\right] =$$

$$= \frac{n!(x-n)!}{(x-n)\cdot x!}[t^n]\frac{1}{1+t} = \frac{(-1)^n}{x-n}\binom{x}{n}^{-1}$$

***(1.44)**

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}\binom{k+1/a}{k}^{-1} = \sum_{k=0}^{n}\frac{1}{ak+1}}$$

(1.45)

$$\boxed{\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k} = H_n}$$

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k} \stackrel{E}{=} [t^n]\frac{1}{1-t}\left[\ln(1+u)\;\middle|\; u=\frac{t}{1-t}\right] = [t^n]\frac{1}{1-t}\ln\frac{1}{1-t} = H_n.$$

(1.46) - The following sum is an S:1/0 or a special instance of an S:1/1. *We use partial fraction expansion to reduce this sum to identity (1.41). There is an error in the formula of Gould.*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{(k+3)(k+4)(k+7)}=\frac{1}{(n+2)(n+1)}\left(\frac{1}{2(n+3)}+\frac{1}{2}\binom{n+7}{5}^{-1}-\binom{n+4}{2}^{-1}\right)$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{1}{4(k+3)}-\frac{1}{3(k+4)}+\frac{1}{12(k+7)}\right)=$$
$$=\frac{1}{12}\sum_k\binom{n}{k}\frac{(-1)^k3}{k+3}-\frac{1}{12}\sum_k\binom{n}{k}\frac{(-1)^k4}{k+4}+\frac{1}{84}\sum_k\binom{n}{k}\frac{(-1)^k7}{k+7}\stackrel{1.41}{=}$$
$$=\frac{1}{12}\binom{n+3}{3}^{-1}-\frac{1}{12}\binom{n+4}{4}^{-1}+\frac{1}{84}\binom{n+7}{7}^{-1}=$$
$$=\frac{1}{(n+2)(n+1)}\left(\frac{1}{2}\binom{n+3}{1}^{-1}-\binom{n+4}{2}+\frac{1}{2}\binom{n+7}{5}\right).$$

(1.47)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k^p}{x+k}=(-1)^px^{p-1}\binom{x+n}{n}^{-1}\qquad(j<n)$$

$$\binom{n}{k}\frac{1}{x+k}=\frac{1}{x}\cdot\frac{n!(x+k-1)!x!k!}{k!(n-k)!(x+k)!(x-1)!k!}=\frac{1}{x}\binom{x+n}{n}^{-1}\binom{x+n}{n-k}\binom{x+k-1}{k}.$$

$$S=\sum_{k=0}^{n}\frac{(-1)^kk^p}{(x+k)}=\frac{1}{x}\binom{x+n}{n}^{-1}\sum_k\binom{x+n}{n-k}\binom{x+k-1}{k}[t^k]\sum_{r=0}^{p}\left\{{p\atop r}\right\}\frac{r!t^r}{(1-t)^{r+1}}=$$
$$=\frac{1}{x}\binom{x+n}{n}^{-1}\sum_{r=0}^{p}\left\{{p\atop r}\right\}r!\sum_{k=0}^{n}\binom{x+n}{n-k}\binom{-r}{k}[t^{k-r}]\frac{1}{(1-t)^{r+1}}=$$
$$=\frac{1}{x}\binom{x+n}{n}^{-1}\sum_{r=0}^{p}\left\{{p\atop r}\right\}r!\sum_{k=0}^{n}\binom{x+n}{n-k}\binom{-x}{k}\binom{k}{r}=$$
$$=\frac{1}{x}\binom{x+n}{n}^{-1}\sum_{r=0}^{p}\left\{{p\atop r}\right\}r!\binom{-x}{r}\sum_{k=0}^{n}\binom{x+n}{n-k}\binom{-x-r}{k-r}.$$

$$\sum_k\binom{x+n}{n-k}\binom{-x-r}{k-r}\stackrel{B}{=}[t^n](1+t)^{n+x}\left[\frac{u^r}{(1+u)^{x+r}}\;\middle|\;u=t\right]=$$
$$=[t^{n-r}](1+t)^{n+x-x-r}=[t^{n-r}](1+t)^{n-r}=1.$$

$$S=\frac{1}{x}\binom{x+n}{n}^{-1}\sum_{r=0}^{p}\left\{{p\atop r}\right\}r!\frac{(-x)^{\underline{r}}}{r!}=\frac{1}{x}\binom{x+n}{n}^{-1}x^p(-1)^p=(-1)^px^{p-1}\binom{x+n}{n}^{-1}.$$

(1.48) - ***(NN)*** *When $x>r$ this sum $S(x,r,n)$ is not natural, being defined for $k\geq r-x$; therefore it should be considered as a difference: $S(x,r,n)-S(x,r,-1)$.*

$$\sum_{k=0}^{n}\binom{k+x}{r}=\binom{n+x+1}{r+1}-\binom{x}{r+1}$$

$$\sum_k \binom{k+x}{r} \stackrel{P}{=} [t^n]\frac{t^{r-x}}{(1-t)^{r+2}} = \binom{-r-2}{n-r+x}(-1)^{n-r+x} = \binom{n+x+1}{r+1}.$$

$$[t^{-1}]\frac{t^{r-x}}{(1-t)^{r+2}} = \binom{x}{r+1}$$

(1.49) - *(NN)* *This is a partial sum*:

$$\boxed{\sum_{k=0}^{n}\binom{x+k}{k} = \binom{x+n+1}{n}}$$

$$\sum_{k=0}^{n}\binom{x+k}{k} \stackrel{P}{=} [t^n]\frac{1}{(1-t)^{x+2}} = \binom{-x-2}{n}(-1)^n = \binom{x+n+1}{n}.$$

(1.50) - *(NN)* *The corresponding natural sum starts from 0; therefore we should subtract the relative value, which is 1. The sum can be generalized to every starting point $r < n$*:

$$\boxed{\sum_{k=1}^{n}\binom{x+k}{k}\frac{1}{x+k} = \frac{1}{x}\left(\binom{x+n}{n} - 1\right)}$$

$$\sum_{k=0}^{n}\frac{1}{x}\left(\binom{x+k}{k} - \binom{x+k-1}{k-1}\right) \stackrel{P}{=} \frac{1}{x}\left([t^n]\frac{1}{(1-t)^{x+2}} - [t^n]\frac{t}{(1-t)^{x+2}}\right) =$$

$$= \frac{1}{x}[t^n]\frac{1}{(1-t)^{x+1}} = \frac{1}{x}\binom{x+n}{n}.$$

(1.51) - *(NN)* *A partial sum, seen as a difference*:

$$\boxed{\sum_{k=a}^{n}\binom{k}{x} = \binom{n+1}{x+1} - \binom{a}{x+1}}$$

$$\sum_{k=0}^{n}\binom{k}{x} \stackrel{P}{=} [t^n]\frac{t^x}{(1-t)^{x+2}} = [t^{n+1}]\frac{t^{x+1}}{(1-t)^{x+2}} = \binom{n+1}{x+1}.$$

(1.52) - *(NN)* *This is just the complete sum relative to (1.51)*:

$$\boxed{\sum_{k=j}^{n}\binom{k}{j} = \binom{n+1}{j+1}}$$

***(1.53)**

$$\boxed{\sum_{k=0}^{(n-a)/r}\binom{n}{a+kr}x^{a+kr} = \frac{1}{r}\sum_{j=1}^{r}(\omega_r^j)^{-a}(1+x\omega_r^j)^n \qquad r-1 \geq a,\ a \in \mathbb{Z},\ \omega_r = e^{2\pi i/r}}$$

***(1.54)**

$$\sum_{k=0}^{(n-a)/r} \binom{n}{a+kr} = \frac{1}{r}\sum_{j=1}^{r}\left(2\cos\frac{\pi j}{r}\right)^n \cos\frac{(n-2a)j\pi}{r} \qquad n \geq a \geq 0,\ r-1 \geq a$$

***(1.55)**

$$\sum_{k=0}^{n/r} \binom{n}{kr} = \frac{1}{r}\sum_{j=1}^{r}(1+\omega_r^j)^n = \frac{2^n}{r}\sum_{j=1}^{r}\left(\cos\frac{\pi j}{r}\right)^n \cos\frac{n\pi j}{r}$$

(1.56)

$$\sum_{k=0}^{n/3} \binom{n}{3k} = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right)$$

$$\sum_{k=0}^{n/3} \binom{n}{3k} \stackrel{A}{=} [t^n]\frac{1}{1-t}\left[\frac{1}{1-u} \;\middle|\; u = \frac{t^3}{(1-t)^3}\right] = [t^n]\frac{(1-t)^2}{1-3t+3t^2-2t^3} =$$

$$= [t^n]\frac{(1-t)^2}{(1-2t)(1-t+t^2)} = [t^n]\frac{1}{3}\left(\frac{1}{1-2t} + \frac{2-t}{1-t+t^2}\right) =$$

$$= \frac{1}{3}\left(2^n + \frac{4}{\sqrt{3}}\sin\frac{(n+1)\pi}{3} - \frac{2}{\sqrt{3}}\sin\frac{n\pi}{3}\right) =$$

$$= \frac{1}{3}\left(2^n + \frac{2}{\sqrt{3}}\left(2\sin\frac{n\pi}{3}\cos\frac{\pi}{3} + 2\cos\frac{n\pi}{3}\sin\frac{\pi}{3} - \sin\frac{n\pi}{3}\right)\right) = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right).$$

(1.57)

$$\sum_{k=0}^{(n-1)/3} \binom{n}{3k+1} = \frac{1}{3}\left(2^n + 2\cos\frac{(n-2)\pi}{3}\right)$$

$$\sum_{k=0}^{(n-1)/3} \binom{n}{3k+1} \stackrel{A}{=} [t^n]\frac{1}{(1-t)^2}\left[\frac{1}{1-u} \;\middle|\; u = \frac{t^3}{(1-t)^3}\right] = [t^{n-1}]\frac{1-t}{1-3t+3t^2-2t^3} =$$

$$= [t^{n-1}]\frac{1-t}{(1-2t)(1-t+t^2)} = [t^{n-1}]\frac{1}{3}\left(\frac{2}{1-2t} + \frac{1+t}{1-t+t^2}\right) = \frac{1}{3}\left(2^n + \frac{2}{\sqrt{3}}\left(\sin\frac{n\pi}{3} + \sin\frac{(n-1)\pi}{3}\right)\right) =$$

$$= \frac{1}{3}\left(2^n + \frac{2}{\sqrt{3}}\left(\sin\frac{n\pi}{3} + \sin\frac{n\pi}{3}\cos\frac{\pi}{3} - \cos\frac{n\pi}{3}\sin\frac{\pi}{3}\right)\right) =$$

$$= \frac{1}{3}\left(2^n + 2\left(\frac{\sqrt{3}}{2}\sin\frac{n\pi}{3} - \frac{1}{2}\cos\frac{n\pi}{3}\right)\right) =$$

$$= \frac{1}{3}\left(2^n + 2\left(\sin\frac{2\pi}{3}\sin\frac{n\pi}{3} + \cos\frac{2\pi}{3}\sin\frac{n\pi}{3}\right)\right) = \frac{1}{3}\left(2^n + 2\cos\frac{(n-2)\pi}{3}\right).$$

(1.58)

$$\sum_{k=0}^{n/4}\binom{n}{4k} = \frac{1}{4}\left(2^n + 2\sqrt{2}^n \cos\frac{n\pi}{4}\right)$$

$$\sum_{k=0}^{n/4}\binom{n}{4k} \stackrel{A}{=} [t^n]\frac{1}{1-t}\left[\frac{1}{1-u}\;\middle|\; u = \frac{t^4}{(1-t)^4}\right] = [t^n]\frac{(1-t)^3}{1-4t+6t^2-4t^3} =$$

$$= [t^n]\frac{(1-t)^3}{(1-2t)(1-2t+2t^2)} = [t^n]\frac{1}{4}\left(\frac{1}{1-2t} + \frac{2-2t}{1-2t+2t^2} + \frac{1}{4}\right) =$$

$$= \frac{1}{4}\left(2^n + 2\left(\sqrt{2}^{n+1}\sin\frac{(n+1)\pi}{4} - \sqrt{2}^n \sin\frac{n\pi}{4}\right)\right) = \frac{1}{4}\left(2^n + 2\sqrt{2}^n\cos\frac{n\pi}{4}\right).$$

(1.59) - *The transformation reveals that this sum is equal to the previous one (1.58), except that, at the end, we should extract the coefficient of* t^{4n}. *This is obtained by substituting* $n \to 4n$ *in the formula of (1.58)*:

$$\sum_{k=0}^{n}\binom{4n}{4k} = \frac{1}{4}\left(16^n + (-1)^n 2^{2n+1}\right)$$

$$\sum_{k=0}^{n}\binom{4n}{4k} \stackrel{A}{=} [t^{4n}]\frac{1}{1-t}\left[\frac{1}{1-u}\;\middle|\; u = \frac{t^4}{(1-t)^4}\right] = [t^{4n}]\frac{(1-t)^3}{1-4t+6t^2-4t^3} =$$

$$= [t^{4n}]\frac{(1-t)^3}{(1-2t)(1-2t+2t^2)} = [t^{4n}]\frac{1}{4}\left(\frac{1}{1-2t} + \frac{2-2t}{1-2t+2t^2} + \frac{1}{4}\right) = \frac{1}{4}\left(16^n + (-1)^n 2^{2n+1}\right).$$

(1.60)

$$\sum_{k=0}^{n/2}(-1)^k\binom{n-k}{k}(xy)^k(x+y)^{n-2k} = \frac{x^{n+1}-y^{n+1}}{x-y}$$

$$\sum_{k=0}^{n/2}\binom{n-k}{n-2k}(x+y)^{n-2k}(-xy)^k \stackrel{A}{=} [t^n]\frac{1}{1-(x+y)t}\left[\frac{1}{1+xyt}\;\middle|\; u = \frac{t^2}{1-(x+y)t}\right] =$$

$$= [t^n]\frac{1}{1-(x+y)t+xyt^2} = [t^n]\frac{1}{(1-xt)(1-yt)} = \frac{x^{n+1}-y^{n+1}}{x-y}$$

(1.61)

$$\sum_{k=0}^{n/2}\binom{n-k}{k}z^k 2^{n-2k} = \frac{x^{n+1}-y^{n+1}}{x-y} \qquad (x,y) = 1 \pm \sqrt{z+1}$$

$$2^n\sum_{k=0}^{n/2}\binom{n-k}{k}\left(\frac{z}{4}\right)^k \stackrel{A}{=} [t^n]\frac{1}{1-t}\left[\frac{1}{1-zu/4}\;\middle|\; u = \frac{t^2}{1-t}\right] = 2^n[t^n]\frac{1}{1-t-zt^2/4} =$$

$$= 2^n[t^n]\frac{1}{\left(1-\frac{1+\sqrt{1+z}}{2}t\right)\left(1-\frac{1-\sqrt{1+z}}{2}t\right)} = \frac{(1+\sqrt{1+z})^{n+1}-(1-\sqrt{1+z})^{n+1}}{2\sqrt{1+z}}.$$

(1.62)

$$\sum_{k=0}^{n/2} \binom{n-k}{k} (2\cos x)^{n-2k} = \frac{\sin(n+1)x}{\sin x}$$

$$(I) \quad \sum_{k=0}^{n/2} \binom{n-k}{n-2k} (2\cos x)^{n-2k} = \sum_{k=0}^{n/2} \binom{-k-1}{n-2k} (-2\cos x)^{n-2k} \stackrel{B}{=}$$

$$= [t^n] \frac{1}{1+t} \left[\frac{1}{1+(2\cos x)u} \;\middle|\; u = \frac{t^2}{1+(2\cos x)t} \right] = [t^n] \frac{1}{1-(2\cos x)t+t^2} =$$

$$= \frac{2\sin(n+1)x}{2\sin x} = \frac{\sin(n+1)x}{\sin x}$$

$$(II) \quad \sum_{k=0}^{n/2} (-1)^k \binom{n-k}{k} (2\cos x)^{n-2k} = [\alpha = 2\cos x] = \alpha^n \sum_k \binom{n-k}{k} \frac{(-1)^k}{\alpha^{2k}} \stackrel{A}{=}$$

$$= \alpha^n [t^n] \frac{1}{1-t} \left[\frac{1}{1+u/\alpha^2} \;\middle|\; u = \frac{t^2}{1-t} \right] = \alpha^n [t^n] \frac{1}{1-t+t^2/\alpha^2} = [t^n] \frac{1}{1-\alpha t+t^2} =$$

$$= [t^n] \frac{1}{(1-e^{ix}t)(1-e^{-ix}t)} = [t^n] \frac{1}{(2i\sin x)t} \left(\frac{1}{1-e^{ix}t} - \frac{1}{1-e^{-ix}t} \right) = \frac{e^{i(n+1)x} - e^{-i(n+1)x}}{2i\sin x} =$$

$$= \frac{\cos(n+1)x + i\sin(n+1)x - \cos(-(n+1)x) - i\sin(-(n+1)x)}{2i\sin x} = \frac{2i\sin(n+1)x}{2i\sin x} = \frac{\sin(n+1)x}{\sin x}.$$

(1.63)

$$\sum_{k=0}^{n/2} (-1)^k \binom{n-k}{k} \frac{(2\cos x)^{n-2k}}{n-k} = \frac{2}{n} \cos nx$$

$$\sum_{k=0}^{n/2} (-1)^k \binom{n-k}{k} \frac{(2\cos x)^{n-2k}}{n-k} = \frac{(2\cos x)^n}{n} \sum_k \left(\binom{n-k}{k} + \binom{n-k-1}{k-1} \right) \frac{(-1)^k}{(2\cos x)^{2k}} =$$

$$= \frac{(2\cos x)^n}{n} [t^n] \frac{2-t}{1-t} \cdot \frac{1-t}{1-t+t^2/(2\cos x)^2} = \frac{1}{n} \cdot \frac{2-(2\cos x)t}{1-(2\cos x)t+t^2} =$$

$$= \frac{2\sin(n+1)x - 2\cos x \sin nx}{n\sin x} = \frac{2\sin nx\cos x + 2\cos nx \sin x - 2\cos x\sin nx}{n\sin x} = \frac{2\cos nx}{n}.$$

***(1.64)**

$$\sum_{k=0}^{n/2} \binom{n-k}{k} \frac{1}{n-k} \left(\frac{z}{4}\right)^k = \frac{1}{n2^{n-1}} \cdot \frac{x^n+y^n}{x+y} \qquad x, y = 1 \pm \sqrt{z+1}$$

(1.65)

$$\sum_{k=0}^{n/2} (-1)^k \binom{n-k}{k} \frac{4^{n-k}}{n-k} = \frac{2^{n+1}}{n} \qquad n \geq 1$$

$$\sum_{k=0}^{n/2} (-1)^k \binom{n-k}{k} \frac{4^{n-k}}{n-k} = \frac{4^n}{n} \sum_k \left(\binom{n-k}{k} + \binom{n-k-1}{k-1} \right) \frac{(-1)^k}{4^k} \stackrel{A}{=}$$

$$= \frac{4^n}{n}[t^n]\frac{2-t}{1-t}\left[\frac{1}{1+u/4} \;\Big|\; u = \frac{t^2}{1-t}\right] = \frac{4^n}{n}[t^n]\frac{2-t}{(1-t/2)^2} =$$

$$= \frac{2\cdot 4^n}{n}[t^n]\frac{1}{1-t/2} = \frac{2\cdot 4^n}{n}\cdot\frac{1}{2^n} = \frac{2^{n+1}}{n}.$$

(1.66)

$$\boxed{\sum_{k=0}^{n/2}(-1)^k\binom{n-k}{k}\frac{(2\cos x)^{n-2k}}{k+1} = \frac{(2\cos x)^{n+2} - 2\cos(n+2)x}{n+2}}$$

$$\sum_{k=0}^{n/2}\binom{n-k}{k}\frac{(-1)^k(2\cos x)^{n-2k}}{k+1} = [\alpha = 2\cos x] = \alpha^n[t^n]\frac{1}{1-t}\left[\frac{-1}{u/\alpha^2}\ln\frac{1}{1+u/\alpha^2} \;\Big|\; u = \frac{t^2}{1-t}\right] =$$

$$= \alpha^n[t^n]\frac{1}{1-t}\left(-\frac{\alpha^2(1-t)}{t^2}\ln\frac{1-t}{1-t+t^2\alpha^2}\right) = \alpha^{n+2}[t^{n+2}]\left(\ln\frac{1}{1-t} - \ln\frac{1}{1-t+t^2/\alpha^2}\right) =$$

$$= [t^{n+2}]\left(\ln\frac{1}{1-(2\cos x)t} - \ln\frac{1}{1-e^{ix}t} - \ln\frac{1}{1-e^{-ix}t}\right) =$$

$$= \frac{(2\cos x)^{n+2}}{n+2} - \frac{\cos(n+2)x + i\sin(n+2)x}{n+2} - \frac{\cos(n+2)x - i\sin(n+2)x}{n+2} =$$

$$= \frac{(2\cos x)^{n+2} - 2\cos(n+2)x}{n+2}.$$

(1.67)

$$\boxed{\sum_{k=0}^{n/2}(-1)^k\binom{n-k}{k}\frac{4^{n-k}}{k+1} = \frac{4^{n+1} - 2^{n+1}}{n+2}}$$

$$4^n\sum_{k=0}^{n/2}\binom{n-k}{k}\frac{(-1/4)^k}{k+1} \stackrel{A}{=} 4^n[t^n]\frac{1}{1-t}\left[-\frac{1}{u/4}\ln\frac{1}{1+u/4} \;\Big|\; u = \frac{t^2}{1-t}\right] =$$

$$= 4^n[t^n]\frac{1}{1-t}\left(\frac{-4(1-t)}{t^2}\right)\ln\frac{1-t}{1-t+t^2/4} = 4^{n+1}[t^{n+2}]\left(\ln\frac{1}{1-t} - 2\ln\frac{1}{1-t/2}\right) =$$

$$= \frac{4^{n+1}}{n+2} - \frac{2\cdot 4^{n+1}}{(n+2)2^{n+2}} = \frac{4^{n+1} - 2^{n+1}}{n+2}.$$

(1.68) - *Egorychev (1.5.1) calls this sum* Hardy's identity. Cambridge Mathematical Tripos, 1932; Hardy, Pure Mathematics, p. 445.

$$\boxed{\sum_{k=0}^{n/2}(-1)^k\binom{n-k}{k}\frac{1}{n-k} = \begin{cases} (-1)^n\cdot 2/n & \text{if} \quad n = 3k \\ (-1)^{n-1}/n & \text{if} \quad n = 3k\pm 1 \end{cases}}$$

$$\sum_{k=0}^{n/2}(-1)^k\binom{n-k}{k}\frac{1}{n-k} = \frac{1}{n}\sum_k\left(\binom{n-k}{k} + \binom{n-k-1}{k-1}\right)(-1)^k =$$

$$= \frac{1}{n}[t^n]\frac{2-t}{1-t}\left[\frac{1}{1+u} \;\Big|\; u = \frac{t^2}{1-t}\right] = \frac{1}{n}[t^n]\frac{2-t}{1-t+t^2} = \frac{2}{n}\cos\frac{n\pi}{3}.$$

$$(\mathbf{n = 3k}) \qquad \frac{2}{n}\cos k\pi = \frac{2}{n}(-1)^k = \frac{2}{n}(-1)^n.$$

$$(\mathbf{n = 3k \pm 1}) \qquad \frac{2}{n}\cos\left(k\pi \pm \frac{\pi}{3}\right) = \frac{2}{n}\left(\cos k\pi \cos\frac{\pi}{3} \mp \sin k\pi \sin\frac{\pi}{3}\right) = \frac{1}{n}(-1)^n.$$

(1.69)

$$\boxed{\binom{n-k}{k}\frac{6^k}{n-k} = \frac{3^n + (-2)^n}{n} \qquad n \geq 1}$$

$$\binom{n-k}{k}\frac{6^k}{n-k} = \frac{1}{n}\sum_k \left(\binom{n-k}{k} + \binom{n-k-1}{k-1}\right)6^k \stackrel{A}{=} \frac{1}{n}[t^n]\frac{2-t}{1-t}\left[\frac{1}{1-6u} \;\Big|\; u = \frac{t^2}{1-t}\right] =$$

$$= \frac{1}{n}[t^n]\frac{2-t}{(1-3t)(1+2t)} = \frac{1}{n}[t^n]\left(\frac{1}{1-3t} + \frac{1}{1+2t}\right) = \frac{3^n + (-2)^n}{n}.$$

(1.70)

$$\boxed{\sum_{k=0}^{n/2}\binom{n-k}{k}z^k = \frac{1}{\sqrt{1+4z}}\left(\left(\frac{1+\sqrt{1+4z}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{1+4z}}{2}\right)^{n+1}\right)}$$

$$\sum_{k=0}^{n/2}\binom{n-k}{k}z^k \stackrel{A}{=} [t^n]\frac{1}{1-t}\left[\frac{1}{1-zu} \;\Big|\; u = \frac{t^2}{1-t}\right] = [t^n]\frac{1}{1-t-zt^2} =$$

$$= [t^n]\frac{1}{\left(1 - \frac{1+\sqrt{1+4z}}{2}t\right)\left(1 - \frac{1-\sqrt{1+4z}}{2}t\right)} = \frac{1}{\sqrt{1+4z}}\left(\left(\frac{1+\sqrt{1+4z}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{1+4z}}{2}\right)^{n+1}\right).$$

$$\text{Ex.}: \quad \sum_{k=0}^{n/2}\binom{n-k}{k}6^k = \frac{3^{n+1} - (-2)^{n+1}}{5}.$$

(1.71) - *This is the same identity as (1.70)*:

$$\boxed{\sum_{k=0}^{n/2}\binom{n-k}{k}z^k = \frac{x^{n+1}-1}{(x-1)(1+x)^n} \quad \text{where}: \quad z = \frac{-x}{(1+x)^2}}$$

(1.72)

$$\boxed{\sum_{k=0}^{n/2}(-1)^k\binom{n-k}{k}2^{n-2k} = n+1}$$

$$2^n\sum_{k=0}^{n/2}\binom{n-k}{k}\left(-\frac{1}{4}\right)^k \stackrel{A}{=} 2^n[t^n]\frac{1}{1-t}\left[\frac{1}{1+u/4} \;\Big|\; u = \frac{t^2}{1-t}\right] =$$

$$= 2^n[t^n]\frac{1}{1-t+t^2/4} = [t^n]\frac{1}{(1-t)^2} = \binom{-2}{n}(-1)^n = n+1.$$

(1.73)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}\frac{1}{4^k} = \frac{2n-1}{4^n}}$$

$$\sum_{k=0}^{n}\binom{2n-k}{k}(-1)^k\frac{1}{4^k}\stackrel{A}{=}[t^{2n}]\frac{1}{1-t}\left[\frac{1}{1+u/4}\;\middle|\;u=\frac{t^2}{1-t}\right]=$$

$$=[t^{2n}]\frac{1}{1-t}\cdot\frac{1-t}{1-t+t^2/4}=[t^{2n}]\frac{1}{(1-t/2)^2}=\binom{-2}{2n}\left(-\frac{1}{2}\right)^{2n}=\frac{2n+1}{4^n}.$$

(1.74) $F_n = n$-th Fibonacci number, where $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$. *we adapted this definition to modern usage.*

$$\boxed{\sum_{k=0}^{n/2}\binom{n-k}{k}=F_{n+1}}$$

$$\sum_{k=0}^{n/2}\binom{n-k}{k}\stackrel{A}{=}[t^n]\frac{1}{1-t}\left[\frac{1}{1-u}\;\middle|\;u=\frac{t^2}{1-t}\right]=[t^n]\frac{1}{1-t-t^2}=[t^{n+1}]\frac{t}{1-t-t^2}=F_{n+1}$$

(1.75)

$$\boxed{\sum_{k=0}^{n/2}(-1)^k\binom{n-k}{k}=\frac{(-1)^{\lfloor n/3\rfloor}+(-1)^{\lfloor (n+1)/3\rfloor}}{2}}$$

$$\sum_{k=0}^{n/2}(-1)^k\binom{n-k}{k}\stackrel{A}{=}[t^n]\frac{1}{1-t}\left[\frac{1}{1+u}\;\middle|\;u=\frac{t^2}{1-t}\right]=[t^n]\frac{1}{1-t+t^2}=$$

$$=\frac{2}{\sqrt{3}}\sin\frac{(n+1)\pi}{3}.$$

(1.76)

$$\boxed{\sum_{k=0}^{n}\binom{n+k}{2k}=\sum_{k=0}^{n}\binom{2n-k}{k}=F_{2n+1}}$$

(1) $$\sum_{k=0}^{n}\binom{n+k}{2k}\stackrel{A}{=}[t^n]\frac{1}{1-t}\left[\frac{1}{1-u}\;\middle|\;u=\frac{t}{(1-t)^2}\right]=[t^n]\frac{1-t}{1-3t+t^2}=F_{2n+1}.$$

(2) $$\sum_{k=0}^{n}\binom{2n-k}{k}\stackrel{A}{=}[t^{2n}]\frac{1}{1-t}\left[\frac{1}{1-u}\;\middle|\;u=\frac{t^2}{1-t}\right]=[t^{2n}]\frac{1}{1-t-t^2}=F_{2n+1}.$$

(1.77)

$$\boxed{\sum_{k=0}^{n}\binom{n+k}{2k}2^{n-k}=\frac{2^{2n+1}+1}{3}}$$

$$\sum_{k=0}^{n}\binom{n+k}{2k}2^{n-k}\stackrel{A}{=}2^n[t^n]\frac{1}{1-t}\left[\frac{1}{1-u/2}\;\middle|\;u=\frac{t}{(1-t)^2}\right]=2^n[t^n]\frac{1-t}{(1-2t)(1-t/2)}=$$

$$=[t^n]\frac{1-2t}{(1-4t)(1-t)}=[t^n]\frac{1}{3}\left(\frac{2}{1-4t}+\frac{1}{1-t}\right)=\frac{2^{2n+1}+1}{3}$$

***(1.78)**

$$\sum_{k=0}^{n} \binom{n+k}{k} \left((1-x)^{n+1}x^k + x^{n+1}(1-x)^k\right) = 1$$

(1.79) - *Egorychev (2.2) Ex. 2 (p. 64)*:

$$\sum_{k=0}^{n} \binom{n+k}{k} 2^{-k} = 2^n$$

$$\frac{1}{2^n} \sum_{k=0}^{n} \binom{n+k}{k} 2^{n-k} \overset{conv}{=} \frac{1}{2^n}[t^n] \frac{1}{(1-t)^{n+1}} \cdot \frac{1}{1-2t} =$$

$$= \frac{1}{2^n}[t^n] \left[\frac{1}{(1-w)(1-2w)} \cdot \frac{1-w}{1-2w} \;\Big|\; w = \frac{t}{1-w} \right] = \frac{1}{2^n} \left[\frac{1}{(1-2w)^2} \;\Big|\; w = \frac{1-\sqrt{1-4t}}{2} \right] =$$

$$= \frac{1}{2^n}[t^n] \frac{1}{1-4t} = \frac{4^n}{2^n} = 2^n.$$

(1.80)

$$\sum_{k=1}^{\infty} \binom{2n+k}{n} 2^{-k} = 4^n$$

$$\sum_{k=-n}^{\infty} \binom{2n+k}{n} 2^{-k} = \left[\frac{1}{t^n(1-t)^{n+1}} \;\Big|\; t = \frac{1}{2} \right] = 2 \cdot 4^n.$$

$$\sum_{k=-n}^{0} \binom{2n+k}{n} 2^{-k} = \sum_{k=0}^{n} \binom{2n-k}{n} 2^k \overset{A}{=} [t^{2n}] \frac{t^n}{(1-t)^{n+1}} \left[\frac{1}{1-2u} \;\Big|\; u = t \right] =$$

$$= [t^n] \frac{1}{(1-2t)(1-t)^{n+1}} = [t^n] \left[\frac{1}{(1-2w)(1-w)} \cdot \frac{1-w}{1-2w} \;\Big|\; w = \frac{t}{1-w} \right] =$$

$$= [t^n] \left[\frac{1}{(1-2w)^2} \;\Big|\; w = \frac{1-\sqrt{1-4t}}{2} \right] = [t^n] \frac{1}{1-4t} = 4^n.$$

$$\sum_{k=1}^{\infty} \binom{2n+k}{n} 2^{-k} = 2 \cdot 4^n - 4^n = 4^n.$$

(1.81)

$$\sum_{k=0}^{n} \binom{2n-k}{n} 2^k = 4^n$$

$$\sum_{k=0}^{n} \binom{2n-k}{n} 2^k \overset{A}{=} [t^{2n}] \frac{t^n}{(1-t)^{n+1}} \left[\frac{1}{1-2u} \;\Big|\; u = t \right] = [t^n] \frac{1}{(1-2t)(1-t)^{n+1}} =$$

$$= [t^n] \left[\frac{1}{(1-w)(1-2w)} \cdot \frac{1-w}{1-2w} \;\Big|\; w = \frac{t}{1-w} \right] = \left[\frac{1}{(1-2w)^2} \;\Big|\; w = \frac{1-\sqrt{1-4t}}{2} \right] = [t^n] \frac{1}{1-4t} = 4^n$$

(1.82)

$$\sum_{k=0}^{n/2} \binom{2n}{n+2k} = 4^{n-1} + \binom{2n-1}{n} \qquad n \geq 1$$

$$\sum_{k=0}^{n} \binom{2n}{n+2k} \stackrel{A}{=} [t^{2n}] \frac{t^n}{(1-t)^{n+1}} \left[\frac{1}{1-u} \;\middle|\; u = \frac{t^2}{(1-t)^2} \right] = [t^n] \frac{1}{(1-2t)(1-t)^{n-1}} =$$

$$= [t^n] \left[\frac{1-w}{1-2w} \cdot \frac{1-w}{1-2w} \;\middle|\; w = \frac{t}{1-w} \right] = \left[\frac{(1-w)^2}{(1-2w)^2} \;\middle|\; w = \frac{1-\sqrt{1-4t}}{2} \right] = [t^n] \frac{1-2t+\sqrt{1-4t}}{2(1-4t)} =$$

$$= [t^n] \left(\frac{1-2t}{2(1-4t)} + \frac{1}{2\sqrt{1-4t}} \right) = \frac{4^n - 2 \cdot 4^{n-1}}{2} + \frac{1}{2} \binom{2n}{n} = 4^{n-1} + \binom{2n-1}{n}.$$

(1.83) - ***(NN)*** *This is a partial sum*:

$$\sum_{k=0}^{n} \binom{2n+1}{k} = 4^n$$

$$\sum_{k=0}^{n} \binom{2n+1}{k} \stackrel{P}{=} [t^n] \frac{(1+t)^{2n+1}}{1-t} = [t^n] \left[\frac{1+w}{1-w} \cdot \frac{1+w}{1-w} \;\middle|\; w = t(1+w)^2 \right] =$$

$$= [t^n] \left[\left(\frac{1+w}{1-w} \right)^2 \;\middle|\; w = \frac{1-2t-\sqrt{1-4t}}{2} \right] = [t^n] \frac{1}{1-4t} = 4^n$$

(1.84) - ***(NN)*** *This is a partial sum. (See (1.82))*:

$$\sum_{k=0}^{n} \binom{2n-1}{k} = 4^{n-1} + \binom{2n-1}{n} \qquad (n \geq 1)$$

$$(I) \quad \sum_{k=0}^{n} \binom{2n-1}{k} \stackrel{P}{=} [t^n] \frac{(1+t)^{2n-1}}{1-t} = [t^n] \left[\frac{1}{(1+w)(1-w)} \cdot \frac{1+w}{1-w} \;\middle|\; w = t(1+w)^2 \right] =$$

$$= [t^n] \left[\frac{1}{(1-w)^2} \;\middle|\; w = \frac{1-2t-\sqrt{1-4t}}{2} \right] = \frac{1}{2} [t^n] \frac{1-2t+\sqrt{1-4t}}{1-4t} = 4^{n-1} + \binom{2n-1}{n}.$$

$$(II) \quad \sum_{k=0}^{n} \binom{2n-1}{k} \stackrel{A}{=} [t^{2n-1}] \frac{1}{1-t} \left[\frac{1-u^{n+1}}{1-u} \;\middle|\; u = \frac{t}{1-t} \right] =$$

$$= [t^{2n-1}] \frac{1}{1-2t} \left(1 - \frac{t^{n+1}}{(1-t)^{n+1}} \right) = [t^{2n-1}] \frac{1}{1-2t} - [t^{n-2}] \frac{1}{(1-2t)(1-t)^{n+1}} =$$

$$= \frac{4^n}{2} - [t^n] \left[\frac{w^2}{(1-2w)(1-w)} \cdot \frac{1-w}{1-2w} \;\middle|\; w = \frac{t}{1-w} \right] =$$

$$= \frac{4^n}{2} - [t^n] \left[\frac{w^2}{(1-2w)^2} \;\middle|\; w = \frac{1-\sqrt{1-4t}}{2} \right] = \frac{4^n}{2} - [t^n] \frac{1-2t-\sqrt{1-4t}}{2(1-4t)} =$$

$$= \frac{4^n}{2} - \frac{4^n}{2} + 4^{n-1} + \frac{1}{2} \binom{2n}{n} = 4^{n-1} + \binom{2n-1}{n}.$$

(1.85) - ***(NN)*** *This is a partial sum*:

$$\sum_{k=0}^{n}\binom{2n}{k} = 2\cdot 4^{n-1}+\binom{2n-1}{n} \qquad (n\geq 1)$$

$$\sum_{k=0}^{n}\binom{2n}{k} \stackrel{P}{=} [t^n]\frac{(1+t)^{2n}}{1-t} = [t^n]\left[\frac{1}{(1-w)}\cdot\frac{1+w}{1-w}\;\Big|\; w=t(1+w)^2\right] =$$

$$= [t^n]\left[\frac{1+w}{(1-w)^2}\;\Big|\; w=\frac{1-2t-\sqrt{1-4t}}{2}\right] = \frac{1}{2}[t^n]\frac{1+\sqrt{1-4t}}{1-4t} = \frac{4^n}{2}+\frac{1}{2}\binom{2n}{n}.$$

(1.86) - ***(NN)*** *This is a partial sum*:

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k} = (-1)^n\binom{2n-1}{n}$$

$$\sum_{k=0}^{n}\binom{2n}{k}(-1)^k = [t^n]\frac{(1-t)^{2n}}{1-t} = [t^n](1-t)^{2n-1} = (-1)^n\binom{2n-1}{n}.$$

(1.87)

$$\sum_{k=0}^{n/2}\binom{n}{2k}x^k = \frac{(1+\sqrt{x})^n+(1-\sqrt{x})^n}{2}$$

$$\sum_{k=0}^{n/2}\binom{n}{2k}x^k \stackrel{A}{=} [t^n]\frac{1}{1-t}\left[\frac{1}{1-xu}\;\Big|\; u=\frac{t^2}{(1-t)^2}\right] = [t^n]\frac{1-t}{1-2t+(1-x)t^2} =$$

$$= \frac{1}{2}[t^n]\left(\frac{1}{1-(1-\sqrt{x})t}+\frac{1}{1-(1+\sqrt{x})t}\right) = \frac{(1+\sqrt{x})^n+(1-\sqrt{x})^n}{2}$$

(1.88)

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{2k} = (-1)^{n/2}2^n\frac{1+(-1)^n}{2}$$

$$\sum_{k=0}^{n}\binom{2n}{2k}(-1)^k \stackrel{A}{=} [t^{2n}]\frac{1}{1-t}\left[\frac{1}{1+u}\;\Big|\; u=\frac{t^2}{(1-t)^2}\right] = [t^{2n}]\frac{1-t}{1-2t+2t^2} =$$

$$= \frac{2\sqrt{2}^{2n+1}\sin(2n+1)\pi/4}{2} - \frac{2\sqrt{2}^{2n}\sin n\pi/2}{2} = 2^n\left(\sqrt{2}\sin\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)-\sin\frac{n\pi}{2}\right) =$$

$$= 2^n\left(\sqrt{2}\sin\frac{n\pi}{2}\cos\frac{\pi}{4}+\sqrt{2}\cos\frac{n\pi}{2}\sin\frac{\pi}{4}-\sin\frac{n\pi}{2}\right) = 2^n\cos\frac{n\pi}{4}$$

(1.89)

$$\sum_{k=0}^{n/2}\binom{n}{2k} = 2^{n-1} \qquad n\geq 1$$

$$\sum_{k=0}^{n/2} \binom{n}{2k} \stackrel{A}{=} [t^n]\frac{1}{1-t}\left[\frac{1}{1-u} \;\Big|\; u = \frac{t^2}{(1-t)^2}\right] = [t^n]\frac{1-t}{1-2t} = 2^n - 2^{n-1} = 2^{n-1}$$

(1.90) - *See identity (1.83)*:

$$\sum_{k=0}^{n/2} (-1)^k \binom{n}{2k} = \frac{(1+i)^n + (1-i)^n}{2} = \sqrt{2}^n \cos\frac{n\pi}{4}$$

$$\sum_{k=0}^{n/2} \binom{n}{2k}(-1)^k \stackrel{A}{=} [t^n]\frac{1}{1-t}\left[\frac{1}{1+u} \;\Big|\; u = \frac{t^2}{(1-t)^2}\right] = [t^n]\frac{1-t}{1-2t+2t^2} =$$

$$= \frac{2\sqrt{2}^{n+1}\sin(n+1)\pi/4}{2} - \frac{2\sqrt{2}^n \sin n\pi/4}{2} = \sqrt{2}^n\left(\sqrt{2}\sin\left(\frac{n\pi}{4} + \frac{\pi}{4}\right) - \sin\frac{n\pi}{4}\right) =$$

$$= \sqrt{2}^n\left(\sqrt{2}\sin\frac{n\pi}{4}\cos\frac{\pi}{4} + \sqrt{2}\cos\frac{n\pi}{4}\sin\frac{\pi}{4} - \sin\frac{n\pi}{4}\right) = \sqrt{2}^n\cos\frac{n\pi}{4}$$

(1.91)

$$\sum_{k=0}^{n} \binom{2n}{2k} = \sum_{k=0}^{n} \binom{2n}{2k+1} = 2^{2n-1} \qquad n \geq 1$$

$$(1) \quad \sum_{k=0}^{n} \binom{2n}{2k} \stackrel{A}{=} [t^{2n}]\frac{1}{1-t}\left[\frac{1}{1-u} \;\Big|\; u = \frac{t^2}{(1-t)^2}\right] = [t^{2n}]\frac{1-t}{1-2t} = 2^{2n-1}.$$

$$(2) \quad \sum_{k=0}^{n} \binom{2n}{2k+1} \stackrel{A}{=} [t^{2n}]\frac{t}{(1-t)^2}\left[\frac{1}{1-u} \;\Big|\; u = \frac{t^2}{(1-t)^2}\right] = [t^{2n-1}]\frac{1}{1-2t} = 2^{2n-1}.$$

(1.92) - *(**NN**) One half of the preceding sum (1.91); the central element is present only if n is even*:

$$\sum_{k=0}^{n/2} \binom{2n}{2k} = 2^{2n-2} + \binom{2n-1}{n}[n \text{ is even}] \qquad (n \geq 1)$$

(1.93) -

$$\sum_{k=0}^{n} \binom{2n+1}{2k} = \sum_{k=0}^{n} \binom{2n+1}{2k+1} = 4^n$$

$$(1) \quad \sum_{k=0}^{n} \binom{2n+1}{2k} \stackrel{A}{=} = [t^{2n+1}]\frac{1}{1-t}\left[\frac{1}{1-u} \;\Big|\; u = \frac{t^2}{(1-t)^2}\right] = [t^{2n+1}]\frac{1-t}{1-2t} = 2^{2n+1} - 2^{2n} = 4^n.$$

$$(2) \quad \sum_{k=0}^{n} \binom{2n+1}{2k+1} \stackrel{A}{=} = [t^{2n+1}]\frac{t}{(1-t)^2}\left[\frac{1}{1-u} \;\Big|\; u = \frac{t^2}{(1-t)^2}\right] = [t^{2n}]\frac{1}{1-2t} = 2^{2n} = 4^n.$$

(1.94) - *See the identity (1.88). Egorychev (2.2) Ex. 1 (p. 63)*:

$$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k} = (-1)^{(n+1)/2} 2^n$$

$$\sum_{k=0}^{n}\binom{2n+1}{2k}(-1)^k \stackrel{A}{=} [t^{2n+1}]\frac{1}{1-t}\left[\frac{1}{1+u}\;\Big|\; u=\frac{t^2}{(1-t)^2}\right] = [t^{2n+1}]\frac{1-t}{1-2t+2t^2} =$$

$$= \frac{2\sqrt{2}^{2n+2}\sin(2n+2)\pi/4}{2} - \frac{2\sqrt{2}^{2n+1}\sin(2n+1)\pi/4}{2} = 2^{n+1}\left(\sin\left(\frac{n\pi}{2}+\frac{\pi}{4}\right) - \frac{\sqrt{2}}{2}\sin\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right) =$$

$$= 2^{n+1}\left(\sqrt{2}\sin\frac{n\pi}{2}\cos\frac{\pi}{2} + \cos\frac{n\pi}{2}\sin\frac{\pi}{2} - \frac{\sqrt{2}}{2}\sin\frac{n\pi}{2}\cos\frac{\pi}{4} - \frac{\sqrt{2}}{2}\cos\frac{n\pi}{2}\sin\frac{\pi}{4}\right) =$$

$$= 2^n\left(\cos\frac{n\pi}{2}\sin\frac{n\pi}{2}\right)$$

(1.95)

$$\boxed{\sum_{k=0}^{(n-1)/2}\binom{n}{2k+1}x^k = \frac{(1+\sqrt{x})^n-(1-\sqrt{x})^n}{2\sqrt{x}} \qquad n\geq 1}$$

$$\sum_{k=0}^{(n-1)/2}\binom{n}{2k+1}x^k \stackrel{A}{=} [t^n]\frac{t}{(1-t)^2}\left[\frac{1}{1-xu}\;\Big|\; u=\frac{t^2}{(1-t)^2}\right] = [t^n]\frac{t}{(1-t)^2}\frac{(1-t)^2}{1-2t+t^2-xt^2} =$$

$$= [t^n]\frac{t}{(1-(1-\sqrt{x})t)(1-(1+\sqrt{x})t)} = [t^n]\frac{1}{2\sqrt{x}}\left(\frac{1}{1-(1+\sqrt{x})t} - \frac{1}{1-(1-\sqrt{x})t}\right) =$$

$$= \frac{(1+\sqrt{x})^n-(1-\sqrt{x})^n}{2\sqrt{x}}.$$

(1.96) - *The special case $x=-1$ of (1.95)*:

$$\boxed{\sum_{k=0}^{(n-1)/2}\binom{n}{2k+1}(-1)^k = \frac{(1+i)^n-(1-i)^n}{2i} = (\sqrt{2})^n\sin\left(\frac{n\pi}{4}\right)}$$

$$[t^n]\frac{t}{1-2t+2t^2} \stackrel{2RF}{=} = (\sqrt{2})^n\sin\left(\frac{n\pi}{4}\right).$$

(1.97) - *The special case $x=1$ of (1.95)*:

$$\boxed{\sum_{k=0}^{(n-1)/2}\binom{n}{2k+1}(-1)^k = \frac{(1+1)^n-(1-1)^n}{2} = 2^{n-1}}$$

(1.98) - ***(NN)*** *One half of the preceding sum (1.91); the central element is present only if n is odd*:

$$\boxed{\sum_{k=0}^{(n-1)/2}\binom{2n}{2k+1} = 4^{n-1} + \binom{2n-1}{n}[n \text{ is odd}]}$$

(1.99)

$$\boxed{\sum_{k=0}^{n-1}(-1)^k\binom{2n}{2k+1} = (-1)^{\lfloor n/2\rfloor}2^n\cdot\frac{1-(-1)^n}{2}}$$

$$\sum_{k=0}^{n-1}(-1)^k\binom{2n}{2k+1} \stackrel{A}{=} [t^{2n}]\frac{t}{(1-t)^2}\left[\frac{1}{1+u}\;\Big|\; u=\frac{t^2}{(1-t)^2}\right] = [t^{2n-1}]\frac{1}{1-2t+2t^2} =$$

$$= \frac{2(\sqrt{2})^{2n}\sin(2n\pi/4)}{2} = 2^n \sin\frac{n\pi}{2}.$$

(1.100)

$$\boxed{\sum_{k=0}^{n}\binom{2n+1}{2k+1}k = (2n-1)4^{n-1}}$$

$$\sum_{k=0}^{n}\binom{2n+1}{2k+1}k \stackrel{A}{=} [t^{2n+1}]\frac{t}{(1-t)^2}\left[\frac{u}{(1-u)^2}\;\Big|\; u=\frac{t^2}{(1-t)^2}\right] = [t^{2n}]\frac{t^2}{(1-t)^4}\cdot\frac{(1-t)^4}{(1-2t)^2} =$$

$$= [t^{2n-2}]\frac{1}{(1-2t)^2} = \binom{-2}{2n-2}(-2)^{2n-2} = (2n-1)4^{n-1}$$

(1.101)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k+1} = (-1)^{\lfloor n/2\rfloor}\cdot 2^n}$$

$$\sum_{k=0}^{n}\binom{2n+1}{2k+1}(-1)^k \stackrel{A}{=} [t^{2n+1}]\frac{t}{(1-t)^2}\left[\frac{1}{1+u}\;\Big|\; u=\frac{t^2}{(1-t)^2}\right] = [t^{2n}]\frac{1}{1-2t+2t^2} =$$

$$= \frac{2\sqrt{2}^{2n+1}\sin((2n+1)\pi/4}{2} = 2^n\left(\sin\frac{n\pi}{4}+\cos\frac{n\pi}{4}\right).$$

***(1.102)** $B^n(x)$ means $B_n(x)$ a Bernoulli polynomial.

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{n+x}{n-k}\frac{1}{k+1} = \binom{B(x)+n}{n} \quad \text{symbolically}}$$

***(1.103)**

$$\boxed{\sum_{k=0}^{n}\binom{x}{k}\frac{k\cdot k!}{x^{k+1}} = 1-\binom{x}{n+1}\frac{(n+1)!}{x^{n+1}}}$$

(1.104)

$$\boxed{\sum_{k=0}^{\infty}\binom{2k}{k}\frac{x^k}{(1-2k)4^k} = \sqrt{1-x} \qquad |x|\le 1}$$

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{x^k}{(1-2k)4^k} = \left[\sqrt{1-xt}\;\Big|\; t=1\right] = \sqrt{1-x}.$$

(1.105)

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{x^k}{4^k}=\frac{1}{\sqrt{1-x}}\qquad -1\le x<1$$

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{x^k}{4^k}=\left[\frac{1}{\sqrt{1-xt}}\;\Big|\; t=1\right]=\frac{1}{\sqrt{1-x}}.$$

(1.106)

$$\sum_{k=1}^{\infty}\binom{2k}{k}\frac{(-1)^{k-1}x^k}{k2^{2k+1}}=\ln\frac{1+\sqrt{1+x}}{2}\qquad |x|<1$$

$$\mathcal{G}\left(-\frac{1}{2}\binom{2k}{k}\frac{1}{k}\left(-\frac{x}{4}\right)^k\right)=-\frac{1}{2}\cdot 2\ln\frac{1-\sqrt{1+xt}}{-xt/2}=\ln\frac{1+\sqrt{1+xt}}{2}.$$

$$\sum_{k=1}^{\infty}\binom{2k}{k}\frac{(-1)^{k-1}x^k}{k2^{2k+1}}=\left[\ln\frac{1+\sqrt{1+xt}}{2}\;\Big|\; t=1\right]=\ln\frac{1+\sqrt{1+x}}{2}.$$

***(1.107)**

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{(2k+1)^2 4^k}=\int_0^1\frac{\arcsin x}{x}\mathrm{d}x=\frac{\pi}{2}\ln 2$$

(1.108)

$$S_r=\sum_{k=0}^{n}\binom{2k}{k}\frac{k^r}{4^k}=\frac{2n+1}{4^n}\binom{2n}{n}\sum_{k=0}^{r}\binom{n}{k}\left\{ r \atop k \right\}\frac{k!}{2k+1}$$

$$S_r=\sum_{k=0}^{n}\binom{2k}{k}\frac{k^r}{4^k}=\sum_{k=0}^{n}\binom{2k}{k}\frac{1}{4^k}[t^k]\sum_{j=0}^{r}\left\{ r \atop j \right\}\frac{j!t^j}{(1-t)^{j+1}}=$$

$$=\sum_{k=0}^{n}\binom{2k}{k}\frac{1}{4^k}\sum_{j=0}^{r}\left\{ r \atop j \right\}j![t^{k-j}]\frac{1}{(1-t)^{j+1}}=\sum_{k=0}^{n}\binom{2k}{k}\frac{1}{4^k}\sum_{j=0}^{r}\left\{ r \atop j \right\}j!\binom{-j-1}{k-j}=$$

$$=\sum_{j=0}^{r}\left\{ r \atop j \right\}j!\sum_{k=0}^{n}\binom{2k}{k}\binom{k}{j}\frac{1}{4^k}.$$

$$f_k=\binom{2k}{k}\binom{k}{j}\frac{1}{4^k};\qquad f_0=0 \text{ for } j>0;\qquad \frac{f_{k+1}}{f_k}=\frac{k+1/2}{k+1-j}.$$

$$(k+1)f_{k+1}-jf_{k+1}=kf_k+\frac{1}{2}f_k;\qquad f'(t)-\frac{j}{t}f(t)=tf'(t)+\frac{1}{2}f(t).$$

$$t(1-t)f'(t)=\left(j+\frac{t}{2}\right)f(t)\qquad f(t)=\binom{2j}{j}\frac{1}{4^j}\cdot\frac{t^j}{(1-t)^{j+1/2}}.$$

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{k}{j}\frac{1}{4^k}\overset{P}{=}[t^n]\binom{2j}{j}\frac{1}{4^j}\cdot\frac{t^j}{(1-t)(1-t)^{j+1/2}}=\binom{2j}{j}\frac{1}{4^j}[t^{n-j}]\frac{1}{(1-t)^{j+3/2}}=$$

$$=\binom{2j}{j}\frac{1}{4^j}\binom{-j-3/2}{n-j}(-1)^{n-j}=\binom{2j}{j}\frac{1}{4^j}\binom{n+1/2}{n-j}\overset{Z46}{=}\frac{2n+1}{2j+1}\binom{2n}{n}\binom{n}{j}\frac{1}{4^n}.$$

$$S_r = \frac{2n+1}{4^n}\binom{2n}{n}\sum_{k=0}^{r}\binom{n}{k}\left\{ {r \atop k} \right\}\frac{k!}{2k+1}.$$

(1.109)

$$\boxed{S_0 = \frac{2n+1}{4^n}\binom{2n}{n} = \binom{n+1/2}{n}}$$

We have: $\left\{ {0 \atop 0} \right\} = 1$, *and the internal sum is 1.*

(1.110)

$$\boxed{S_1 = \frac{n}{3}S_0}$$

We have: $\left\{ {1 \atop j} \right\} = 1,\ \forall j$, *and therefore the internal sum is* $n/3$.

(1.111)

$$\boxed{S_2 = \frac{n(3n+2)}{3\cdot 5}S_0}$$

$$\left(\frac{n}{3} + \frac{2}{5}\cdot\frac{n(n-1)}{2}\right)S_0 = \frac{n(3n+2)}{3\cdot 5}S_0.$$

(1.112)

$$\boxed{S_3 = \frac{n(15n^2+18n+2)}{3\cdot 5\cdot 7}S_0}$$

$$\left(\frac{n}{3} + \frac{3n(n-1)}{5} + \frac{n(n-1)(n-2)}{7}\right)S_0 = \frac{n(15n^2+18n+2)}{3\cdot 5\cdot 7}S_0.$$

***(1.113)**

$$\boxed{\sum_{k=0}^{n}\binom{2k}{k}\frac{k\cdot k!}{2^k} = \binom{2n+2}{n+1}\frac{(n+1)!}{2^{n+2}} - \frac{1}{2}}$$

***(1.114)** Bernstein polynomials:

$$\boxed{B_n^f(x) = \sum_{k=0}^{n}\binom{n}{k}(1-x)^{n-k}x^k f\left(\frac{k}{n}\right)}$$

***(1.115)** Generalized Laguerre polynomials (ordinary when $a=0$):

$$\boxed{L_n^{(a)} = \sum_{k=0}^{n}\binom{n+a}{n-k}\frac{x^k}{k!} = \frac{1}{n!}\cdot\frac{e^x}{x^a}D_x^n\left(\frac{x^{a+n}}{e^x}\right)}$$

(1.116)

$$\sum_{k=0}^{n} \binom{p-k}{n-k} x^k = x^{p+1}(x-1)^{n-p-1} \quad \text{provided} : \quad 0 \le p \le n-1$$

If $0 \le p \le n-1$, we perform the change of variable: $h = k-p-1$ that is $k = h+p+1$:

$$\sum_{k=0}^{n} \binom{p-k}{n-k} x^k = \sum_{h=0}^{n-p-1} \binom{-h-1}{n-h-p-1} x^{h+p+1} = x^{p+1} \sum_{h=0}^{n-p-1} \binom{n-p-1}{h} (-x)^h \stackrel{E}{=}$$

$$= x^{p+1}[t^{n-p-1}]\frac{1}{1-t}\left[\frac{1}{1+xu} \;\middle|\; u = \frac{t}{1-t}\right] = x^{p+1}[t^{n-p-1}]\frac{1}{1-(1-x)t} = x^{p+1}(1-x)^{n-p-1}.$$

(1.117) This and the next two are special case of a general formula due to Abel.

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} (x+k)^{n-k}(k+1)^{k-1} = (x-1)^n$$

$$S_n = n! \sum_{k=0}^{n} \frac{(x+k)^{n-k}}{(n-k)!} \cdot \frac{(k+1)^{k-1}}{k!}(-1)^k.$$

$$\frac{(x+k)^{n-k}}{(n-k)!} = [t^{n-k}]e^{xt}e^{kt} \rightsquigarrow \mathcal{R}\left(e^{xt}, e^t\right).$$

$$\mathcal{G}\left(\frac{(-1)^k(k+1)^{k-1}}{k!}\right) = \mathcal{G}\left([t^k]e^{-(k+1)t} + [t^{k-1}]e^{-(k+1)t}\right) = \mathcal{G}\left([t^k](1+t)e^{-t}e^{-kt}\right) =$$

$$= \left[\frac{(1+w)e^{-w}}{1+w} \;\middle|\; w = te^{-w}\right] = \left[e^{-w} \;\middle|\; w = te^{-w}\right].$$

$$S_n = n![t^n]e^{xt}\left[\left[e^{-w} \;\middle|\; w = ue^{-w}\right] \;\middle|\; u = te^t\right] =$$

$$= n![t^n]e^{xt}\left[e^w \;\middle|\; w = te^te^{-w}\right] = n![t^n]e^{(x-1)t} = (x-1)^n.$$

(1.118) *Special case of a general formula due to Abel*:

$$\sum_{k=1}^{n} \binom{n}{k} \frac{1}{k} (yk)^{n-k}(x-yk)^k = \frac{x^n}{n} \qquad n \ge 2$$

$$S_n = n! \sum_{k=0}^{n} \frac{(yk)^{n-k}}{k \cdot (n-k)!} \cdot \frac{(x-yk)^k}{k!}.$$

$$\frac{(yk)^{n-k}}{k \cdot (n-k)!} = y \cdot \frac{(yk)^{n-k-1}}{(n-k)!} = \frac{1}{n}\left(\frac{(yk)^{n-k}}{(n-k)!} + y \cdot \frac{(yk)^{n-k-1}}{(n-k-1)!}\right) =$$

$$= \frac{1}{n}[t^{n-k}](1+yt)e^{ykt} \rightsquigarrow \mathcal{R}\left((1+yt), e^{yt}\right).$$

$$\mathcal{G}\left(\frac{(x-yk)^k}{k!}\right) = \mathcal{G}\left([t^k]e^{xt}e^{-ykt}\right) = \left[\frac{e^{xw}}{1+yw} \;\middle|\; w = te^{-yw}\right].$$

$$S_n = \frac{n!}{n}[t^n](1+yt)\left[\left[\frac{e^{xw}}{1+yw} \;\middle|\; w = ue^{-yw}\right] \;\middle|\; u = te^{yt}\right] = \frac{n!}{n}[t^n](1+yt) \cdot \frac{e^{xt}}{1+yt} = \frac{x^n}{n}.$$

(1.119) *Special case of a general formula due to Abel*:

$$\sum_{k=1}^{n} \binom{n}{k} (yk)^{n-k}(x-yk)^{k-1} = x^{n-1}$$

$$S_n = n! \sum_{k=0}^{n} \frac{(yk)^{n-k}}{(n-k)!} \cdot \frac{(x-yk)^{k-1}}{k!}.$$

$$\frac{(yk)^{n-k}}{(n-k)!} = [t^{n-k}]e^{ykt} \rightsquigarrow \mathcal{R}\left(1, e^{yt}\right).$$

$$\mathcal{G}\left(\frac{(x-yk)^{k-1}}{k!}\right) = \frac{1}{x}\mathcal{G}\left([t^k]e^{(x-yk)t} + y[t^{k-1}]e^{(x-yk)t}\right) =$$

$$= \frac{1}{x}\mathcal{G}\left([t^k](1+yt)e^{xt}e^{-ykt}\right) = \frac{1}{x}\left[\frac{(1+yw)e^{xw}}{1+yw} \;\middle|\; w = te^{-yw}\right] = \frac{1}{x}\left[e^{xw} \;\middle|\; w = te^{-yw}\right].$$

$$S_n = \frac{n!}{x}[t^n]\left[\left[e^{xw} \;\middle|\; w = ue^{-yw}\right] \;\middle|\; u = te^{yt}\right] = \frac{n!}{x}[t^n]e^{xt} = x^{n-1}.$$

(1.120) Polya and Szegö.

$$S = \sum_{k=0}^{\infty} \binom{a+bk}{k} z^k = \frac{x^{a+1}}{(1-b)x+b} \qquad z = \frac{x-1}{x^b}, |z| < \left|\frac{(b-1)^{b-1}}{b^b}\right|$$

From (GO2), the second generating function of Gould, we get:

$$S = \left[\left[\frac{(1+w)^{a+1}}{1-(b-1)w} \;\middle|\; w = zt(1+w)^b\right] \;\middle|\; t = 1\right].$$

This implies $z = w/(1+w)^b$ or, by setting $w = x-1$, $z = (x-1)/x^b$. Finally, we substitute $w = x-1$ in the formula of the generating function and obtain the desired result.

(1.121) - Same conditions as above:

$$S = \sum_{k=0}^{\infty} \frac{a}{a+bk} \binom{a+bk}{k} z^k = x^a \qquad z = \frac{x-1}{x^b}, |z| < \left|\frac{(b-1)^{b-1}}{b^b}\right|$$

From (GO1), the first generating function of Gould, we get:

$$S = \left[\left[(1+w)^a \;\middle|\; w = zt(1+w)^b\right] \;\middle|\; t = 1\right].$$

This implies $z = w/(1+w)^b$ or, by setting $w = x-1$, $z = (x-1)/x^b$. Finally, we substitute $w = x-1$ in the formula of the generating function and obtain the desired result.

***(1.122)** A discussion of this sum has been given by D. Dickinson:

$$\sum_{k=-\infty}^{+\infty} \binom{n+ak}{b+ck}$$

***(1.123)** This sum was studied by S. Ramanujan, in whose collected works one may find evaluations of this:

$$\boxed{\sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{(2k+1)^n 4^k}}$$

(1.124)

$$\boxed{\sum_{k=0}^{n}\binom{n}{k}(x+kz)^k(y-kz)^{n-k} = n!\sum_{k=0}^{n}\frac{(x+y)^k}{k!}\cdot z^{n-k}}$$

$$S_n = n!\sum_{k=0}^{n}\frac{(x+kz)^k}{k!}\cdot\frac{(y-kz)^{n-k}}{(n-k)!}.$$

$$[t^{n-k}]e^{(y-kz)t} = [t^{n-k}]e^{yt}(e^{-zt})^k \rightsquigarrow \mathcal{R}(e^{yt}, e^{-zt}).$$

$$\mathcal{G}\left(\frac{(x+kz)^k}{k!}\right) = \mathcal{G}\left([t^k]e^{(x+kz)t}\right) = \left[\frac{e^{xw}}{1-zw}\;\middle|\; w = te^{zw}\right].$$

$$S_n = n![t^n]e^{yt}\left[\left[\frac{e^{xw}}{1-zw}\;\middle|\; w = ue^{zw}\right]\;\middle|\; u = te^{-zt}\right] = n![t^n]e^{yt}\left[\frac{e^{xw}}{1-zw}\;\middle|\; w = te^{-zt}e^{zw}\right] =$$

$$= n![t^n]e^{yt}\frac{e^{xt}}{1-zt} = n![t^n]\frac{e^{(x+y)t}}{1-zt} = n!\sum_{k=0}^{n}\frac{(x+y)^k}{k!}\cdot z^{n-k}.$$

(1.125) This is a generalized Abel Convolution. Compare with (3.142) - (3.148).

$$\boxed{\sum_{k=0}^{n}\frac{x(x+kz)^{k-1}}{k!}\cdot\frac{y(y+(n-k)z)^{n-k-1}}{(n-k)!}(p+qk) = \frac{p(x+y)+qnz}{x+y}\cdot\frac{(x+y)(x+y+nz)^{n-1}}{n!}}$$

$$S = \sum_{k=0}^{n}\frac{x(x+kz)^{k-1}}{k!}\cdot\frac{y(y+(n-k)z)^{n-k-1}}{(n-k)!}(p+qk).$$

$$\frac{x(x+kz)^{k-1}}{k!} = \frac{(x+kz)^k}{k!} - z\frac{(x+kz)^{k-1}}{(k-1)!} = [t^k](1-zt)e^{(x+kz)t}.$$

$$\mathcal{G}\left((p+qk)\frac{x(x+kz)^{k-1}}{k!}\right) = p(1-zt)e^{(x+kz)t} + qxte^{(x+kz)t} =$$

$$= (p-pzt+qxt)e^{(x+kz)t} = \left[\frac{(p-pzw+qxw)e^{xw}}{1-zw}\;\middle|\; w = te^{zw}\right].$$

$$[t^{n-k}](1-zt)e^{(y+(n-k)z)t} = [t^{n-k}](1-zt)e^{(y+nz)t}e^{-kzt} \rightsquigarrow \mathcal{R}\left((1-zt)e^{(y+nz)t}, e^{-zt}\right).$$

$$S = [t^n](1-zt)e^{(y+nz)t}\left[\left[\frac{(p-pzw+qxw)e^{xw}}{1-zw}\;\middle|\; w = ue^{zw}\right]\;\middle|\; u = e^{-zt}\right] =$$

$$= [t^n](1-zt)e^{(y+nz)t}\cdot\frac{(p-pzt+qxt)e^{xt}}{1-zt} = [t^n](p-pzt+qxt)e^{(x+y+nz)t} =$$

$$= p\frac{(x+y+nz)^n}{n!} + (qx-pz)\frac{(x+y+nz)^{n-1}}{(n-1)!} = (px+py+qnx)\frac{(x+y+nz)^{n-1}}{n!} =$$

$$= \frac{p(x+y)+qnz}{x+y}\cdot\frac{(x+y)(x+y+nz)^{n-1}}{n!}.$$

(1.126)

$$\sum_{k=0}^{n}\binom{n}{k}k^r x^k = (1+x)^n \sum_{j=0}^{r}\binom{n}{j}\frac{(-x)^j}{(1+x)^j}\sum_{k=0}^{j}(-1)^k\binom{j}{k}k^r$$

$$\sum_{k=0}^{j}(-1)^k\binom{j}{k}k^r \stackrel{E}{=} [t^j]\frac{1}{1-t}\left[\sum_{k=0}^{j}\left\{ {r \atop k} \right\}\frac{k!(-1)^k u^k}{(1+u)^{k+1}}\;\Big|\; u=\frac{t}{1-t}\right] =$$

$$= [t^j]\frac{1}{1-t}\sum_{k=0}^{r}\left\{ {r \atop k} \right\}\frac{k!(-1)^k t^k(1-t)^{k+1}}{(1-t)^k} = [t^j]\sum_{k=0}^{r}\left\{ {r \atop k} \right\}(-1)^k t^k = \left\{ {r \atop j} \right\}j!(-1)^j.$$

(1) $$x^n\sum_{k=0}^{n}\binom{n}{n-k}\frac{k^r}{x^{n-k}} \stackrel{E}{=} x^n[t^n]\frac{1}{1-t/x}\sum_{j=0}^{r}\left\{ {r \atop j} \right\}\frac{j!t^j(1-t/x)^{j+1}}{(1-t/x)^j(1-(1+x)t/x)^{j+1}} =$$

$$= x^n[t^n]\sum_{j=0}^{r}\left\{ {r \atop j} \right\}\frac{j!t^j}{(1-(1+x)t/x)^{j+1}} = x^n\sum_{j=0}^{r}\left\{ {r \atop j} \right\}j![t^{n-j}]\left(1-\frac{1+x}{x}t\right)^{-j-1} =$$

$$= x^n\sum_{j=0}^{r}\left\{ {r \atop j} \right\}j!\binom{-j-1}{n-j}\left(-\frac{1+x}{x}\right)^{n-j} = (1+x)^n\sum_{j=0}^{r}\left\{ {r \atop j} \right\}j!\binom{n}{j}\frac{x^j}{(1+x)^j}.$$

(2) $$(1+x)^n\sum_{j=0}^{r}\binom{n}{j}\left(\frac{-x}{1+x}\right)^j\left\{ {r \atop j} \right\}j!(-1)^j = (1+x)^n\sum_{j=0}^{r}\left\{ {r \atop j} \right\}j!\binom{n}{j}\left(\frac{x}{1+x}\right)^j.$$

***(1.127)** Definition of Eulerian numbers:

$$x^n = \sum_{k=0}^{n}\binom{x+k-1}{n}A_{n,k}$$

and explicitly (Cf. Carlitz, Math. Mag. 32 (1959), p. 247):

$$A_{n.k} = \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k-j)^n.$$

These numbers of Euler (also Worpitzky) are a special case of numbers due to Nielsen.

***(1.128)** The numbers of Nielsen can be defined as follows:

$$B^n_{r,q} = \sum_{k=0}^{r}(-1)^k\binom{q}{k}(r-k)^n$$

From this it follows that, for all integers $m \geq n$:

$$x^n = (-1)^{m+n}\sum_{k=0}^{m+1}B^n_{k,m+1}\binom{x+k-1}{m}.$$

(1.129) - Schwatt [72], p. 48. - **(NN)**

$$\sum_{k=0}^{n}\binom{2k+1}{j} = \frac{(-1)^{j-1}}{2^{j+2}}\sum_{k=0}^{j+1}\binom{2n+3}{k}(-2)^k + \frac{(-1)^{j-1}}{2^{j+2}}$$

$$\sum_{k=0}^{n}\binom{2k+1}{j} \overset{B}{=} [t^j](1+t)\left[\frac{1-u^{n+1}}{1-u} \;\Big|\; u=(1+t)^2\right] = [t^j]\frac{1+t}{-2t(1+t/2)} - [t^j]\frac{(1+t)^{2n+3}}{-2t(1+t/2)}.$$

$$\frac{1}{2}\left([t^{j+1}]\frac{1}{1+t/2} + [t^j]\frac{1}{1+t/2}\right) = \frac{1}{2}\frac{(-1)^{j+1}}{2^{j+1}} + \frac{1}{2}\frac{(-1)^j}{2^j} = \frac{(-1)^j}{2^{j+2}}.$$

$$\frac{1}{2}[t^{j+1}]\frac{(1+t)^{2n+3}}{1+t/2} \overset{conv}{=} \frac{1}{2}\sum_{k=0}^{j+1}\binom{2n+3}{k}\left(-\frac{1}{2}\right)^{j+1-k} = \frac{(-1)^{j+1}}{2^{j+2}}\sum_{k=0}^{j+1}\binom{2n+3}{k}(-2)^k.$$

(1.130) - Schwatt [72], p. 51.

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{j+k}{j} = \frac{(-1)^{n+j}}{2^{j+1}}\sum_{k=0}^{j}\binom{n+j+1}{k}(-2)^k + \frac{1}{2^{j+1}}}$$

$$\sum_{k=0}^{n}(-1)^k\binom{j+k}{j} \overset{B}{=} [t^j](1+t)^j\left[\frac{1+(-1)^{n+1}u^{n+1}}{1+u} \;\Big|\; u=1+t\right] =$$

$$= \frac{1}{2}[t^j]\frac{(1+t)^j}{1+t/2} - \frac{(-1)^n}{2}[t^j]\frac{(1+t)^{j+n+1}}{1+t/2}.$$

$$[t^j]\frac{(1+t)^j}{1+t/2} \overset{LIF}{=} [t^j]\left[\frac{1+w}{1+w/2} \;\Big|\; w=t(1+w)\right] = [t^j]\frac{1}{1-t}\cdot\frac{1-t}{1-t/2} = \frac{1}{2^j}.$$

$$\frac{1}{2}[t^j]\frac{(1+t)^{j+n+1}}{1+t/2} \overset{conv}{=} \frac{1}{2}\sum_{k=0}^{j}\binom{j+n+1}{k}\left(-\frac{1}{2}\right)^{j-k} = \frac{(-1)^j}{2^{j+1}}\sum_{k=0}^{j}\binom{j+n+1}{k}(-2)^k.$$

Schwatt gives various other more complicated examples of this, such as formulas for:

$$\sum_{k=0}^{n}(-1)^k\binom{2k+1}{j} \quad\text{and}\quad \sum_{k=0}^{n}(-1)^k\binom{3k-2}{j} \quad\text{etc.}$$

***(1.131)** Gould.

$$\boxed{\sum_{j=k}^{n}\binom{n-a+1}{j-a+1}(1-x)^{n-j}x^j = \sum_{j=k}^{n}\binom{j-a}{k-a}(1-x)^{j-k}x^k}$$

(1.132) Chung.

$$\boxed{\sum_{k=m}^{n-1}\binom{k-1}{m-1}\frac{1}{n-k} = \binom{n-1}{m-1}\sum_{k=m}^{n-1}\frac{1}{k} = \binom{n-1}{m-1}(H_{n-1}-H_{m-1})}$$

$$\sum_{k=m}^{n}\binom{k-1}{m-1}\frac{1}{n-k} = \sum_{k=0}^{n}\binom{-m}{k-m}\frac{(-1)^{k-m}}{n-k} \overset{conv}{=} [t^n]\frac{t^m}{(1-t)^m}\ln\frac{1}{1-t} =$$

$$= [t^{n-m}]\frac{1}{(1-t)^m}\ln\frac{1}{1-t} \overset{1ZV}{=} (H_{m-1+n-m}-H_{m-1})\binom{m-1+n-m}{m-1} = (H_{n-1}-H_{m-1})\binom{n-1}{m-1}.$$

(1.133)

$$\boxed{\sum_{k=0}^{n}\binom{x-k}{n-k}=\binom{x+1}{n}}$$

$$\sum_{k=0}^{n}\binom{x-k}{n-k}\overset{B}{=}[t^n](1+t)^x\left[\frac{1}{1-u}\;\middle|\;u=\frac{t}{1+t}\right]=[t^n](1+t)^{x+1}=\binom{x+1}{n}.$$

***(1.134)** Saalschütz.

$$\boxed{\sum_{k=a}^{n}(-1)^{k-a}\binom{n}{k}\frac{1}{k}=\sum_{k=a}^{n}\binom{k-1}{a-1}\frac{1}{k}}$$

***(1.135)** Paul Bruckman.

$$\boxed{\left(\sum_{k=0}^{n}\binom{-1/2}{k}\right)^2=\binom{-1/2}{n}\sum_{k=0}^{n}\binom{-1/2}{n-k}\frac{2n+1}{2k+1}}$$

2.2 Table 2: summations of the form S:0/1

This table contains 26 identities.

(2.1)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x}{k}^{-1}=\frac{x+1}{x+2}\left(1+(-1)^n\binom{x+1}{n+1}^{-1}\right)}$$

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}^{-1}=\sum_{k=0}^{n}(-1)^k(x+1)\int_0^1 y^k(1-y)^{x-k}\mathrm{d}y=(x+1)\int_0^1(1-y)^x\left(\sum_{k=0}^{n}\left(\frac{-y}{1-y}\right)^k\right)\mathrm{d}y=$$

$$=(x+1)\int_0^1(1-y)^x\left(1-\frac{(-1)^{n+1}y^{n+1}}{(1-y)^{n+1}}\right)\Big/\left(1+\frac{y}{1-y}\right)\mathrm{d}y=$$

$$=(x+1)\int_0^1(1-y)^{x+1}\mathrm{d}y+(x+1)\int_0^1(-1)^n y^{n+1}(1-y)^{x-n}\mathrm{d}y=$$

$$=\frac{x+1}{x+2}\binom{x+1}{0}^{-1}+\frac{x+1}{x+2}(-1)^n\binom{x+1}{n+1}^{-1}=\frac{x+1}{x+2}\left(1+(-1)^n\binom{x+1}{n+1}^{-1}\right).$$

(2.2)

$$\boxed{\sum_{k=0}^{n}\binom{x+k}{k}^{-1}=\frac{x}{x-1}\left(1-\binom{x+n}{n+1}^{-1}\right)}$$

$$\sum_{k=0}^{n}\binom{x+k}{k}^{-1}=\sum_{k=0}^{n}(x+k+1)\int_0^1 y^k(1-y)^x\mathrm{d}y=\int_0^1(1-y)^x\left((x+1)\sum_{k=0}^{n}y^k+\sum_{k=0}^{n}ky^k\right)\mathrm{d}y=$$

$$=(x+1)\int_0^1(1-y)^x\frac{1-y^{n+1}}{1-y}\mathrm{d}y+\int_0^1(1-y)^x\left(\frac{y-y^{n+2}}{(1-y)^2}-(n+1)\frac{y^{n+1}}{1-y}\right)\mathrm{d}y=$$

$$=(x+1)\int_0^1(1-y)^{x-1}\mathrm{d}y-(x+1)\int_0^1 y^{n+1}(1-y)^{x-1}+\int_0^1 y(1-y)^{x-2}\mathrm{d}y-$$

$$-\int_0^1 y^{n-2}(1-y)^{x-2}\mathrm{d}y-(n+1)\int_0^1 y^{n+1}(1-y)^{x-1}\mathrm{d}y=$$

$$=\frac{x+1}{x}\binom{x-1}{0}^{-1}-\frac{x+1}{n+x+1}\binom{x+n}{n+1}^{-1}+\frac{1}{x}\binom{x-1}{1}^{-1}-\frac{1}{x+n+1}\binom{x+n}{n+2}-\frac{n+1}{x+n+1}\binom{x+n}{n+1}=$$

$$=\frac{x}{x-1}-\frac{x}{x-1}\binom{x+n}{n+1}^{-1}=\frac{x}{x-1}\left(1-\binom{x+n}{n+1}^{-1}\right).$$

***(2.3)**

$$\boxed{\sum_{k=p}^{n}\binom{k}{p}^{-1}=\begin{cases}H_p, & p=1\\ \frac{p}{p-1}\left(1-\binom{n}{p-1}\right), & p\neq 1\end{cases}}$$

***(2.4)**

$$\boxed{\sum_{k=0}^{n}x^k\binom{n}{k}^{-1}=(n+1)\left(\frac{x}{1+x}\right)^{n+1}\cdot\sum_{k=1}^{n+1}\frac{1}{k}\cdot\frac{1+x^k}{1+x}\left(\frac{1+x}{x}\right)^k=S_n(x)}$$

***(2.5)**

$$\boxed{\left(1+\frac{1}{x}\right)S_{n+1}(x)=\frac{n+2}{n+1}S_n(x)+x^{n+1}+\frac{1}{x}}$$

***(2.6)**

$$\boxed{\sum_{k=1}^{r}(n-2k)\binom{n}{k}^{-1}=1-(n+1)\binom{n+1}{r+1}^{-1}}$$

(2.7)

$$\boxed{\sum_{k=1}^{n}(-1)^{k-1}\binom{2n}{k}^{-1}=\frac{1}{2(n+1)}+\frac{(-1)^{n-1}}{2}\binom{2n}{n}^{-1}}$$

$$\sum_{k=1}^{n}(-1)^{k-1}(2n+1)\int_0^1 y^k(1-y)^{2n-k}\mathrm{d}y=(2n+1)\int_0^1(1-y)^{2n}\left(\sum_{k=1}^{n}\frac{(-1)^k y^k}{(1-y)^k}\right)\mathrm{d}y=$$

$$=(2n+1)\int_0^1(1-y)^{2n}\frac{y}{1-y}\left(1-\frac{(-1)^n y^n}{(1-y)^n}\right)\Big/\left(1-\frac{-y}{1-y}\right)\mathrm{d}y=$$

$$= (2n+1)\int_0^1 y(1-y)^{2n}\mathrm{d}y + (-1)^{n+1}(2n+1)\int_0^1 y^{n+1}(1-y)^n\mathrm{d}y =$$

$$= \frac{2n+1}{2n+2}\binom{2n+1}{1}^{-1} + (-1)^{n+1}\frac{2n+1}{2n+2}\binom{2n+1}{n+1}^{-1} =$$

$$= \frac{2n+1}{2n+2}\frac{1}{2n+1} + (-1)^{n+1}\frac{2n+1}{2n+2}\frac{n+1}{2n+1}\binom{2n}{n}^{-1} = \frac{1}{2n+2} + \frac{(-1)^{n+1}}{2}\binom{2n}{n}^{-1}.$$

(2.8)

$$\boxed{\sum_{k=1}^{2n-1}(-1)^{k-1}k\binom{2n}{k}^{-1} = \frac{n}{n+1}}$$

$$\sum_{k=1}^{2n-1}(-1)^{k-1}k(2n+1)\int_0^1 y^k(1-y)^{2n-k}\mathrm{d}y = (2n+1)\int_0^1(1-y)^{2n}\left(\sum_{k=1}^{2n-1}(-1)^{k-1}k\left(\frac{y}{1-y}\right)^k\right)\mathrm{d}y =$$

$$= (2n+1)\int_0^1(1-y)^{2n}\left(\left(\frac{y}{1-y} + \frac{(-y)^{2n+1}}{(1-y)^{2n+1}}\right)\Big/\left(1+\frac{y}{1-y}\right)^2 - 2n\left(\frac{-y}{1-y}\right)^{2n}\Big/\left(1+\frac{y}{1-y}\right)\right)\mathrm{d}y =$$

$$= (2n+1)\int_0^1(1-y)^{2n+2}\left(\frac{y}{1-y} - \frac{y^{2n+1}}{(1-y)^{2n+1}}\right)\mathrm{d}y + 2n(2n+1)\int_0^1(1-y)^{2n+1}\frac{y^{2n}}{(1-y)^{2n}}\mathrm{d}y =$$

$$= (2n+1)\int_0^1 y(1-y)^{2n+1}\mathrm{d}y - (2n+1)\int_0^1 y^{2n+1}(1-y)\mathrm{d}y + 2n(2n+1)\int_0^1 y^{2n}(1-y)\mathrm{d}y =$$

$$= \frac{2n+1}{2n+3}\binom{2n+2}{2n+1}^{-1} - \frac{2n+1}{2n+3}\binom{2n+2}{2n+1}^{-1} + \frac{2n(2n+1)}{2n+2}\binom{2n+1}{1} =$$

$$= \frac{n(2n+1)}{(n+1)(2n+1} = \frac{n}{n+1}.$$

(2.9)

$$\boxed{\sum_{k=1}^{n}\frac{4^k}{2k}\binom{2k}{k}^{-1} = 4^n\binom{2n}{n}^{-1} - 1}$$

By integrating (IC0) we obtain the generating function:

$$\mathcal{G}\left(\frac{4^k}{k}\binom{2k}{k}^{-1}\right) = 2\sqrt{\frac{t}{1-t}}\arctan\sqrt{\frac{t}{1-t}};$$

therefore:

$$\sum_{k=1}^{n}\frac{4^k}{2k}\binom{2k}{k}^{-1} = [t^n]\frac{1}{1-t}\sqrt{\frac{t}{1-t}}\arctan\sqrt{\frac{t}{1-t}} =$$

$$= [t^n]\sqrt{\frac{t}{(1-t)^3}}\arctan\sqrt{\frac{t}{1-t}} + \frac{1}{1-t} - \frac{1}{1-t} \stackrel{IC0}{=} 4^n\binom{2n}{n}^{-1} - 1.$$

(2.10)

$$\boxed{\sum_{k=n}^{\infty}\binom{k}{r}^{-1} = \frac{n}{r-1}\binom{n}{r}^{-1}, \qquad r > 1}$$

$$\sum_{k=n}^{\infty}\binom{k}{r}^{-1}=\lim_{m\to\infty}\sum_{k=n}^{m}(k+1)\int_0^1 y^r(1-y)^{k-r}\mathrm{d}y.$$

$$\int_0^1\frac{y^r}{(1-y)^r}\left(\sum_{k=n}^{m}k(1-y)^k+\sum_{k=n}^{m}(1-y)^k\right)\mathrm{d}y=$$

$$=\int_0^1\frac{y^r}{(1-y)^r}\left(\frac{(1-y)-(1-y)^{m+2}}{(1-(1-y))^2}-(m+1)\frac{(1-y)^{m+1}}{1-(1-y)}-\right.$$

$$\left.-\frac{(1-y)-(1-y)^{n+1}}{(1-(1-y))^2}+n\frac{(1-y)^n}{1-(1-y)}+\frac{1-(1-y)^{m+1}}{1-(1-y)}-\frac{1-(1-y)^n}{1-(1-y)}\right)\mathrm{d}y=$$

Four terms annihilate:

$$=-\int_0^1 y^{r-2}(1-y)^{n-r}\mathrm{d}y-(m+1)\int_0^1 y^{r-1}(1-y)^{m-r+1}\mathrm{d}y+\int_0^1 y^{r-2}(1-y)^{n-r+1}\mathrm{d}y+$$

$$+n\int_0^1 y^{r-1}(1-y)^{n-r}\mathrm{d}y-\int_0^1 y^{r-1}(1-y)^{m-r+1}\mathrm{d}y+\int_0^1 y^{r-1}(1-y)^{n-r}\mathrm{d}y=$$

$$=-\frac{1}{m+1}\binom{m}{r-1}^{-1}-\frac{m+2}{m+1}\binom{m}{r-1}^{-1}+\frac{1}{n}\binom{n-1}{r-2}^{-1}+\frac{n+1}{n}\binom{n-1}{r-1}^{-1}=$$

The two terms in m tend to 0 when $m\to\infty$, and so:

$$=\frac{n-r+1}{r(r-1}\binom{n}{r}^{-1}+\frac{n+1}{r}\binom{n}{r}^{-1}=\frac{nr}{r(r-1}\binom{n}{r}^{-1}=\frac{n}{r-1}\binom{n}{r}^{-1}.$$

***(2.11)**

$$\sum_{k=1}^{\infty}\frac{1}{k}\binom{k+n}{k}=\frac{1}{n}\quad(n\geq 1)$$

***(2.12)**

$$\sum_{k=1}^{\infty}\frac{1}{k^2}\binom{k+n}{k}=\frac{\pi^2}{6}-\sum_{k=1}^{n}\frac{1}{k^2}\quad(n\geq 1)$$

***(2.13)**

$$\sum_{k=0}^{\infty}x^{k+1}\binom{z}{k}^{-1}=\sum_{k=0}^{\infty}\frac{z+1}{z+1-k}\left(\frac{x}{x+1}\right)^{k+1}\quad \Re(x)<\frac{1}{2},z\neq -1,0,1,2,\ldots$$

***(2.14)**

$$\sum_{k=0}^{\infty}x^{k+1}\binom{z+k}{k}^{-1}=\sum_{k=0}^{\infty}(-1)^k\frac{z}{z+k}\left(\frac{x}{1-x}\right)^{k+1}\quad \Re(x)<\frac{1}{2},z\neq 0,-1,-2,\ldots$$

***(2.15)**

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \binom{x+k}{k}^{-1} = x \sum_{k=0}^{\infty} \frac{k^k}{k!} \frac{e^{-k}}{x+k}$$

***(2.16)**

$$\sum_{k=0}^{\infty} \frac{1}{k!} \binom{x+k}{k}^{-1} = x \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{x+k}$$

***(2.17)**

$$\sum_{k=1}^{n} \frac{1}{k^2} \left(1 - \binom{r+k}{k}^{-1}\right) = \sum_{k=1}^{r} \frac{1}{k^2} \left(1 - \binom{n+k}{k}\right)$$

***(2.18)**

$$\sum_{k=1}^{2n-1} (-1)^{k-1} H_k \binom{2n}{k}^{-1} = \frac{n}{2(n+1)^2} + \frac{H_{2n}}{2(n+1)}$$

***(2.19)** - Ljunggren:

$$\sum_{k=0}^{2n-1} \frac{(-1)^k}{k+1} \binom{2n}{k}^{-1} = (2n+1) \sum_{k=1}^{n} \frac{1}{(k+n)^2} = (2n+1)\left(H_{2n}^{(2)} - H_n^{(2)}\right)$$

(2.20)

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \binom{2k}{k}^{-1} = \frac{\pi^2}{18}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \binom{2k}{k}^{-1} = \left[2\left(\arctan \sqrt{\frac{4t}{1-4t}}\right)^2 \;\Big|\; t=1 \right] = 2 \cdot \left(\arctan \frac{\sqrt{3}}{3}\right) = 2 \cdot \left(\frac{\pi}{6}\right)^2 = \frac{\pi^2}{18}.$$

***(2.21)**

$$\sum_{k=1}^{\infty} \frac{2^{2k-1}}{k(2k+1)} \binom{2k}{k}^{-1} x^{2k+1} = x - \sqrt{1-x^2}\arcsin(x) \quad (-1 \le x \le 1)$$

***(2.22)**

$$\sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1}}{k(2k+1)} \binom{2k}{k}^{-1} x^{2k+1} = x - \sqrt{1+x^2}\ln(x + \sqrt{1+x^2} \quad (-1 \le x \le 1)$$

***(2.23)**

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \binom{2k}{k}^{-1} = 4 \sum_{k=0}^{\infty} \frac{(\sqrt{5}-2)^{2k+1}}{(2k+1)^2} = \frac{\pi^2}{6} - 3\ln(\phi)^2$$

***(2.24)** - E. H. Clarke:

$$\sum_{k=1}^{\infty} \binom{kn}{n}^{-1} = n \int_0^1 \frac{(1-x)^{n-1}}{1-x^n} \mathrm{d}x = n! \sum_{k=0}^{\infty} \frac{1}{(kn+1)(kn+2)\cdots(kn+n)} =$$

$$= \sum_{k=1}^{n-1} (-\omega_k)(1-\omega_k)^{n-1} \ln\left(\frac{1-\omega_k}{-\omega_k}\right) \qquad \omega_k = \exp\left(\frac{ik\pi}{n}\right)$$

***(2.25)** - Tor B. Staver. Also special case of (2.4):

$$\sum_{k=0}^{n} \binom{n}{k}^{-1} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}$$

***(2.26)** - Natural extension of (2.2); Gould:

$$\sum_{k=0}^{n} \binom{x+k}{r}^{-1} = \frac{r}{r-1} \left(\binom{x-1}{r-1}^{-1} - \binom{n+x}{r-1}^{-1} \right)$$

2.3 Table 3: summations of the form S:2/0

This table contains 183 identities.

(3.1) - Vandermonde convolution:

$$\sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}$$

$$\sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k} \stackrel{conv}{=} [t^n](1+t)^x (1+t)^y = [t^n](1+t)^{x+y} \stackrel{BC1}{=} \binom{x+y}{n}$$

(3.2)

$$\boxed{\sum_{k=0}^{n}\binom{x+k}{k}\binom{y+n-k}{n-k}=\binom{x+y+n+1}{n}}$$

$$\sum_{k=0}^{n}\binom{x+k}{k}\binom{y+n-k}{n-k}\overset{BC4,conv}{=}[t^n]\frac{1}{(1-t)^{x+1}}\cdot\frac{1}{(1-t)^{y+1}}=[t^n]\frac{1}{(1-t)^{x+y+2}}\overset{BC4}{=}$$
$$=\binom{-x-y-2}{n}(-1)^n=\binom{x+y+n+1}{n}.$$

(3.3) - *The constants r and s should be positive integers*:

$$\boxed{\sum_{k=r}^{n-s}\binom{k}{r}\binom{n-k}{s}=\binom{n+1}{r+s+1}}$$

$$\sum_{k=r}^{n-s}\binom{k}{r}\binom{n-k}{s}\overset{BC5,conv}{=}[t^n]\frac{t^r}{(1-t)^{r+1}}\cdot\frac{t^s}{(1-t)^{s+1}}=[t^{n+1}]\frac{t^{r+s+1}}{(1-t)^{r+s+2}}\overset{BC5}{=}$$
$$=\binom{-r-s-2}{n-r-s}(-1)^{n-r-s}=\binom{n+1}{r+s+1}.$$

(3.4) - *The constant x should be a positive integer*:

$$\boxed{\sum_{k=0}^{n}\binom{n}{k}\binom{x+y-n}{x-k}=\binom{x+y}{x}}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{x+y-n}{x-k}\overset{B}{=}[t^x](1+t)^{x+y-n}\left[(1+u)^n \;\middle|\; u=t\right]=[t^x](1+t)^{x+y}=\binom{x+y}{x}.$$

(3.5) - (NN)

$$\boxed{\sum_{k=0}^{n}\binom{x}{k}\binom{y}{k}=\sum_{k=0}^{n}(-1)^k\binom{n-x}{k}\binom{y+n-k}{n}}$$

(3.6) - *(NN)* *The second sum is symmetric with respect to $-n\le k\le n$ and therefore we can equivalently compute the complete sum*:

$$\boxed{\sum_{k=0}^{n}\binom{x}{k}\binom{x}{2n-k}=\sum_{k=0}^{n}\binom{x}{n-k}\binom{x}{n+k}=\frac{1}{2}\left(\binom{2x}{2n}+\binom{x}{n}^2\right)}$$

(1) → (2) by the change of variable $k\to n-k$.

$$(2)\qquad \sum_{k=-n}^{n}\binom{x}{n-k}\binom{x}{n+k}\overset{B}{=}[t^n](1+t)^x\left[\frac{(1+u)^x}{u^n}\;\middle|\; u=t\right]=[t^{2n}](1+t)^{2x}=\binom{2x}{2n}.$$

We now should take one half of this value and add one half of the middle element ($k=0$).

(3.7) - *(NN)* *The second sum is symmetric with respect to $-n \le k \le n-1$ and therefore we can equivalently compute the complete sum:*

$$\sum_{k=0}^{n-1}\binom{x}{k}\binom{x}{2n-1-k} = \sum_{k=0}^{n-1}\binom{x}{n-1-k}\binom{x}{n+k} = \frac{1}{2}\binom{2x}{2n-1}$$

(1) → (2) by the change of variable $k \to n-1-k$.

$$(2)\qquad \sum_{k=-n}^{n-1}\binom{x}{n-1-k}\binom{x}{n+k} \stackrel{B}{=} [t^{n-1}](1+t)^x\left[\frac{(1+u)^x}{u^n}\;\middle|\; u=t\right] = [t^{2n-1}](1+t)^{2x} = \binom{2x}{2n-1}.$$

Now, we should take one half of this value. There is no central element.

(3.8)

$$\sum_{k=0}^{n/2}\binom{x}{2k}\binom{x}{n-2k} = \frac{1}{2}\binom{2x}{n} + \frac{(-1)^{n/2}}{2}\binom{x}{n/2}$$

$$\sum_{k=0}^{n/2}\binom{x}{2k}\binom{x}{n-2k} \stackrel{B}{=} [t^n](1+t)^x\left[\frac{(1+\sqrt{u})^x+(1-\sqrt{u})^x}{2}\;\middle|\; u=t^2\right] =$$

$$= [t^n]\frac{(1+t)^{2x}+(1-t^2)^x}{2} = \frac{1}{2}\binom{2x}{n} + \frac{(-1)^{n/2}}{2}\binom{x}{n/2}.$$

(3.9)

$$\sum_{k=0}^{n}\binom{x}{2k}\binom{x}{2n-2k} = \frac{1}{2}\binom{2x}{2n} + \frac{(-1)^n}{2}\binom{x}{n}$$

$$\sum_{k=0}^{n}\binom{x}{2k}\binom{x}{2n-2k} \stackrel{B}{=} [t^{2n}](1+t)^x\left[\frac{(1+\sqrt{u})^x+(1-\sqrt{u})^x}{2}\;\middle|\; u=t^2\right] =$$

$$= [t^{2n}]\frac{(1+t)^{2x}+(1-t^2)^x}{2} = \frac{1}{2}\binom{2x}{2n} + \frac{(-1)^n}{2}\binom{x}{n}.$$

(3.10)

$$\sum_{k=0}^{(n-1)/2}\binom{x}{2k+1}\binom{x}{n-2k-1} = \frac{1}{2}\binom{2x}{n} + \frac{(-1)^{n/2}}{2}\binom{x}{n/2}$$

$$\sum_{k=0}^{(n-1)/2}\binom{x}{2k+1}\binom{x}{n-2k-1} \stackrel{B}{=} [t^{n-1}](1+t)^x\left[\frac{(1+\sqrt{u})^x-(1-\sqrt{u})^x}{2\sqrt{u}}\;\middle|\; u=t^2\right] =$$

$$= [t^n]\frac{(1+t)^{2x}-(1-t^2)^x}{2} = \frac{1}{2}\binom{2x}{n} - \frac{(-1)^{n/2}}{2}\binom{x}{n/2}.$$

(3.11)

$$\sum_{k=0}^{n-1}\binom{x}{2k+1}\binom{x}{2n-2k-1} = \frac{1}{2}\binom{2x}{2n} - \frac{(-1)^n}{2}\binom{x}{n}$$

$$\sum_{k=0}^{n-1}\binom{x}{2k+1}\binom{x}{2n-2k-1} \overset{B}{=} [t^{2n-1}](1+t)^x\left[\frac{(1+\sqrt{u})^x-(1-\sqrt{u})^x}{2\sqrt{u}}\;\Big|\; u=t^2\right]=$$

$$=[t^{2n}]\frac{(1+t)^{2x}-(1-t^2)^x}{2}=\frac{1}{2}\binom{2x}{2n}-\frac{(-1)^n}{2}\binom{x}{n}$$

(3.12)

$$\sum_{k=0}^{n/2}\binom{x}{2k}\binom{2n-x}{n-2k}=\frac{1}{2}\left(\binom{2n}{n}+(-4)^n\binom{(x-1)/2}{n}\right)$$

$$\sum_{k=0}^{n/2}\binom{x}{2k}\binom{2n-x}{n-2k}\overset{B}{=}\frac{1}{2}[t^n](1+t)^{2n-x}\left[(1+\sqrt{u})^x+(1-\sqrt{u})^x\;\Big|\; u=t^2\right]=$$

$$=\frac{1}{2}[t^n](1+t)^{2n-x}\left((1+t)^x+(1-t)^x\right)=\frac{1}{2}[t^n](1+t)^{2n}+\frac{1}{2}[t^n](1+t)^{2n}\left(\frac{1-t}{1+t}\right)^x.$$

We only need to extract the second coefficient:

$$[t^n](1+t)^{2n}\left(\frac{1-t}{1+t}\right)^x=[t^n]\left[\left(\frac{1-w}{1+w}\right)^x\cdot\frac{1+w}{1-w}\;\Big|\; w=t(1+w)^2\right]=$$

$$=[t^n]\left[\left(\frac{1-w}{1+w}\right)^{x-1}\;\Big|\; w=\frac{1-2t-\sqrt{1-4t}}{2t}\right]=[t^n](1-4t)^{(x-1)/2}=\binom{(x-1)/2}{n}(-4)^n.$$

(3.13)

$$\sum_{k=0}^{(n-1)/2}\binom{x}{2k+1}\binom{2n-x}{n-2k-1}=\frac{1}{2}\left(\binom{2n}{n}-(-4)^n\binom{(x-1)/2}{n}\right)$$

$$\sum_{k=0}^{(n-1)/2}\binom{x}{2k+1}\binom{2n-x}{n-2k-1}\overset{B}{=}[t^{n-1}](1+t)^{2n-x}\left[\frac{(1+\sqrt{u})^x-(1-\sqrt{u})^x}{2\sqrt{u}}\;\Big|\; u=t^2\right]=$$

$$=\frac{1}{2}[t^n](1+t)^{2n-x}\left((1+t)^x-(1-t)^x\right)=\frac{1}{2}[t^n](1+t)^{2n}-\frac{1}{2}[t^n](1+t)^{2n}\left(\frac{1-t}{1+t}\right)^x.$$

The coefficients are extracted as in (3.12).

(3.14) - *(NN)* - Erik Sparre Andersen:

$$\sum_{k=0}^{r}\binom{x}{k}\binom{-x}{n-k}=-\frac{x-r}{n}\binom{x}{r}\binom{-x-1}{n-r-1}=\frac{n-r}{n}\binom{x-1}{r}\binom{-x}{n-r}$$

$$=-\sum_{k=r+1}^{n}\binom{x}{k}\binom{-x}{n-k}=\left[(-1)^{n+r}\binom{x+n-r-1}{n}\binom{n-1}{r}\right]\qquad(n\geq 1,\ 0\leq r\leq n)$$

(3.15) - *(NN)* - Erik Sparre Andersen:

$$\sum_{k=0}^{r}\binom{x}{k}\binom{1-x}{n-k}=\frac{(n-1)(1-x)-r}{n(n-1)}\binom{x-1}{r}\binom{-x}{n-r-1}\qquad(n\geq 2,\ 0<r\leq n-1)$$

(3.16) - *(NN)*

$$\sum_{k=0}^{n} \binom{-1/2}{k}\binom{1/2}{k} = \binom{-1/2}{n}\binom{-3/2}{n} = (2n+1)\binom{-1/2}{n}^2$$

(3.17)

$$\sum_{k=0}^{n} \binom{n}{k}\binom{x}{k} z^k = \sum_{k=0}^{n} \binom{n}{k}\binom{x+n-k}{n}(z-1)^k$$

$$(1) \quad \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k} z^k \overset{E}{=} [t^n]\frac{1}{1-t}\left[(1+zu)^x \;\middle|\; u = \frac{t}{1-t}\right] = [t^n]\frac{(1+(z-1)t)^x}{(1-t)^{x+1}}.$$

$$(2) \quad \sum_{k=0}^{n} \binom{n}{n-k}\binom{x+k}{n}(z-1)^{n-k} \overset{B^*}{=} (z-1)^n[t^n](1+t)^x\left[\left(1+\frac{u}{z-1}\right)^n \;\middle|\; u = 1+t\right] =$$

$$= (z-1)^n[t^n](1+t)^x\frac{(z+t)^n}{(z-1)^n} = z^n[t^n](1+t)^x(1+t/z)^n = [t^n](1+zt)^x(1+t)^n \overset{LIF}{=}$$

$$= [t^n]\left[\frac{(1+zw)^x}{1-w/(1+w)} \;\middle|\; w = t(1+w)\right] = [t^n]\left(1+\frac{zt}{1-t}\right)^x\left(1+\frac{t}{1-t}\right) = [t^n]\frac{(1+(z-1)t)^x}{(1-t)^{x+1}}.$$

(3.18) Ljunggren. *Egorychev (2.1.2) Ex. 5. Formula (2.8) p. 48*:

$$\sum_{k=0}^{n} \binom{n}{k}\binom{z+k}{k}(x-y)^{n-k}y^k = \sum_{k=0}^{n} \binom{n}{k}\binom{z}{k} x^{n-k}y^k$$

$$(1) \quad \sum_{k=0}^{n} \binom{n}{k}\binom{z+k}{k}(x-y)^{n-k}y^k \overset{E}{=} [t^n]\frac{1}{1-(x-y)t}\left[\frac{1}{(1-yu)^{z+1}} \;\middle|\; u = \frac{t}{1-(x-y)t}\right] =$$

$$= [t^n]\frac{(1-(x-y)t)^z}{(1-xt)^{z+1}}.$$

$$(2) \quad \sum_{k=0}^{n} \binom{n}{k}\binom{z}{k} x^{n-k}y^k \overset{E}{=} [t^n]\frac{1}{1-xt}\left[(1+yu)^z \;\middle|\; u = \frac{1}{1-xt}\right] = [t^n]\frac{(1-(x-y)t)^z}{(1-xt)^{z+1}}.$$

(3.19)

$$\sum_{k=0}^{n} \binom{n}{k}\binom{x}{k} 2^k = \sum_{k=0}^{n} \binom{n}{k}\binom{x+k}{n}$$

$$(1) \quad \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k} 2^k \overset{E,BC1}{=} [t^n]\frac{1}{1-t}\left[(1+2u)^x \;\middle|\; u = \frac{t}{1-t}\right] = [t^n]\frac{(1+t)^x}{(1-t)^{x+1}}.$$

$$(2) \quad \sum_{k=0}^{n} \binom{x+k}{n}\binom{n}{k} \overset{BX}{=} \sum_{k=0}^{n} \binom{x+k}{k}\binom{x}{n-k} \overset{conv}{=} [t^n]\frac{(1+t)^x}{(1-t)^{x+1}}.$$

(3.20)

$$\boxed{\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k+r}=\binom{n+x}{n+r}}$$

$$\sum_{k=0}^{n}\binom{x}{k+r}\binom{n}{k}\stackrel{A}{=}[t^x]\frac{t^r}{(1-t)^{r+1}}\left[(1+u)^n \;\middle|\; u=\frac{t}{1-t}\right]=$$

$$=[t^{x-r}]\frac{1}{(1-t)^{r+n+1}}=\binom{-r-n-1}{x-r}(-1)^{x-r}=\binom{n+x}{x-r}=\binom{n+x}{n+r}$$

(3.21) - Karl Goldberg:

$$\boxed{\sum_{k=0}^{n}\binom{x}{k}\binom{y+k}{n-k}4^k=\sum_{k=0}^{n}\binom{2x}{k}\binom{y}{n-k}2^k=\sum_{k=0}^{n}\binom{2x}{k}\binom{y+2x-k}{n-k}}$$

(1) $$\sum_{k=0}^{n}\binom{x}{k}\binom{y+k}{n-k}4^k\stackrel{B}{=}[t^n](1+t)^y\left[(1+4u)^x \;\middle|\; u=t(1+t)\right]=[t^n](1+t)^y(1+2t)^{2x}.$$

(2) $$\sum_{k=0}^{n}\binom{2x}{k}\binom{y}{n-k}2^k\stackrel{B}{=}[t^n](1+t)^y\left[(1+2u)^{2x} \;\middle|\; u=t\right]=[t^n](1+t)^y(1+2t)^{2x}.$$

(3) $$\sum_{k=0}^{n}\binom{2x}{k}\binom{y+2x-k}{n-k}\stackrel{B}{=}[t^n](1+t)^{y+2x}\left[(1+u)^{2x} \;\middle|\; u=\frac{t}{1+t}\right]=$$

$$=[t^n](1+t)^{y+2x}\frac{(1+2t)^{2x}}{(1+t)^{2x}}=[t^n](1+t)^y(1+2t)^{2x}.$$

(3.22)

$$\boxed{\sum_{k=0}^{n/2}\binom{x}{k}\binom{x-k}{n-2k}4^{n-k}=\sum_{k=n/2}^{n}\binom{x}{k}\binom{k}{n-k}4^k=2^n\binom{2x}{n}}$$

(1) $$4^n\sum_{k=0}^{n/2}\binom{x}{k}\binom{x-k}{n-2k}\frac{1}{4^k}\stackrel{B}{=}4^n[t^n](1+t)^x\left[\left(1+\frac{u}{4}\right)^x \;\middle|\; u=\frac{t^2}{1+t}\right]=$$

$$=4^n[t^n](1+t)^x\frac{1}{4^x}\left(\frac{4+4t+t^2}{1+t}\right)^x=4^n[t^n](1+t)^x\frac{(1+t/2)^{2x}}{(1+t)^x}=4^n[t^n](1+t/2)^{2x}=2^n\binom{2x}{n}.$$

(2) $$\sum_{k=n/2}^{n}\binom{x}{k}\binom{k}{n-k}4^k\stackrel{B}{=}[t^n]\left[(1+4u)^x \;\middle|\; u=t(1+t)\right]=[t^n](1+2t)^{2x}=2^n\binom{2x}{n}.$$

(3.23)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{n+x}{n-k}\binom{k+x+1}{k}=\delta_{n,0}-\delta_{n,1}}$$

$$\sum_{k=0}^{n}\binom{n+x}{n-k}\binom{k+x+1}{k}(-1)^k\stackrel{B,BC2}{=}[t^n](1+t)^{n+x}\left[\frac{1}{(1+u)^{x+2}} \;\middle|\; u=t\right]=$$

$$= [t^n](1+t)^{n-2} = \delta_{n,0} - \delta_{n,1}$$

(3.24)

$$\sum_{k=0}^{n} \binom{x}{2k} \binom{x+n-k-1}{n-k} = \binom{x+2n-1}{2n}$$

$$\sum_{k=0}^{n} \binom{x}{2k} \binom{x+n-k-1}{n-k} \overset{conv}{=} [t^n] \frac{(1+\sqrt{t})^x + (1-\sqrt{t})^x}{2} \cdot \frac{1}{(1-t)^x} =$$

$$= [t^n] \frac{1}{2} \left(\frac{1}{(1-\sqrt{t})^x} + \frac{1}{(1+\sqrt{t})^x} \right) = [t^{2n}] \frac{1}{(1-t)^x} = \binom{x+2n-1}{2n}$$

(3.25) - *Analogous to the previous sum (3.24); the bisection with odd elements is considered instead*:

$$\sum_{k=0}^{n} \binom{x}{2k+1} \binom{x+n-k-1}{n-k} = \binom{x+2n}{2n+1}$$

(3.26)

$$\sum_{k=0}^{n} \binom{2x}{2k} \binom{x-k}{n-k} = \frac{x}{x+n} \binom{x+n}{2n} 4^n = \frac{4^n}{(2n)!} \prod_{k=0}^{n-1} (x^2 - k^2)$$

$$S = \sum_{k=0}^{n} \binom{2x}{2k} \binom{x-k}{n-k} \overset{B}{=} [t^n](1+t)^x \left[\frac{(1+\sqrt{u})^{2x} + (1-\sqrt{u})^{2x}}{2} \;\middle|\; u = \frac{t}{1+t} \right] =$$

$$= \frac{1}{2} [t^n] \left((1+2t+2\sqrt{t(1+t)})^x + (1-2t-2\sqrt{t(1+t)})^x \right).$$

By setting $1+2t+2\sqrt{t(1+t)} = 1+2\sqrt{y}$ *we find* $y = t(1+2\sqrt{y})$*. The two expressions give the same value, so the value of the first one is our final result*:

$$S = [t^n] \left[(1+2\sqrt{y})^x \;\middle|\; y = t(1+2\sqrt{y}) \right] = \frac{1}{n} [t^n] x \frac{(1+2\sqrt{t})^{x-1}}{\sqrt{t}} (1+2\sqrt{t})^n =$$

$$= \frac{x}{n} [w^{2n-1}] (1+2w)^{x+n-1} = \frac{x}{n} 2^{2n-1} \binom{x+n-1}{2n-1} = 4^n \frac{x}{x+n} \binom{x+n}{2n}.$$

(3.27)

$$\sum_{k=0}^{n} \binom{2x+1}{2k+1} \binom{x-k}{n-k} = \frac{2x+1}{2n+1} \binom{x+n}{2n} 4^n = \frac{2x+1}{(2n+1)!} \prod_{k=0}^{n-1} ((2x+1)^2 - (2k+1)^2)$$

$$S = \sum_{k=0}^{n} \binom{2x+1}{2k+1} \binom{x-k}{n-k} = [t^n](1+t)^x \left[\frac{(1+\sqrt{u})^{2x+1} - (1-\sqrt{u})^{2x+1}}{2\sqrt{u}} \;\middle|\; u = \frac{t}{1+t} \right] =$$

$$= [t^n] \frac{(1+t)^{x+1/2}}{2\sqrt{t}} \left(\frac{(1+2t+2\sqrt{t(1+t)})^{x+1/2} - (1+2t-2\sqrt{t(1+t)})^{x+1/2}}{(1+t)^{x+1/2}} \right) =$$

$$= \frac{1}{2} [t^n] \left(\sqrt{\frac{1+t}{t}} + 1 \right) \left(1+2t+2\sqrt{t(1+t)} \right)^x - \frac{1}{2} [t^n] \left(\sqrt{\frac{1+t}{t}} + 1 \right) \left(1+2t-2\sqrt{t(1+t)} \right)^x.$$

By setting $1+2t+2\sqrt{t(1+t)} = 1+2\sqrt{y}$ *we find* $y = t(1+2\sqrt{y})$*. The two expressions give the same value, so the value of the first one is our final result:*

$$\sqrt{\frac{1+t}{t}}+1 \to \sqrt{\left(1+\frac{y}{1+2\sqrt{y}}\right)\frac{1+2\sqrt{y}}{y}}+1 = \frac{1+\sqrt{y}}{\sqrt{y}}+1 = \frac{1+2\sqrt{y}}{\sqrt{y}}.$$

$$S = [t^n]\left[\frac{1+2\sqrt{y}}{\sqrt{y}}(1+2\sqrt{y})^x \;\middle|\; y = t(1+2\sqrt{y})\right] = \frac{1}{n}[t^{n-1}]\frac{x2\sqrt{t}-1}{2t\sqrt{t}}(1+2\sqrt{t})^{x+n} =$$

$$= \frac{x}{n}[w^{2n}](1+2w)^{x+n} - \frac{1}{2n}[w^{2n+1}](1+2w)^{x+n} = \frac{x4^n}{n}\binom{x+n}{2n} - \frac{2\cdot 4^n}{2n}\binom{x+n}{2n+1} =$$

$$= 4^n\binom{x+n}{2n}\left(\frac{x}{n}-\frac{x-n}{n(2n+1)}\right) = 4^n\frac{2x+1}{2n+1}\binom{x+n}{2n}.$$

(3.28)

$$\boxed{\sum_{k=0}^{n}\binom{n}{k}\binom{n+2x}{k+x} = \binom{2x+2n}{x+n}}$$

$$\sum_{k=0}^{n}\binom{n+2x}{x+n-k}\binom{n}{k} \overset{E}{=} [t^n]\frac{1}{1-t}\left[\frac{(1+u)^{n+2x}}{u^x} \;\middle|\; u = \frac{t}{1-t}\right] = [t^n]\frac{1}{1-t}\cdot\frac{1}{(1-t)^{n+2x}}\cdot\frac{(1-t)^x}{t^x} =$$

$$= [t^{n+x}]\frac{1}{(1-t)^{n+x+1}} = \binom{-n-x-1}{n+x}(-1)^{n+x} = \binom{2x+2n}{x+n}.$$

(3.29)

$$\boxed{\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k-r} = \binom{n+x}{n-r}}$$

$$\sum_{k=0}^{n}\binom{x}{k-r}\binom{n}{k} \overset{E}{=} [t^n]\frac{1}{1-t}\left[u^r(1+u)^x \;\middle|\; u = \frac{t}{1-t}\right] = [t^n]\frac{t^r}{(1-t)^{x+r+1}} =$$

$$= \binom{-x-r-1}{n-r}(-1)^{n-r} = \binom{n+x}{n-r}$$

(3.30)

$$\boxed{\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}k = n\binom{x+n-1}{n}}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}k = \frac{1}{x}\sum_{k}\binom{n}{k}\binom{x-1}{k-1} \overset{E}{=} \frac{1}{x}[t^n]\frac{1}{1-t}\left[u(1+u)^{x-1} \;\middle|\; u = \frac{t}{1-t}\right] =$$

$$= \frac{1}{x}[t^n]\frac{t}{(1-t)^2}\cdot\frac{1}{(1-t)^{x-1}} = \frac{1}{x}[t^{n-1}]\frac{1}{(1-t)^{x+1}} = \frac{1}{x}\binom{-x-1}{n-1}(-1)^{n-1} = n\binom{x+n-1}{n}$$

(3.31)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{y}{n-k} = \sum_{k=0}^{n/2}(-1)^k\binom{x}{k}\binom{y-x}{n-2k}}$$

$$(1)\quad \sum_{k=0}^{n}\binom{y}{n-k}\binom{x}{k}(-1)^k \stackrel{B}{=} [t^n](1+t)^y\left[(1-u)^x \;\middle|\; u=t\right] = [t^n](1+t)^y(1-t)^x.$$

$$(2)\quad \sum_{k=0}^{n/2}\binom{y-x}{n-2k}\binom{x}{k}(-1)^k \stackrel{B}{=} [t^n](1+t)^{y-x}\left[(1-u)^x \;\middle|\; u=t^2\right] = [t^n](1+t)^y(1-t)^x.$$

(3.32) - *Special case of (3.31) with $y = x$; the result is 0 if n is odd*:

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{x}{n-k} = (-1)^{n/2}\binom{x}{n/2} = \binom{x}{n}\frac{2^n(x-n)!\sqrt{\pi}}{(x-n/2)!(-n/2-1/2)!}}$$

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{x}{n-k} \stackrel{conv}{=} [t^n](1-t^2)^x = (-1)^{n/2}\binom{x}{n/2}.$$

(3.33) - *In this case, for n odd the result is 0*:

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{k}{r}\binom{n-k}{r} = (-1)^r\binom{n/2}{r}}$$

$$\sum_{k=0}^{n}(-1)^k\binom{k}{r}\binom{n-k}{r} \stackrel{conv,BC3}{=} [t^n]\frac{(-1)^r t^r}{(1+t)^{r+1}}\frac{t^r}{(1-t)^{r+1}} =$$

$$= (-1)^r[t^n]\frac{t^{2r}}{(1-t^2)^{r+1}} = (-1)^r[t^{n/2}]\frac{t^r}{(1-t)^{r+1}} = (-1)^r\binom{-r-1}{n/2-r}(-1)^{n/2-r} = (-1)^r\binom{n/2}{r}$$

(3.34)

$$\boxed{\sum_{k=0}^{2n}(-1)^k\binom{x}{k}\binom{x}{2n-k} = (-1)^n\binom{x}{n}}$$

$$\sum_{k=0}^{2n}\binom{x}{2n-k}\binom{x}{k}(-1)^k \stackrel{B}{=} [t^{2n}](1+t)^x\left[(1-u)^x \;\middle|\; u=t\right] =$$

$$= [t^{2n}](1-t^2)^x = [t^n](1-t)^x = \binom{x}{n}(-1)^n.$$

(3.35) - *(**NN**) The first part of the previous identity; the central element is always present:*

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{x}{2n-k} = \frac{(-1)^n}{2}\left(\binom{x}{n}+\binom{x}{n}^2\right)}$$

(3.36)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x+k}{k}\binom{x+n-k}{n-k} = \binom{x+n/2}{n/2}}$$

$$\sum_{k=0}^{n}(-1)^k\binom{x+k}{k}\binom{x+n-k}{n-k} \stackrel{conv,BC2}{=} [t^n]\frac{1}{(1+t)^{x+1}}\cdot\frac{1}{(1-t)^{x+1}} =$$

$$= [t^n]\frac{1}{(1-t^2)^{x+1}} = \binom{-x-1}{n/2}(-1)^{n/2} = \binom{x+n/2}{n/2}.$$

(3.37) - ***(NN)*** *This is just (3.35) with the change of variable $k \to n-k$:*

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x}{n-k}\binom{x}{n+k} = \frac{(-1)^n}{2}\left(\binom{x}{n}+\binom{x}{n}^2\right)}$$

(3.38) - K. v. Szily

$$\boxed{\sum_{k=-n}^{n}(-1)^k\binom{2n}{n-k}\binom{2r}{r-k} = \binom{2n}{n}\binom{2r}{r}\binom{n+r}{n}^{-1} = \frac{(2n)!(2r)!}{(n+r)!n!r!}}$$

$$S = \sum_{k=-n}^{n}(-1)^k\binom{2n}{n-k}\binom{2r}{r-k} \stackrel{B}{=} [t^n](1+t)^{2n}\left[\frac{(-1)^r(1-u)^{2r}}{u^r}\;\middle|\; u=t\right] =$$

$$= (-1)^r[t^n](1+t)^{2n}\frac{(1-t)^{2r}}{t^r} = (-1)^r[t^n]\left[\frac{(1-w)^{2r}}{w^r}\cdot\frac{1+w}{1-w}\;\middle|\; w=t(1+w)^2\right].$$

$$\frac{(1-w)^2}{w} = \frac{1-4t}{t}; \qquad \frac{1+w}{1-w} = \frac{1}{\sqrt{1-4t}}.$$

$$S = (-1)^r[t^n](1-4t)^{r-1/2} = (-1)^r\binom{r-1/2}{n+r}(-4)^{n+r} = (-1)^n\binom{r-1/2}{n+r}4^{n+r}.$$

$$\binom{r-1/2}{k+r} = \frac{(r-1/2)(r-3/2)\cdots(-n+1/2)}{(n+r)!} = \frac{(2r-1)(2r-3)\cdots(-2n+1)}{2^{n+r}(n+r)!} = \frac{(2r)!(2n)!(-1)^n}{4^{n+r}(n+r)!}.$$

$$S = \frac{(2r)!(2n)!}{r!n!(n+r)!} = \binom{2r}{r}\binom{2n}{n}\binom{n+r}{n}^{-1}.$$

(3.39)

$$\boxed{\sum_{k=0}^{\infty}(-1)^k\binom{x}{k}\binom{-x}{k} = \frac{\sin(\pi x)}{\pi}\int_0^1 u^{x-1}\left(\frac{1+u}{1-u}\right)^x \mathrm{d}u}$$

(3.40)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{-x}{k}\binom{x}{n-k} = \sum_{k=0}^{n-1}\binom{x}{k+1}\binom{n-1}{k}2^{k+1}}$$

$$(1) \quad \sum_{k=0}^{n}(-1)^k\binom{-x}{k}\binom{x}{n-k} \stackrel{conv}{=} [t^n]\frac{(1+t)^x}{(1-t)^x}.$$

$$(2) \quad \sum_{k=0}^{n-1}\binom{n-1}{k}\binom{x}{k+1}2^{k+1} \stackrel{A}{=} [t^{n-1}]\frac{1}{1-t}\left[\frac{(1+2u)^x}{u}\;\middle|\; u=\frac{t}{1-t}\right] =$$

$$= [t^{n-1}]\frac{(1+t)^x}{t(1-t)^x} = [t^n]\frac{(1+t)^x}{(1-t)^x}.$$

(3.41)

$$\boxed{\sum_{k=0}^{n/2}(-1)^k\binom{x}{k}\binom{-x}{n-2k} = (-1)^n\binom{x}{n}}$$

$$\sum_{k=0}^{n/2}\binom{-x}{n-2k}\binom{x}{k}(-1)^k \stackrel{B}{=} [t^n](1+t)^{-x}\left[(1-u)^x \;\middle|\; u=t^2\right] = [t^n]\frac{(1+t)^x(1-t)^x}{(1+t)^x} = (-1)^n\binom{x}{n}.$$

(3.42)

$$\boxed{\begin{gathered}\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{2n-x}{n-k} = \sum_{k=0}^{n/2}(-1)^k\binom{x}{k}\binom{2n-2x}{n-2k} = \\ = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{2n-k}{n-k}\binom{2n-x}{k}2^k = (-4)^n\binom{(x-1)/2}{n}\end{gathered}}$$

$$(1)\quad \sum_{k=0}^{n}\binom{2n-x}{n-k}\binom{x}{k}(-1)^k \stackrel{B}{=} [t^n](1+t)^{2n-x}\left[(1-u)^x \;\middle|\; u=t\right] = [t^n]\left(\frac{1-t}{1+t}\right)^x(1+t)^{2n}.$$

$$(2)\quad \sum_{k=0}^{n/2}\binom{2n-2x}{n-2k}\binom{x}{k}(-1)^k \stackrel{B}{=} [t^n](1+t)^{2n-2x}\left[(1-u)^x \;\middle|\; u=t^2\right] =$$

$$= [t^n]\frac{(1-t)^x(1+t)^x}{(1+t)^{2x}}(1+t)^{2n} = [t^n]\left(\frac{1-t}{1+t}\right)^x(1+t)^{2n}.$$

$$(3)\quad (-1)^n\sum_{k=0}^{n}(-1)^k\binom{2n-k}{n-k}\binom{2n-x}{k}2^k \stackrel{B}{=} (-1)^n[t^n](1+t)^{2n}\left[(1-2u)^{2n-x} \;\middle|\; u=\frac{t}{1+t}\right] =$$

$$= (-1)^n[t^n](1+t)^{2n}\left(\frac{1-t}{1+t}\right)^{2n-x} = (-1)^n[t^n](1-t)^{2n}\left(\frac{1+t}{1-t}\right)^x = [t^n]\left(\frac{1-t}{1+t}\right)^x(1+t)^{2n}.$$

The explicit value:

$$S = [t^n]\left(\frac{1-t}{1+t}\right)^x(1+t)^{2n} = [t^n]\left[\left(\frac{1-w}{1+w}\right)^x\frac{1+w}{1-w} \;\middle|\; w=t(1+w)^2\right] =$$

$$= [t^n]\left[\left(\frac{1-w}{1+w}\right)^{x-1} \;\middle|\; w=\frac{1-2t-\sqrt{1-4t}}{2t}\right].$$

$$\frac{1-w}{1+w} = \sqrt{1-4t}.$$

$$S = [t^n](1-4t)^{(x-1)/2} = \binom{(x-1)/2}{n}(-4)^n.$$

(3.43)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{y-2k}{n-k}2^k = \sum_{k=0}^{n/2}\binom{x}{k}\binom{y-2x}{n-2k}}$$

$$(1)\quad \sum_{k=0}^{n} \binom{y-2k}{n-k}\binom{x}{k}(-2)^k \stackrel{B}{=} [t^n](1+t)^y \left[(1-2u)^x \;\middle|\; u=\frac{t}{(1+t)^2}\right] =$$

$$= [t^n](1+t)^y \left(1-\frac{2t}{(1+t)^2}\right)^x = [t^n](1+t)^y \frac{(1+t^2)^x}{(1+t)^{2x}}.$$

$$(2)\quad \sum_{k=0}^{n/2} \binom{y-2x}{n-2k}\binom{x}{k} \stackrel{B}{=} [t^n](1+t)^{y-2x} \left[(1+u)^x \;\middle|\; u=t^2\right] = [t^n](1+t)^{y-2x}(1+t^2)^x.$$

(3.44)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{2x-2k}{n-k}4^k = (-1)^n\binom{2x}{n}}$$

$$\sum_{k=0}^{n} \binom{2x-2k}{n-k}\binom{x}{k}(-4)^k \stackrel{B}{=} [t^n](1+t)^{2x} \left[(1-4u)^x \;\middle|\; u=\frac{t}{(1+t)^2}\right] =$$

$$= [t^n](1+t)^{2x}\frac{(1-t)^{2x}}{(1+t)^{2x}} = [t^n](1-t)^{2x} = \binom{2x}{n}(-1)^n.$$

(3.45)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{4x-2k}{n-k}4^k = (-1)^{n/2}\binom{2x}{n/2}}$$

$$\sum_{k=0}^{n} \binom{4x-2k}{n-k}\binom{x}{k}(-4)^k \stackrel{B}{=} [t^n](1+t)^{4x} \left[(1-4u)^x \;\middle|\; u=\frac{t}{(1+t)^2}\right] =$$

$$= [t^n](1-t^2)^{2x} = \binom{2x}{n/2}(-1)^{n/2}.$$

(3.46)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{2x+1}{k}\binom{2n-2x-1}{n-k} = (-4)^n\binom{x}{n}}$$

$$\sum_{k=0}^{n} \binom{2n-2x-1}{n-k}\binom{2x+1}{k}(-1)^k \stackrel{B}{=} [t^n](1+t)^{2n-2x-1} \left[(1-u)^{2x+1} \;\middle|\; u=t\right] =$$

$$= [t^n]\left(\frac{1-t}{1+t}\right)^{2x+1}(1+t)^{2n} = [t^n]\left[\left(\frac{1-w}{1+w}\right)^{2x+1}\frac{1+w}{1-w} \;\middle|\; w=t(1+w)^2\right] =$$

$$= [t^n]\left[\left(\frac{1-w}{1+w}\right)^{2x} \;\middle|\; w=\frac{1-2t-\sqrt{1-4t}}{2t}\right] = [t^n](1-4t)^x = \binom{x}{n}(-4)^n.$$

(3.47)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{r} = (-1)^n\binom{x}{r-n}}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{x+k}{r}(-1)^k \stackrel{E}{=} [t^n]\frac{1}{1-t}\left[\frac{(-u)^{r-x}}{(1+u)^{r+1}} \;\middle|\; u=\frac{t}{1-t}\right] =$$

$$= [t^n]\frac{1}{1-t}\cdot\frac{(-1)^{r-x}t^{r-x}}{(1-t)^{r-x}}\cdot(1-t)^{r+1} = (-1)^{r-x}[t^{n+x-r}](1-t)^x = (-1)^n\binom{x}{r-n}.$$

(3.48)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{r+k} = (-1)^n\binom{x}{n+r}}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{x+k}{x-r}(-1)^k \stackrel{E}{=} [t^n]\frac{1}{1-t}\left[\frac{(-u)^{-r}}{(1+u)^{x-r+1}} \;\middle|\; u=\frac{t}{1-t}\right] =$$

$$= [t^n]\frac{1}{1-t}\cdot\frac{(-1)^r(1-t)^r}{t^r}(1-t)^{x-r+1} = (-1)^r[t^{r+n}](1-t)^x = (-1)^n\binom{x}{r+n}.$$

(3.49)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x-k}{r} = \binom{x-n}{r-n}}$$

$$\sum_{k=0}^{n}\binom{x-k}{r}\binom{n}{k}(-1)^k \stackrel{A}{=} [t^x]\frac{t^r}{(1-t)^{r+1}}\left[(1-u)^n \;\middle|\; u=t\right] =$$

$$= [t^{x-r}](1-t)^{n-r-1} = \binom{n-r-1}{x-r}(-1)^{x-r} = \binom{x-n}{r-n}.$$

(3.50)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{2n-k}{n} = \binom{2n-x}{n}}$$

$$\sum_{k=0}^{n}\binom{2n-k}{n}\binom{x}{k}(-1)^k \stackrel{A}{=} [t^{2n}]\frac{t^n}{(1-t)^{n+1}}\left[(1-u)^x \;\middle|\; u=t\right] =$$

$$= [t^n](1-t)^{x-n-1} = \binom{x-n-1}{n}(-1)^n = \binom{2n-x}{n}$$

(3.51)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{y-2k}{n-k}3^k = \sum_{k=0}^{n/3}\binom{x}{k}\binom{y-3x}{n-3k}}$$

(1) $$\sum_{k=0}^{n}\binom{y-2k}{n-k}\binom{x}{k}(-3)^k \stackrel{B}{=} [t^n](1+t)^y\left[(1-3u)^x \;\middle|\; u=\frac{t}{(1+t)^2}\right] =$$

$$= [t^n](1+t)^y\frac{(1+2t+t^2-3t)^{2x}}{(1+t)^{2x}} = [t^n](1+t)^{y-2x}(1-t+t^2)^{2x}.$$

(2) $$\sum_{k=0}^{n/3}\binom{y-3x}{n-3k}\binom{x}{k} \stackrel{B}{=} [t^n](1+t)^{y-3x}\left[(1+u)^x \;\middle|\; u=t^3\right] = [t^n](1+t)^{y-3x}(1+t^3)^x =$$

$$= [t^n](1+t)^{y-3x}(1+t)^x(1-t+t^2)^x = [t^n](1+t)^{y-2x}(1-t+t^2)^x.$$

(3.52)

$$\boxed{\sum_{k=0}^{n}(-3)^k\binom{x}{k}\binom{3x-2k}{n-k} = (-1)^n\binom{x}{n/3}}$$

$$\sum_{k=0}^{n}\binom{3x-2k}{n-k}\binom{x}{k}(-3)^k \stackrel{B}{=} [t^n](1+t)^{3x}\left[(1-3u)^x \;\middle|\; u = \frac{t}{(1+t)^2}\right] =$$

$$= [t^n](1+t)^{3x}\left(\frac{1+2t+t^2-3t}{(1+t)^2}\right)^x = [t^n](1+t)^x(1-t+t^2)^x = [t^n](1+t^3)^x = (-1)^n\binom{x}{n/3}$$

(3.53)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{2n-k}{n}\frac{k2^k}{2n-k} = 4^n\binom{n-1-x/2}{n} = \left[\binom{x/2}{n}(-4)^n\right]}$$

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{2n-k}{n}\frac{k2^k}{2n-k} = -\frac{2x}{n}\sum_{k=0}^{n}\binom{2n-1-k}{n-1}\binom{x-1}{k-1}(-2)^{k-1} = -\frac{2x}{n}S_n$$

$$S_n \stackrel{A}{=} [t^{2n-1}]\frac{t^{n-1}}{(1-t)^n}\left[u(1-2u)^{x-1} \;\middle|\; u = t\right] = [t^n]\frac{t(1-2t)^{x-1}}{(1-t)^n} =$$

$$= [t^n]\left[\frac{w(1-2w)^{x-1}}{1-w/(1-w)} \;\middle|\; w = \frac{t}{1-w}\right] = [t^n]\left[w(1-w)(1-2w)^{x-2} \;\middle|\; w = \frac{1-\sqrt{1-4t}}{2}\right] =$$

$$= [t^n]\frac{1-\sqrt{1-4t}}{2}\cdot\frac{1+\sqrt{1-4t}}{2}\sqrt{1-4t}^{x-2} = [t^{n-1}](1-4t)^{x/2-1} = \binom{x/2-1}{n-1}(-4)^{n-1}$$

$$-\frac{2x}{n}S_n = \frac{x/2}{n}\binom{x/2-1}{n-1}(-4)^n = \binom{x/2}{n}4^n.$$

(3.54)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x}{n-k}\binom{n+k}{k}\frac{n-k}{n+k}\cdot 2^{-n-k} = \binom{x/2}{n}}$$

$$S_n = \sum_{k=0}^{n}(-1)^k\binom{x}{n-k}\left(\binom{n+k}{k} - 2\binom{n+k-1}{k-1}\right)2^{-n-k} =$$

$$= \sum_{k=0}^{n}(-1)^k\binom{x}{n-k}\binom{n+k}{k}2^{-n-k} - \sum_{k=0}^{n}(-1)^k\binom{x}{n-k}\binom{n+k-1}{k-1}2^{-n-k+1} = T_{n,k} - T_{n,k-1}.$$

$$T_{n,k} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{x}{n-k}\binom{n+k}{k}(-2)^{-k} \stackrel{B}{=} \frac{1}{2^n}[t^n](1+t)^x\left[\frac{1}{(1+u/2)^{n+1}} \;\middle|\; u = t\right] =$$

$$= [t^n]\left(1+\frac{t}{2}\right)^x\frac{1}{(1+t/4)^{n+1}}.$$

$$T_{n,k} - T_{n,k-1} = [t^n]\left(1+\frac{t}{2}\right)^{x+1}\frac{1}{(1+t/4)^{n+1}} = [t^n]\left[\frac{(1+w/2)^{x+1}}{1+w/4}\cdot\frac{4(1+w/4)}{4(1+w/2)} \;\middle|\; w = \frac{t}{1+w/4}\right] =$$

$$= [t^n]\left[\left(1+\frac{w}{2}\right)^x \;\middle|\; w = \sqrt{1+t}-1\right] = [t^n](1+\sqrt{1+t}-1)^x = \binom{x/2}{n}.$$

(3.55) Generalizes a formula proposed by B. C. Wong. *The natural sum arrives to $k = n+1$ and we use an appropriate generating function:*

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{n+1}{k}\binom{2n-2k+x}{n} = \binom{n-x+1}{n}}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n+1}{k}\binom{2n-2k+x}{n} \overset{B^*}{=} [t^n](1+t)^{2n+x}\left[(1-u)^{n+1}-(-1)^{n+1}u^{n+1} \;\middle|\; u=\frac{1}{(1+t)^2}\right] =$$

$$= [t^n](1+t)^{2n+x}\left(\frac{(1+2t+t^2-1)^{n+1}}{(1+t)^{2n+2}} - \frac{(-1)^{n+1}}{(1+t)^{2n+2}}\right) =$$

$$= [t^n](1+t)^{x-2}t^{n+1}(2+t)^{n+1} - [t^n](-1)^{n+1}(1+t)^{x-2} =$$

$$= [t^{-1}](1+t)^{x-2}(2+t)^{n+1} + (-1)^n\binom{x-2}{n} = 0 + \binom{-x+2+n-1}{n} = \binom{n-x+1}{n}.$$

(3.56) - B.C. Wong:

$$\boxed{\sum_{k=0}^{n/2}(-1)^k\binom{n+1}{k}\binom{2n-2k}{n} = n+1}$$

$$\sum_{k=0}^{n/2}\binom{2n-2k}{n}\binom{n+1}{k}(-1)^k \overset{A}{=} [t^{2n}]\frac{t^n}{(1-t)^{n+1}}\left[(1-u)^{n+1} \;\middle|\; u=t^2\right] = [t^n](1+t)^{n+1} = n+1$$

(3.57)

$$\boxed{\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2n+2x}{k+x} = (-1)^n\binom{2n}{n}\binom{2n+2x}{x}\binom{n+x}{n}^{-1} = \left[\binom{n+x-1/2}{2n+x}(-4)^{2n+x}\right]}$$

$$S = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2n+2x}{k+x} \overset{B}{=} [t^{2n}](1+t)^{2n}\left[\frac{(1-u)^{2n+2x}}{u^x} \;\middle|\; u=t\right] = [t^{2n}](1-t^2)^{2n}\frac{(1-t)^{2x}}{t^x} =$$

$$\overset{m=2n}{=} [t^m](1-t^2)^m\frac{(1-t)^{2x}}{t^x} = [t^m]\left[\left(\frac{(1-w)^2}{w}\right)^x\cdot\frac{1-w^2}{1+w^2} \;\middle|\; w=t(1-w^2)\right] = S.$$

$$tw^2+w-t=0; \quad w=\frac{\sqrt{1+4t^2}-1}{2t}; \quad \frac{(1-w^2)}{w} = \frac{\sqrt{1+4t^2}-2t}{t}; \quad \frac{1-w^2}{1+w^2} = \frac{1}{\sqrt{1+4t^2}}.$$

$$S = [t^{2n}]\frac{1}{\sqrt{1+4t^2}}\left(\frac{\sqrt{1+4t^2}-2t}{t}\right)^x = [t^{2n+x}]\frac{(\sqrt{1+4t^2}-2t)^x}{\sqrt{1+4t^2}}.$$

$$F=\sqrt{1+4t^2}-2t; \quad \phi F+2w=\sqrt{\phi^2+4w^2}; \quad \phi=\frac{4wF}{1-F^2}; \quad F(0)=1;\ \phi=F;\ 1-\phi^2=4w.$$

$$\phi=\sqrt{1-4w}=F; \quad w=t\sqrt{1-4w}; \quad \sqrt{1+4t^2}=F+2t=\sqrt{1-4w}+\frac{2w}{\sqrt{1-4w}}.$$

$$S=[t^{2n+x}]\left[\frac{(\sqrt{1-4w})^x}{\sqrt{1-4w}+2w/\sqrt{1-4w}} \;\middle|\; w=t\sqrt{1-4w}\right] \overset{2LF}{=}$$

$$= [t^{2n+x}]\frac{(\sqrt{1-4t})^x}{\sqrt{1-4t}+2t/\sqrt{1-4t}}(\sqrt{1-4t})^{2n+x-1}\left(\sqrt{1-4t}+\frac{2t}{\sqrt{1-4t}}\right) =$$

$$= [t^{2n+x}](\sqrt{1-4t})^{2n+2x-1} = \binom{n+x-1/2}{2n+x}(-4)^{2n+x}.$$

(3.58)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2x-n}{x-k} = (-1)^{n/2}\binom{x}{n/2}\binom{2x}{x}\binom{2x}{n}^{-1} = \left[(-1)^{n+x}\binom{(n-1)/2}{x}4^x\right]}$$

$$S = \sum_{k=0}^{n}(-1)^k\binom{n}{n-k}\binom{2x-n}{x-n+k}(-1)^k \stackrel{B}{=} [t^n](1+t)^n\left[(1-u)^{2x-n}(-u)^{n-x}\;\Big|\; u=t\right] =$$

$$= [t^n](1+t)^n(1-t)^{2x-n}(-t)^{n-x} = (-1)^{n+x}[t^x]\left(\frac{1+t}{1-t}\right)^n(1-t)^{2x} =$$

$$= (-1)^{n+x}[t^x]\left[\left(\frac{1+w}{1-w}\right)^n\cdot\frac{1-w}{1+w}\;\Big|\; w=t(1-w)^2\right] = S.$$

$$tw^2-(1+2t)w+t=0; \quad w=\frac{1+2t-\sqrt{1+4t}}{2t}; \quad \frac{1+w}{1-w}=\sqrt{1+4t}.$$

$$S = (-1)^{n+x}[t^x](1+4t)^{(n-1)/2} = (-1)^{n+x}\binom{(n-1)/2}{x}4^x.$$

(3.59)

$$\boxed{\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2x-2n}{x-k} = (-1)^n\binom{x}{n}\binom{2x}{x}\binom{2x}{2n}^{-1} = \left[\binom{n-1/2}{x}(-4)^x\right]}$$

$$\sum_{k=0}^{2n}\binom{2n}{2n-k}\binom{2x-2n}{x-2n+k}(-1)^k \stackrel{B}{=} [t^{2n}](1+t)^{2n}\left[(1-u)^{2x-2n}(-u)^{2n-x}\;\Big|\; u=t\right] =$$

$$= (-1)^x[t^x](1+t)^{2n}(1-t)^{2x-2n} = (-1)^x[t^x]\left[\left(\frac{1+w}{1-w}\right)^{2n}\cdot\frac{1-w}{1+w}\;\Big|\; w=t(1-w)^2\right] =$$

$$= (-1)^x[t^x]\left(\sqrt{1+4t}\right)^{2n-1} = (-1)^x\binom{n-1/2}{x}4^x = \binom{n-1/2}{x}(-4)^x.$$

(3.60) - *This is one half of the preceding sum, plus one half of the central term:*

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{2n}{k}\binom{2x-2n}{x-k} = \frac{(-1)^n}{2}\left(\binom{x}{n}+\binom{x}{n}^2\right)\binom{2x}{x}\binom{2x}{2n}^{-1}}$$

(3.61)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{2x-k}{n-k}2^k = (-1)^{n/2}\binom{x}{n/2}}$$

$$\sum_{k=0}^{n}\binom{2x-k}{n-k}\binom{x}{k}(-2)^k \stackrel{B}{=} [t^n](1+t)^{2x}\left[(1-2u)^x\;\Big|\; u=\frac{t}{1+t}\right] = [t^n](1+t)^{2x}\left(\frac{1-t}{1+t}\right)^x =$$

$$= [t^n](1-t^2)^x = (-1)^{n/2}\binom{x}{n/2}$$

(3.62)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{2x-2k}{n-k}2^k = \binom{x}{n/2}}$$

$$\sum_{k=0}^{n}\binom{2x-2k}{n-k}\binom{x}{k}(-2)^k \stackrel{B}{=} [t^n](1+t)^{2x}\left[(1-2u)^x \;\middle|\; u=\frac{t}{(1+t)^2}\right] = [t^n](1+t)^{2x}\frac{(1+t^2)^x}{(1+t)^{2x}} =$$

$$= [t^n](1+t^2)^x = \binom{x}{n/2}.$$

(3.63)

$$\boxed{\sum_{k=0}^{n/2}(-1)^k\binom{x}{k}\binom{2x-2k}{n-2k} = \binom{x}{n}2^n}$$

$$\sum_{k=0}^{n/2}\binom{2x-2k}{n-2k}\binom{x}{k}(-1)^k \stackrel{B}{=} [t^n](1+t)^{2x}\left[(1-u)^x \;\middle|\; u=\frac{t^2}{(1+t)^2}\right] = [t^n](1+t)^{2x}\frac{(1+2t)^x}{(1+t)^{2x}} =$$

$$= [t^n](1+2t)^x = \binom{x}{n}2^n.$$

(3.64)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{j} = (-1)^n\binom{n}{j-n}2^{2n-j}}$$

$$\sum_{k=0}^{n}\binom{2k}{j}\binom{n}{k}(-1)^k \stackrel{B^*}{=} [t^j](1+t)^0\left[(1-u)^n \;\middle|\; u=(1+t)^2\right] = [t^j](2t+t^2)^n(-1)^n =$$

$$= [t^{j-n}](-2)^n(1+t/2)^n = (-2)^n\binom{n}{j-n}\frac{1}{2^{j-n}} = (-1)^n\binom{n}{j-n}2^{2n-j}.$$

(3.65)

$$\boxed{\sum_{k=0}^{n}\binom{n}{k}^2x^k = \sum_{k=0}^{n}\binom{n}{k}\binom{2n-k}{n}(x-1)^k}$$

(1) $$\sum_{k=0}^{n}\binom{n}{n-k}\binom{n}{k}x^k \stackrel{B}{=} [t^n](1+t)^n\left[(1+xu)^n \;\middle|\; u=t\right] = [t^n](1+t)^n(1+xt)^n.$$

(2) $$\sum_{k=0}^{n}\binom{2n-k}{n-k}\binom{n}{k}(x-1)^k \stackrel{B}{=} [t^n](1+t)^{2n}\left[(1+(x-1)u)^n \;\middle|\; u=\frac{t}{1+t}\right] =$$

$$= [t^n](1+t)^{2n}\left(\frac{1+t+xt-t}{1+t}\right)^n = [t^n](1+t)^n(1+xt)^n.$$

(3.66) - *This is a particular case of the Vandermonde convolution (3.1); it is also the case $x = 1$ of identity (3.65):*

$$\boxed{\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}}$$

(3.67) - *(**NN**) This is the first half of the previous sum; if n is even, the central element is to be subtracted:*

$$\boxed{\sum_{k=0}^{(n-1)/2}\binom{n}{k}^2 = \frac{1}{2}\binom{2n}{n} - \frac{1}{2}\binom{n}{n/2}^2}$$

(3.68) - *(**NN**) Analogous to the previous one; the central element is always present:*

$$\boxed{\sum_{k=0}^{n}\binom{2n}{k}^2 = \frac{1}{2}\binom{4n}{2n} + \frac{1}{2}\binom{2n}{n}^2}$$

(3.69) - *(**NN**) Analogous to the previous one; the central element is never present:*

$$\boxed{\sum_{k=0}^{n}\binom{2n+1}{k}^2 = \frac{1}{2}\binom{4n+2}{2n+1}}$$

(3.70)

$$\boxed{\sum_{k=0}^{n/2}\binom{n}{2k}^2 = \frac{1}{2}\binom{2n}{n} + \frac{(-1)^{n/2}}{2}\binom{n}{n/2}}$$

$$\sum_{k=0}^{n/2}\binom{n}{n-2k}\binom{n}{2k} \stackrel{B}{=} [t^n](1+t)^n\left[\frac{(1+\sqrt{u})^n + (1-\sqrt{u})^n}{2} \;\middle|\; u = t^2\right] =$$

$$= [t^n]\frac{(1+t)^{2n} + (1-t^2)^n}{2} = \frac{1}{2}\binom{2n}{n} + \frac{(-1)^{n/2}}{2}\binom{n}{n/2}$$

(3.71)

$$\boxed{\sum_{k=0}^{n}\binom{2n}{2k}^2 = \frac{1}{2}\binom{4n}{2n} + \frac{(-1)^n}{2}\binom{2n}{n}}$$

$$\sum_{k=0}^{n}\binom{2n}{2n-2k}\binom{2n}{2k} \stackrel{B}{=} [t^{2n}](1+t)^{2n}\left[\frac{(1+\sqrt{t})^{2n} + (1-\sqrt{t})^{2n}}{2} \;\middle|\; u = t^2\right] =$$

$$= [t^{2n}](1+t)^{2n}\frac{(1+t)^{2n} + (1-t)^{2n}}{2} = \frac{1}{2}[t^{2n}]\left((1+t)^{4n} + (1-t^2)^{2n}\right) =$$

$$= \frac{1}{2}\binom{4n}{2n} + \frac{(-1)^n}{2}\binom{2n}{n}$$

(3.72) - *(**NN**) Analogous to the previous one; the central element is always present:*

$$\sum_{k=0}^{n/2}\binom{2n}{2k}^2=\frac{1}{4}\binom{4n}{2n}+\frac{(-1)^n}{4}\binom{2n}{n}+\frac{(-1)^{n/2}}{4}\binom{n}{n/2}$$

(3.73)

$$\sum_{k=0}^{(n-1)/2}\binom{n}{2k+1}^2=\frac{1}{2}\binom{2n}{n}-\frac{(-1)^{n/2}}{2}\binom{n}{n/2}$$

$$\sum_{k=0}^{(n-1)/2}\binom{n}{n-2k-1}\binom{n}{2k+1}\stackrel{B}{=}[t^{n-1}](1+t)^n\left[\frac{(1+\sqrt{t})^n-(1-\sqrt{t})^n}{2\sqrt{t}}\;\Big|\;u=t^2\right]=$$

$$=[t^{n-1}](1+t)^n\frac{(1+t)^n-(1-t)^n}{2t}=\frac{1}{2}[t^n]\left((1+t)^{2n}-(1-t^2)^n\right)=\frac{1}{2}\binom{2n}{n}-\frac{(-1)^{n/2}}{2}\binom{n}{n/2}.$$

(3.74)

$$\sum_{k=0}^{n-1}\binom{2n}{2k+1}^2=\frac{1}{2}\binom{4n}{2n}-\frac{(-1)^n}{2}\binom{2n}{n}$$

$$S=\sum_{k=0}^{n-1}\binom{2n}{2k+1}^2=[t^{2n}]\frac{t}{(1-t)^2}\left[\frac{(1+\sqrt{u})^{2n}(1-\sqrt{u})^{2n}}{2\sqrt{u}}\;\Big|\;u=\frac{t^2}{(1-t)^2}\right]=$$

$$=\frac{1}{2}[t^{2n}]\frac{1}{1-t}\left(\left(1+\frac{t}{1-t}\right)^{2n}-\left(1-\frac{1}{1-t}\right)^{2n}\right)=$$

$$=\frac{1}{2}[t^{2n}]\frac{1}{(1-t)^{2n+1}}-\frac{1}{2}[t^{2n}]\frac{1}{1-t}\left(\frac{1-2t}{1-t}\right)^{2n}=$$

$$=\frac{1}{2}\binom{-2n-1}{2n}-\frac{T}{2}=\frac{1}{2}\binom{4n}{n}-\frac{T}{2}.$$

$$T=[t^{2n}]\left[\frac{1}{1-w}\frac{(1-2w)(1-w)}{1-2w+2w^2}\;\Big|\;w=t\frac{1-2w}{1-w}\right]=\frac{1}{2}[t^{2n}]\left[\frac{1-2w}{1-2w+2w^2}\;\Big|\;w=\frac{1+2t-\sqrt{1+4t^2}}{2}\right]=$$

$$=[t^{2n}]\frac{1}{\sqrt{1+4t^2}}=[t^n]\frac{1}{\sqrt{1+4t}}=(-1)^n\binom{2n}{n}.$$

$$S=\frac{1}{2}\binom{4n}{2n}-\frac{(-1)^n}{2}\binom{2n}{n}.$$

(3.75) - *This is one half of the preceding sum; the central element is only present when n is odd:*

$$\sum_{k=0}^{(n-1)/2}\binom{2n}{2k+1}^2=\frac{1}{4}\binom{4n}{2n}-\frac{(-1)^n}{4}\binom{2n}{n}+\frac{1-(-1)^n}{4}\binom{2n}{n}^2$$

(3.76) - ***(NN)***:

$$\boxed{\sum_{k=0}^{(n-1)/2} \binom{n}{k}^2 (n-2k)^2 = n\binom{2n-2}{n-1}}$$

$$S = \sum_{k=0}^{(n-1)/2} \binom{n}{k}^2 (n-2k)^2 = \frac{1}{2}\sum_{k=0}^{n}(n^2 - 4kn + 4k^2) =$$

$$= \frac{1}{2}\left(n^2\sum_{k=0}^{n}\binom{n}{k}^2 - 4n\sum_{k=0}^{n}\binom{n}{k}^2 + 4\sum_{k=0}^{n}\binom{n}{k}^2 k^2\right).$$

This sum is symmetric, since it is invariant with respect to the transformation $k \mapsto n-k$:

$$\sum_{k=0}^{n}\binom{n}{k}^2 = [t^n]\frac{1}{1-t}\left[(1+u)^n \;\middle|\; u = \frac{t}{1-t}\right] = [t^n]\frac{1}{(1-t)^{n+1}} = \binom{2n}{n}.$$

$$\sum_{k=0}^{n}\binom{n}{k}^2 k = n\sum_{k=0}^{n}\binom{n}{k}\binom{n-1}{k-1} = n[t^{n-1}]\frac{1}{t}\left[(1+u)^n \;\middle|\; u = \frac{t}{1-t}\right] =$$

$$= n[t^n]\frac{1}{(1-t)^n} = n\binom{2n-1}{n}.$$

$$\sum_{k=0}^{n}\binom{n}{k}^2 k^2 = n^2\sum_{k=0}^{n-1}\binom{n-1}{k-1}^2 = n^2[t^{n-1}]\frac{1}{t}\left[u(1+u)^{n-1} \;\middle|\; u = \frac{t}{1-t}\right] =$$

$$= n^2[t^n]\frac{t}{1-t}\frac{1}{(1-t)^{n-1}} = n^2[t^{n-1}]\frac{1}{(1-t)^n} = \binom{2n-2}{n-1}.$$

$$S = \frac{n^2}{2}\binom{2n}{n} - 2n^2\binom{2n-1}{n} + 2n^2\binom{2n-2}{n-1} = \left(\frac{n^2 2n(2n-1)}{n^2} - 2n^2\frac{2n-1}{n} + 2n^2\right)\binom{2n-2}{n-1} =$$

$$= (2n^2 - n - 4n^2 + 2n + 2n^2)\binom{2n-2}{n-1} = n\binom{2n-2}{n-1}.$$

(3.77)

$$\boxed{S_r = \sum_{k=0}^{n}\binom{n}{k}^2 \cdot k^r = \sum_{k=0}^{r}\binom{n}{k}\binom{2n-k}{n}\left\{{r \atop k}\right\}k!}$$

$$\sum_{k=0}^{n}\binom{n}{k}^2 \cdot k^r = \sum_{k=0}^{n}\binom{n}{k}^2[t^k]\sum_{j=0}^{r}\left\{{r \atop j}\right\}\frac{j!t^j}{(1-t)^{j+1}} =$$

$$= \sum_{k=0}^{n}\binom{n}{k}^2\sum_{j=0}^{r}\left\{{r \atop j}\right\}j![t^{k-j}]\frac{1}{(1-t)^{j+1}} = \sum_{k=0}^{n}\binom{n}{k}^2\sum_{j=0}^{r}\left\{{r \atop j}\right\}j!\binom{k}{j} =$$

$$= \sum_{j=0}^{r}\left\{{r \atop j}\right\}j!\sum_{k=0}^{n}\binom{n}{k}\binom{n}{k}\binom{k}{j} = \sum_{j=0}^{r}\left\{{r \atop j}\right\}j!\binom{n}{j}\sum_{k=0}^{n}\binom{n}{k}\binom{n-j}{k-j} \overset{E}{=}$$

$$= \sum_{j=0}^{r}\left\{{r \atop j}\right\}j!\binom{n}{j}[t^n]\frac{1}{1-t}\left[(1+u)^{n-j}u^j \;\middle|\; u = \frac{t}{1-t}\right] = \sum_{j=0}^{r}\left\{{r \atop j}\right\}j!\binom{n}{j}[t^n]\frac{t^j}{(1-t)^{n+1}} =$$

$$= \sum_{j=0}^{r}\left\{{r \atop j}\right\}j!\binom{n}{j}\binom{2n-j}{n-j} = \sum_{k=0}^{r}\binom{n}{k}\binom{2n-k}{n}\left\{{r \atop k}\right\}k!.$$

(3.78)

$$S_0 = \binom{2n}{n} \qquad S_1 = \frac{n}{2}\binom{2n}{n} = (2n-1)\binom{2n-2}{n-1}$$

(1) $$\sum_{k=0}^{n}\binom{n}{k}^2 \stackrel{E}{=} [t^n]\frac{1}{1-t}\left[(1+u)^n \;\middle|\; u = \frac{t}{1-t}\right] = [t^n]\frac{1}{(1-t)^{n+1}} = \binom{-n-1}{n}(-1)^n = \binom{2n}{n}.$$

(2) $$\sum_{k=0}^{n}\binom{n}{k}^2 k = n\sum_{k=0}^{n}\binom{n}{k}\binom{n-1}{k-1} \stackrel{E}{=} n[t^n]\frac{1}{1-t}\left[u(1+u)^{n-1} \;\middle|\; u = \frac{t}{1-t}\right] =$$

$$= n[t^n]\frac{t}{(1-t)^2}\cdot\frac{1}{(1-t)^{n-1}} = n[t^{n-1}]\frac{1}{(1-t)^{n+1}} = n\binom{2n-1}{n-1} = \frac{n}{2}\binom{2n}{n}.$$

(3.79) *I have to add the proof of the second part:*

$$S_2 = n^2\binom{2n-2}{n-1} \qquad S_3 = \frac{n^2(n+1)}{2}\binom{2n-2}{n-1}$$

(1) $$\sum_{k=0}^{n}\binom{n}{k}^2 k^2 \stackrel{E}{=} n[t^n]\frac{1}{1-t}\cdot\frac{t}{1-t}\left(1+\frac{nt}{1-t}\right)\frac{1}{(1-t)^{n-2}} = n[t^{n-1}]\frac{1+(n-1)t}{(1-t)^{n+1}} =$$

$$= n\binom{2n-1}{n-1} + n(n-1)\binom{2n-2}{n-1} = \left(\frac{2n-1}{n}n + \frac{n(n-1)^2}{n}\right)\binom{2n-2}{n-1} = n^2\binom{2n-2}{n-1}.$$

(3.80)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n}2^k = (-1)^{n/2}\binom{n}{n/2}$$

(1) $$\sum_{k=0}^{n}\binom{n}{k}^2(-1)^k \stackrel{E}{=} [t^n]\frac{1}{1-t}\left[(1-u)^n \;\middle|\; u = \frac{t}{1-t}\right] = [t^n]\frac{(1-2t)^n}{(1-t)^{n+1}} =$$

$$= \sum_{k=0}^{n}\binom{n}{k}(-2)^k\binom{-n-1}{n-k}(-1)^{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n}2^k.$$

(2) $$\sum_{k=0}^{n}\binom{n}{k}^2(-1)^k = [t^n]\frac{(1-2t)^n}{(1-t)^{n+1}} = [t^n]\left[\frac{1}{1-w}\cdot\frac{(1-2w)(1-w)}{1-2w+2w^2} \;\middle|\; w = t\frac{1-2w}{1-w}\right] =$$

$$= [t^n]\left[\frac{1-2w}{1-2w+2w^2} \;\middle|\; w = \frac{1+2t-\sqrt{1+4t^2}}{2}\right] = [t^n]\frac{\sqrt{1+4t^2}-2t}{1+4t^2-2t\sqrt{1+4t^2}} =$$

$$= [t^n]\frac{1}{\sqrt{1+4t^2}} = (-1)^{n/2}\binom{n}{n/2}.$$

(3.81)

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^2 = (-1)^n\binom{2n}{n}$$

$$\sum_{k=0}^{2n}\binom{2n}{2n-k}\binom{2n}{k}(-1)^k \stackrel{B}{=} [t^{2n}](1+t)^{2n}\left[(1-u)^{2n} \;\middle|\; u = t\right] = [t^{2n}](1-t^2)^{2n} = (-1)^n\binom{2n}{n}$$

(3.82) - *(**NN**) Analogous to the previous one; the central element is always present:*

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k}^2 = \frac{(-1)^n}{2}\left(\binom{2n}{n}+\binom{2n}{n}^2\right)$$

(3.83)

$$\sum_{k=0}^{n}\binom{n}{k}^2 2^k = \sum_{k=0}^{n/2}(-1)^k\binom{n}{k}\binom{3n-2k}{2n}$$

$$(1)\quad \sum_{k=0}^{n}\binom{n}{k}^2 2^k \stackrel{E}{=} [t^n]\frac{1}{1-t}\left[(1+2u)^n \;\middle|\; u=\frac{t}{1-t}\right] = [t^n]\frac{(1+t)^n}{(1-t)^{n+1}}.$$

$$(2)\quad \sum_{k=0}^{n/2}\binom{3n-2k}{2n}\binom{n}{k}(-1)^k \stackrel{A}{=} [t^{3n}]\frac{t^{2n}}{(1-t)^{2n+1}}\left[(1-u)^n \;\middle|\; u=t^2\right] =$$

$$= [t^n]\frac{(1-t)^n(1+t)^n}{(1-t)^{2n+1}} = [t^n]\frac{(1+t)^n}{(1-t)^{n+1}}.$$

Observe the following derivation of the left hand member:

$$\sum_{k=0}^{n}\binom{n}{n-k}\binom{n}{k}2^k \stackrel{B}{=} [t^n](1+t)^n\left[(1+2u)^n \;\middle|\; u=t\right] = [t^n](1+t)^n(1+2t)^n.$$

Both expressions correspond to the g.f.: $1/\sqrt{1-6t+t^2}$ *as can be shown by an application of the LIF.*

(3.84)

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\left(-\frac{z}{4}\right) = \sum_{k=0}^{n}\binom{n}{k}\binom{-1/2}{k}z^k = \frac{1}{4^n}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}(1-z)^k$$

$$(1)\quad \sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\left(-\frac{z}{4}\right) \stackrel{E}{=} [t^n]\frac{1}{1-t}\left[\frac{1}{\sqrt{1+zu}} \;\middle|\; u=\frac{t}{1-t}\right] = [t^n]\frac{1}{\sqrt{(1-t)(1-(1-z)t)}}.$$

$$(2)\quad \sum_{k=0}^{n}\binom{n}{k}\binom{-1/2}{k}z^k \stackrel{E}{=} [t^n]\frac{1}{1-t}\left[\frac{1}{\sqrt{1+zu}} \;\middle|\; u=\frac{t}{1-t}\right] = [t^n]\frac{1}{\sqrt{(1-t)(1-(1-z)t)}}.$$

$$(3)\quad \frac{1}{4^n}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}(1-z)^k \stackrel{conv}{=} \frac{1}{4^n}[t^n]\frac{1}{\sqrt{1-4t}}\frac{1}{\sqrt{1-4(1-z)t}} = [t^n]\frac{1}{\sqrt{(1-t)(1-(1-z)t)}}.$$

(3.85)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\frac{1}{4^k} = \binom{2n}{n}\frac{1}{4^n}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\frac{(-1)^k}{4^k} \stackrel{E}{=} [t^n]\frac{1}{1-t}\left[\frac{1}{\sqrt{1+u}} \;\middle|\; u=\frac{t}{1-t}\right] = [t^n]\frac{1}{\sqrt{1-t}} = \binom{2n}{n}\frac{1}{4^n}.$$

(3.86) - *This sum is related to trinomial coefficients:*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k} = (-1)^n\sum_{k=0}^{n/2}\binom{n}{2k}\binom{2k}{k}$$

$$(1)\quad \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k} \overset{E}{=} [t^n]\frac{1}{1-t}\left[\frac{1}{\sqrt{1+4u}}\;\Big|\; u=\frac{t}{1-t}\right] =$$

$$= [t^n]\frac{1}{\sqrt{(1-t)(1+3t)}} = [t^n]\frac{1}{\sqrt{1+2t-3t^2}}.$$

$$(2)\quad (-1)^n\sum_{k=0}^{n/2}\binom{n}{2k}\binom{2k}{k} \overset{A}{=} (-1)^n[t^n]\frac{1}{1-t}\left[\frac{1}{\sqrt{1-4u}}\;\Big|\; u=\frac{t^2}{(1-t)^2}\right] =$$

$$= (-1)^n[t^n]\frac{1}{1-t}\sqrt{\frac{(1-t)^2}{1-2t-3t^2}} = [t^n]\frac{1}{\sqrt{1+2t-3t^2}}.$$

(3.87)

$$4^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k} = \sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}(-3)^k$$

$$(1)\quad 4^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k} \overset{E}{=} 4^n[t^n]\frac{1}{1-t}\left[\frac{1}{\sqrt{1+4u}}\;\Big|\; u=\frac{t}{1-t}\right] =$$

$$= 4^n[t^n]\frac{1}{\sqrt{(1-t)(1-3t)}} = [t^n]\frac{1}{\sqrt{(1-4t)(1-12t)}}.$$

$$(2)\quad \sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}(-3)^k \overset{conv}{=} [t^n]\frac{1}{\sqrt{1-4t}}\cdot\frac{1}{\sqrt{1-12t}}.$$

(3.88)

$$4^n\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k} = \sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}5^k$$

$$(1)\quad 4^n\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k} \overset{E}{=} 4^n[t^n]\frac{1}{1-t}\left[\frac{1}{\sqrt{1-4u}}\;\Big|\; u=\frac{t}{1-t}\right] =$$

$$= 4^n[t^n]\frac{1}{\sqrt{(1-t)(1-5t)}} = [t^n]\frac{1}{\sqrt{(1-4t)(1-20t)}}.$$

$$(2)\quad \sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}5^k \overset{conv}{=} [t^n]\frac{1}{\sqrt{1-4t}}\cdot\frac{1}{\sqrt{1-20t}}.$$

(3.89)

$$\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\binom{2k}{k}\frac{2k+1}{4^k} = \binom{2n}{n}\frac{1}{(2n-1)4^n}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\frac{(-1)^{k+1}(2k+1)}{4^k}\overset{E}{=}-[t^n]\frac{1}{1-t}\left[\frac{1}{(1+u)^{3/2}}\;\Big|\;u=\frac{t}{1-t}\right]=-[t^n](1-t)^{1/2}=$$

$$=-\binom{1/2}{n}(-1)^n=\frac{(-1)^n(-1)^n}{4^n(2n-1)}=\binom{2n}{n}\frac{1}{4^n(2n-1)}.$$

(3.90)

$$\boxed{\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}=4^n}$$

$$\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\overset{conv}{=}[t^n]\left(\frac{1}{\sqrt{1-4t}}\right)^2=[t^n]\frac{1}{1-4t}=4^n.$$

(3.91)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{2n-2k}{n-k}\binom{2k}{k}=\binom{n}{n/2}2^n}$$

$$\sum_{k=0}^{n}(-1)^k\binom{2k}{k}\binom{2n-2k}{n-k}\overset{conv}{=}[t^n]\frac{1}{\sqrt{1+4t}}\cdot\frac{1}{\sqrt{1-4t}}=[t^n]\frac{1}{\sqrt{1-16t^2}}=\binom{n}{n/2}4^{n/2}.$$

(3.92)

$$\boxed{\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\frac{1}{2k-1}=-\delta_{n,0}}$$

$$\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\frac{1}{2k-1}\overset{conv}{=}[t^n]\frac{1}{\sqrt{1-4t}}(-\sqrt{1-4t})=[t^n]-1=-\delta_{n,0}.$$

(3.93)

$$\boxed{\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\frac{1}{(2k-1)(2n-2k-1)}=\delta_{n,0}-4\delta_{n,1}}$$

$$\sum_{k=0}^{n}\binom{2n-2k}{n-k}\frac{1}{2n-2k-1}\binom{2k}{k}\frac{1}{2k-1}\overset{conv}{=}[t^n](-\sqrt{1-4t})^2=[t^n]1-4t=\delta_{n,0}-4\delta_{n,1}.$$

(3.94)

$$\boxed{\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\frac{1}{(2k-1)(2n-2k+1)}=\frac{16^n}{2n(2n+1)}\binom{2n}{n}^{-1}\qquad n\geq 1}$$

$$\sum_{k=0}^{n}\binom{2n-2k}{n-k}\frac{1}{2n-2k+1}\binom{2k}{k}\frac{1}{2k-1}\overset{conv}{=}[t^n]-\sqrt{1-4t}\frac{1}{\sqrt{4t}}\arctan\sqrt{\frac{4t}{1-4t}}=$$

$$=[t^n]-\sqrt{\frac{1-4t}{4t}}\arctan\sqrt{\frac{4t}{1-4t}}=\frac{16^n}{2n(2n+1)}\binom{2n}{n}^{-1}\qquad n>0.$$

***(3.95)**

$$\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\frac{x}{x+k}=4^n\binom{n+x-1/2}{n}\binom{x+n}{n}^{-1}\quad(n\geq 0)$$
$$=2^n\frac{(2x+1)(2x+3)\cdots(2x+2n-1)}{(x+1)\cdots(x+n)}\qquad(n\geq 1)$$

(3.96)

$$\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\frac{1}{2k+1}=\frac{16^n}{2n+1}\binom{2n}{n}^{-1}$$

$$\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}\frac{1}{2k+1}\overset{conv}{=}[t^n]\frac{1}{\sqrt{1-4t}}\frac{1}{\sqrt{4t}}\arctan\sqrt{\frac{4t}{1-4t}}=$$
$$=[t^n]-\frac{1}{\sqrt{4t(1-4t)}}\arctan\sqrt{\frac{4t}{1-4t}}=\frac{16^n}{2n+1}\binom{2n}{n}^{-1}.$$

(3.97)

$$\sum_{k=0}^{n}\binom{4n-4k}{2n-2k}\binom{4k}{2k}=\frac{1}{2}\left(\binom{2n}{n}4^n+16^n\right)$$

$$\sum_{k=0}^{n}\binom{4n-4k}{2n-2k}\binom{4k}{2k}\overset{conv}{=}[t^n]\left(\frac{1}{2\sqrt{1-4\sqrt{t}}}+\frac{1}{2\sqrt{1+4\sqrt{t}}}\right)^2=$$
$$=[t^n]\left(\frac{1}{4(1-4\sqrt{t})}+\frac{1}{4(1+4\sqrt{t})}+\frac{1}{2\sqrt{1-16t}}\right)=$$
$$=[t^n]\frac{1}{2(1-16t)}+[t^n]\frac{1}{2\sqrt{1-16t}}=\frac{4^{2n}}{2}+\frac{4^n}{2}\binom{2n}{n}.$$

(3.98)

$$\sum_{k=0}^{n-1}\binom{4n-4k-2}{2n-2k-1}\binom{4k+2}{2k+1}=\frac{1}{2}\left(16^n-\binom{2n}{n}4^n\right)\quad n>0$$

$$\sum_{k=0}^{n-1}\binom{4n-4k-2}{2n-2k-1}\binom{4k+2}{2k+1}\overset{conv}{=}[t^{n-1}]\left(\frac{1}{2\sqrt{t}\sqrt{1-4\sqrt{t}}}-\frac{1}{2\sqrt{t}\sqrt{1+4\sqrt{t}}}\right)^2=$$
$$=[t^{n-1}]\left(\frac{1}{4t(1-4\sqrt{t})}+\frac{1}{4t(1+4\sqrt{t})}-\frac{1}{2t\sqrt{1-16t}}\right)=$$
$$=[t^n]\frac{1}{2(1-16t)}-[t^n]\frac{1}{2\sqrt{1-16t}}=\frac{4^{2n}}{2}-\frac{4^n}{2}\binom{2n}{n}.$$

(3.99)

$$\sum_{k=0}^{n/2}\binom{n}{2k}\binom{2k}{k}2^{n-2k}=\binom{2n}{n}$$

$$2^n \sum_{k=0}^{n/2} \binom{n}{2k}\binom{2k}{k}\frac{1}{4^k} \stackrel{A}{=} 2^n[t^n]\frac{1}{1-t}\left[\frac{1}{\sqrt{1-u}} \;\middle|\; u = \frac{t^2}{(1-t)^2}\right] =$$

$$= 2^n[t^n]\frac{1}{\sqrt{1-2t}} = [t^n]\frac{1}{\sqrt{1-4t}} = \binom{2n}{n}$$

(3.100)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{n+k}{2k}\binom{2k}{k}\frac{x}{x+k} = (-1)^n\binom{x-1}{n}\binom{x+n}{n}^{-1}}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n+k}{2k}\binom{2k}{k}\frac{x}{x+k} = \sum_{k=0}^{n}(-1)^k\binom{n+k}{k}\binom{n}{k}\frac{x}{x+k} =$$

$$= \binom{n+x}{n}^{-1}\sum_{k=0}^{n}\binom{n+k}{k}\binom{n+x}{n-k}\binom{x+k-1}{k}(-1)^k = \binom{n+x}{n}^{-1}\sum_{k=0}^{n}\binom{-x}{k}\binom{n+x}{n-k}\binom{n+k}{n} \stackrel{T_+}{=}$$

$$= \binom{n+x}{n}^{-1}\binom{n}{0}\binom{n-x}{n} = (-1)^n\binom{x-1}{n}\binom{n+x}{n}^{-1}.$$

(3.101)

$$\boxed{\sum_{k=r}^{n}(-1)^k\binom{n}{k}\binom{2k}{k-r}2^{n-k} = \frac{(-1)^k+(-1)^n}{2}\binom{n}{(n-r)/2}}$$

$$S = \sum_{k=r}^{n}(-1)^k\binom{n}{k}\binom{2k}{k-r}2^{n-k} = (-1)^n\sum_k\binom{n}{k}(-2)^{n-k}\binom{2k}{k-r}.$$

$$\mathcal{G}\left(\binom{2k}{k-r}\right) = \mathcal{G}\left([t^k]t^r(1+t)^{2k}\right) = \left[\frac{w^r(1+w)}{1-w} \;\middle|\; w = t(1+w)^2\right].$$

$$S \stackrel{E}{=} (-1)^n[t^n]\frac{1}{1+2t}\left[\left[\frac{w^r(1+w)}{1-w} \;\middle|\; w = u(1+w)^2\right] \;\middle|\; u = \frac{t}{1+2t}\right] =$$

$$= (-1)^n[t^n]\frac{1}{1+2t}\left[\frac{w^r(1+w)}{1-w} \;\middle|\; w = \frac{t(1+w)^2}{1+2t}\right].$$

$$w = \frac{1-\sqrt{1-4t^2}}{2t}; \qquad \frac{1+w}{1-w} = \frac{1+2t}{\sqrt{1-4t^2}}.$$

$$S = (-1)^n[t^n]\frac{1}{1+2t}\frac{1+2t}{\sqrt{1-4t^2}}\left(\frac{1-\sqrt{1-4t^2}}{2t}\right)^r = (-1)^n[t^{n-r}]\frac{1}{\sqrt{1-4t^2}}\left(\frac{1-\sqrt{1-4t^2}}{2t^2}\right)^r =$$

$$\stackrel{y=t^2}{=} (-1)^n[y^{(n-r)/2}]\frac{1}{\sqrt{1-4y}}\left(\frac{1-\sqrt{1-4y}}{2y}\right)^r = (-1)^n[y^{(n+r)/2}]\frac{1}{\sqrt{1-4y}}\left(\frac{1-\sqrt{1-4y}}{2}\right)^r =$$

$$= (-1)^n\binom{n+r-r}{(n+r)/2-r} = (-1)^n\binom{n}{(n-r)/2}.$$

(3.102)

$$\boxed{\sum_{k=0}^{n/2}(-1)^k\binom{n}{k}\binom{2n-2k}{n+r} = 2^{n-r}\binom{n}{r}}$$

$$\sum_{k=0}^{n/2}(-1)^k\binom{n}{k}\binom{2n-2k}{n+r} \stackrel{A}{=} [t^{2n}]\frac{t^{n+r}}{(1-t)^{n+r+1}}\left[(1-u)^n \;\middle|\; u=t^2\right] = [t^n]\frac{t^r(1-t)^n(1+t)^n}{(1-t)^{n+r+1}} =$$

$$= [t^n]\left[\frac{w^r}{(1-w)^{r+1}}(1+w) \;\middle|\; w=t(1+w)\right] = [t^n]\frac{t^r}{(1-t)^r}\cdot\frac{(1-t)^{r+1}}{(1-2t)^{r+1}}\cdot\frac{1}{1-t} =$$

$$= [t^{n-r}]\frac{1}{(1-2t)^{r+1}} = \binom{-r-1}{n-r}(-2)^{n-r} = \binom{n}{r}2^{n-r}.$$

(3.103) - Grosswald:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+2r+k}{n+r}2^{n-k} = (-1)^{n/2}\binom{n+r}{n/2}\binom{n+2r}{r}\binom{n+r}{n}^{-1}$$

$$(-1)^n\sum_{k=0}^{n}\binom{n}{k}\binom{n+2r+k}{n+r}(-2)^{n-k} \stackrel{E}{=} (-1)^n[t^n]\frac{1}{1+2t}\left[\frac{1}{u^r(1-u)^{n+r+1}} \;\middle|\; u=\frac{t}{1+2t}\right] =$$

$$= (-1)^n[t^n]\frac{1}{1+2t}\cdot\frac{(1+2t)^r}{t^t}\cdot\frac{(1+2t)^{n+r+1}}{(1+t)^{n+r+1}} = (-1)^n[t^{n+r}]\frac{(1+2t)^{n+2r}}{(1+t)^{n+r+1}} =$$

$$= (-1)^n[t^{n+r}]\frac{(1+2t)^r}{(1+t}\left(\frac{1+2t}{1+t}\right)^{n+r} = (-1)^n[t^{n+r}]\left[\frac{(1+2w)^r}{1+w}\cdot\frac{(1+2w)(1+w)}{1+2w+2w^2} \;\middle|\; w=t\frac{1+2w}{1+w}\right].$$

$$1-t\phi'(w) = 1-\frac{w(1+w)}{1+2w}\cdot\frac{2(1+w)-(1+2w)}{(1+w)^2} = \frac{1+2w+2w^2}{(1+2w)(1+w)}.$$

$$w+w^2 = t+2tw; \qquad w^2+(1-2t)w-t=0; \qquad w=\frac{2t-1\pm\sqrt{1+4t^2}}{2}; \qquad 1+2w=2t+\sqrt{1+4t^2};$$

$$1+2w+2w^2 = 2t+\sqrt{1+4t^2}+1-2t+4t^2-(1-2t)\sqrt{1+4t^2} = 1+4t^2+2t\sqrt{1+4t^2} = \sqrt{1+4t^2}(2t+\sqrt{1+4t^2}.$$

$$S = (-1)^n[t^{n+r}]\left[\frac{(1+2w)^{r+1}}{1+2w+2w^2} \;\middle|\; w=\frac{\sqrt{1+4t^2}+2t}{2}\right] = (-1)^n[t^{n+r}]\frac{(2t+\sqrt{1+4t^2})^r}{\sqrt{1+4t^2}}.$$

$$F=\frac{2w}{\phi}+\sqrt{1+\frac{4w^2}{\phi^2}}; \qquad (\phi F-2w)^2=\phi^2+4w^2; \qquad \phi=\frac{4wF}{(F-1)(F+1)}; \qquad F(0)=1;$$

$$\phi=F; \qquad 4w=F^2-1; \qquad F=\sqrt{1+4w}=\phi; \qquad t=\frac{w}{\sqrt{1+4w}};$$

$$\sqrt{1+4t^2}=\sqrt{1+4w}-\frac{2w}{\sqrt{1+4w}}=W; \qquad \phi-t\phi'=\sqrt{1+4w}-\frac{2w}{\sqrt{1+4w}}=W.$$

$$S=(-1)^n[t^{n+r}]\frac{(\sqrt{1+4w})^r}{W}(\sqrt{1+4w})^{n+r-1}W=$$

$$=(-1)^n[t^{n+r}](\sqrt{1+4w})^{n+2r-1} = (-1)^n\binom{r+(n-1)/2}{n+r}4^{n+r}.$$

This expression seems to be equivalent to the formula given by Gould.

(3.103) - Grosswald:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+2r+k}{n+r}2^{n-k} = (-1)^{n/2}\binom{n+r}{n/2}\binom{n+2r}{r}\binom{n+r}{n}^{-1}$$

$$S = (-1)^n \sum_{k=0}^{n} \binom{n}{k} \binom{n+2r+k}{n+r} (-2)^{n-k} \stackrel{E}{=} [t^n] \frac{1}{1+2t} \left[\frac{1}{u^r(1-u)^{n+r+1}} \;\Big|\; u = \frac{t}{1+2t} \right] =$$

$$= (-1)^n [t^n] \frac{1}{1+2t} \frac{(1+2t)^r}{t^r} \frac{(1+2t)^{n+r+1}}{(1+t)^{n+r+1}} = (-1)^n [t^{n+r}] \frac{(1+2t)^{n+2r}}{(1+t)^{n+r+1}} =$$

$$= (-1)^n [t^{n+r}] \frac{(1+2t)^r}{1+t} \left(\frac{1+2t}{1+t} \right)^{n+r} = (-1)^n [t^{n+r}] \left[\frac{(1+2w)^r}{1+w} \frac{(1+2w)(1+w)}{1+2w+2w^2} \;\Big|\; w = t\frac{1+2w}{1+w} \right].$$

In fact:

$$1 - t\phi'(w) = 1 - \frac{w(1+w)}{1+2w} \frac{2(1+w)-(1+2w)}{(1+w)^2} = \frac{1+2w+2w^2}{(1+2w)(1+w)}.$$

$$w = t\frac{1+2w}{1+w} \quad \text{has solutions} \quad w = \frac{2t-1 \pm \sqrt{1+4t^2}}{2}.$$

If we set $1 + 2w = 2t + \sqrt{1+4t^2}$*:*

$$1 + 2w + 2w^2 = 1 + 4t^2 + 2t\sqrt{1+4t^2} = \sqrt{1+4t^2}(2t + \sqrt{1+4t^2}).$$

$$S = (-1)^n [t^{n+r}] \left[\frac{(1+2w)^{r+1}}{1+2w+2w^2} \;\Big|\; w = \frac{\sqrt{1+4t^2}+2t}{2} \right] = (-1)^n [t^{n+r}] \frac{(2t+\sqrt{1+4t^2})^r}{\sqrt{1+4t^2}}.$$

We apply the LIF:

$$F = \frac{2w}{\phi} + \sqrt{1 + \frac{4w^2}{\phi^2}} \qquad (\phi F - 2w)^2 = \phi^2 + 4w^2 \qquad \phi = \frac{4wF}{(F-1)(F+1)}.$$

Since we should have $F(0) = 1$*, we choose* $\phi = F$*, and so:*

$$4w = F^2 - 1 \qquad F = \sqrt{1+4w} = \phi \qquad t = \frac{w}{\sqrt{1+4w}}.$$

We now set:

$$W = \sqrt{1+4t^2} = \sqrt{1+4w} - \frac{2w}{\sqrt{1+4w}} \quad \text{so :} \quad \phi - t\phi' = \sqrt{1+4w} - \frac{2w}{\sqrt{1+4w}} = W.$$

$$S = (-1)^n [w^{n+r}] \frac{(\sqrt{1+4w})^r}{W} (\sqrt{1+4w})^{n+r-1} W = (-1)^n [w^{n+r}] (\sqrt{1+4w})^{n+2r-1} =$$

$$= (-1)^n \binom{r+(n-1)/2}{n+r} 4^{n+r}$$

which seems a better formula than that of Gould.

***(3.104)** - Grosswald:

$$\sum_{k=0}^{n-r} (-1)^k \binom{n}{k+r} \binom{n+r+k}{k} 2^{n-r-k} = (-1)^{(n-r)/2} \binom{n}{(n-r)/2} [n-r \text{ is even}]$$

***(3.105)** - Grosswald:

$$\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \binom{n+r+k}{n} 2^{n-r-k} = (-1)^{(n-r)/2} \binom{n}{(n-r)/2} \binom{n+r}{n} \binom{n}{r}^{-1} [n-r \text{ is even}]$$

***(3.106)**

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2n+2k}{n+k}3^{2n-k}=\binom{2n}{n}$$

(3.107)

$$\sum_{k=r}^{n/2}\binom{n-r}{k-r}\binom{n-k}{k}2^{n-2k}=\binom{2n-2r}{n}$$

$$\sum_{k=r}^{n/2}\binom{n-r}{k-r}\binom{n-k}{n-2k}2^{n-2k}\overset{B}{=}[t^n](1+2t)^n\left[u^r(1+u)^{n-r}\;\middle|\;u=\frac{t^2}{1+2t}\right]=$$

$$=[t^n](1+2t)^n\frac{t^{2r}}{(1+2t)^r}\cdot\frac{(1+2t+t^2)^{n-r}}{(1+2t)^{n-r}}=[t^n]t^{2r}(1+t)^{2n-2r}=\binom{2n-2r}{n-2r}.$$

(3.108) - *(NN)

$$\sum_{k=0}^{n}\binom{k+j}{k}\binom{k+j+m+1}{m}=\sum_{k=0}^{m}\binom{k+j}{k}\binom{k+j+n+1}{n}$$

(3.109)

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-k}{n}4^{n-k}=\binom{4n+1}{2n}$$

$$4^n\sum_{k=0}^{n}\binom{n+(n-k)}{n-k}\binom{2k}{k}\frac{1}{4^k}\overset{conv}{=}4^n[t^n]\frac{1}{(1-t)^{n+1}}\cdot\frac{1}{\sqrt{1-t}}=4^n\binom{-n-3/2}{n}=$$

$$=4^n\binom{2n+1/2}{n}\overset{Z45}{=}\binom{4n+1}{2n}$$

By applying the LIF, we obtain the generating function:

$$\mathcal{G}\left(\binom{4n+1}{2n}\right)=\frac{\sqrt{2}\sqrt{1-\sqrt{1-16t}}}{4\sqrt{t}\sqrt{1-16t}}.$$

(3.110)

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-k}{n}\frac{k}{(2n-k)2^k}=(-4)^n\binom{-1/4}{n}$$

$$\sum_{k=0}^{n}\left(2\binom{2n-k-1}{n-1}-\binom{2n-k}{k}\right)\binom{2k}{k}\frac{1}{2^k}\overset{A}{=}$$

$$=\left(2[t^{2n-1}]\frac{t^{n-1}}{(1-t)^n}-[t^{2n}]\frac{t^n}{(1-t)^{n+1}}\right)\left[\frac{1}{\sqrt{1-2u}}\;\middle|\;u=t\right]=[t^n]\frac{1-2t}{(1-t)^{n+1}}\cdot\frac{1}{\sqrt{1-2t}}=$$

$$= [t^n]\left[\frac{\sqrt{1-2w}}{1-w}\cdot\frac{1-w}{1-2w}\;\Big|\; w=\frac{t}{1-w}\right] = [t^n]\left[\frac{1}{\sqrt{1-2w}}\;\Big|\; w=\frac{1-\sqrt{1-4t}}{2}\right] =$$

$$= [t^n]\frac{1}{\sqrt[4]{1-4t}} = \binom{-1/4}{n}(-4)^n.$$

(3.111)

$$\boxed{\sum_{k=0}^{n/2}(-1)^k\binom{n}{k}\binom{2n-2k-1}{n-1} = 1 \qquad n\ge 1}$$

$$\sum_{k=0}^{n/2}\binom{2n-2k-1}{n-1}\binom{n}{k}(-1)^k \stackrel{A}{=} [t^{2n-1}]\frac{t^{n-1}}{(1-t)^n}\left[(1-u)^n \;\Big|\; u=t^2\right] =$$

$$= [t^n]\frac{(1-t^2)^n}{(1-t)^n} = [t^n](1+t)^n = 1.$$

(3.112)

$$\boxed{\sum_{k=0}^{n/2}(-1)^k\binom{n+1}{k}\binom{2n-2k-1}{n} = \frac{n(n+1)}{2} \qquad n\ge 1}$$

$$\sum_{k=0}^{n/2}\binom{2n-2k-1}{n}\binom{n+1}{k}(-1)^k \stackrel{A}{=} [t^{2n-1}]\frac{t^n}{(1-t)^{n+1}}\left[(1-u)^{n+1} \;\Big|\; u=t^2\right] =$$

$$= [t^{n-1}](1+t)^{n+1} = \binom{n+1}{n-1} = \frac{n(n+1)}{2}.$$

***(3.113)**

$$\boxed{\sum_{k=0}^{(r-1)n/r}(-1)^k\binom{n+1}{k}\binom{rn-rk}{n} = \binom{n+r-1}{n}}$$

(3.114) - *This is a fine example:*

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{2n}{n-k}\binom{2n+2k+1}{2k} = (-1)^n(n+1)4^n}$$

$$\sum_{k=0}^{n}\binom{2n+2k+1}{2n+1}\binom{2n}{n-k}(-1)^k \stackrel{B^*}{=} [t^{2n+1}](1+t)^{2n+1}\left[\frac{(1-u)^{2n}}{(-1)^n u^n} \;\Big|\; u=(1+t)^2\right] =$$

$$= [t^{2n+1}](1+t)^{2n+1}\frac{(-1)^n(-2)^{2n}t^{2n}(1+t/2)^{2n}}{(1+t)^{2n}} =$$

$$= (-4)^n\left([t^1](1+t/2)^{2n} + [t^0](1+t/2)^{2n}\right) = (-4)^n(n+1).$$

(3.115)

$$\boxed{\sum_{k=0}^{n}\binom{4n+1}{2n-2k}\binom{k+n}{n} = 4^n\binom{3n}{n}}$$

$$\sum_{k=0}^{n} \binom{4n+1}{2n+1+2k} \binom{n+k}{n} \stackrel{A}{=} [t^{4n+1}] \frac{t^{2n+1}}{(1-t)^{2n+2}} \left[\frac{1}{(1-u)^{n+1}} \;\middle|\; u = \frac{t^2}{(1-t)^2} \right] =$$

$$= [t^{2n}] \frac{1}{(1-2t)^{n+1}} = \binom{-n-1}{2n} (-2)^{2n} = 4^n \binom{3n}{n}$$

(3.116)

$$\boxed{\sum_{k=0}^{n} \binom{4n}{2n-2k} \binom{k+n}{n} = \frac{2}{3} 4^n \binom{3n}{n}}$$

$$\sum_{k=0}^{n} \binom{4n}{2n+2k} \binom{n+k}{n} \stackrel{A}{=} [t^{4n}] \frac{t^{2n}}{(1-t)^{2n+1}} \left[\frac{1}{(1-u)^{n+1}} \;\middle|\; u = \frac{t^2}{(1-t)^2} \right] =$$

$$= [t^{2n}] \frac{1-t}{(1-2t)^{n+1}} = \binom{-n-1}{2n} (-2)^{2n} - \binom{-n-1}{2n-1} (-2)^{2n-1} =$$

$$= \binom{3n}{n} 4^n - \binom{3n-1}{n} \frac{4^n}{2} = \frac{2}{3} 4^n \binom{3n}{n}$$

(3.117)

$$\boxed{\sum_{k=0}^{n/2} (-1)^k \binom{n}{k} \binom{2n-2k}{n} = 2^n}$$

$$\sum_{k=0}^{n/2} \binom{2n-2k}{n} \binom{n}{k} (-1)^k \stackrel{A}{=} [t^{2n}] \frac{t^n}{(1-t)^{n+1}} \left[(1-u)^n \;\middle|\; u = t^2 \right] = [t^n] \frac{(1+t)^n}{1-t} =$$

$$= [t^n] \left[\frac{1+w}{1-w} \;\middle|\; w = t(1+w) \right] = [t^n] \frac{1}{1-t} \cdot \frac{1-t}{1-2t} = [t^n] \frac{1}{1-2t} = 2^n$$

(3.118)

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} \binom{k}{j} x^k = \binom{n}{j} x^j (1+x)^{n-j}}$$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{k}{j} x^k \stackrel{E}{=} [t^n] \frac{1}{1-t} \left[\frac{x^j u^j}{(1-xu)^{j+1}} \;\middle|\; u = \frac{t}{1-t} \right] =$$

$$= [t^n] \frac{1}{1-t} \cdot \frac{x^j t^j}{(1-t)^j} \cdot \frac{(1-t)^{j+1}}{(1-(x+1)t)^{j+1}} = x^j [t^{n-j}] \frac{1}{(1-(x+1)t)^{j+1}} =$$

$$= x^j \binom{-j-1}{n-j} (-1)^{n-j} (x+1)^{n-j} = x^j \binom{n}{n-j} (x+1)^{n-j} = \binom{n}{j} x^j (1+x)^{n-j}.$$

(3.119)

$$\boxed{\sum_{k=j}^{n} (-1)^k \binom{n}{k} \binom{k}{j} = (-1)^n \delta_{n,j}}$$

$$\binom{n}{j} \sum_{k=j}^{n} (-1)^k \binom{n-j}{n-k} \stackrel{B}{=} \binom{n}{j} [t^n] (1+t)^{n-j} \left[\frac{1}{1+u} \;\middle|\; u = t \right] =$$

$$= \binom{n}{j}[t^n](1+t)^{n-j-1} = (-1)^n \delta_{n,j}.$$

(3.120)

$$\boxed{\sum_{k=j}^{n/2} \binom{n}{2k}\binom{k}{j} = 2^{n-2j-1}\binom{n-j}{j}\frac{n}{n-j}}$$

$$\sum_{k=j}^{n/2} \binom{n}{n-2k}\binom{k}{j} \stackrel{B}{=} [t^n](1+t)^n \left[\frac{u^j}{(1-u)^{j+1}} \;\Big|\; u = t^2\right] = [t^n]\frac{t^{2j}(1+t)^n}{(1-t^2)^{j+1}} =$$

$$= [t^n]\left[\frac{w^{2j}(1+w)}{(1-w^2)^{j+1}} \;\Big|\; w = t(1+w)\right] = [t^n]\frac{t^{2j}}{(1-t)^{2j}} \cdot \frac{(1-t)^{2j+2}}{(1-2t)^{j+1}} \cdot \frac{1}{1-t} = [t^{n-2j}]\frac{1-t}{(1-2t)^{j+1}} =$$

$$= \binom{-j-1}{n-2j}(-2)^{n-2j} - \binom{-j-1}{n-2j-1}(-2)^{n-2j-1} =$$

$$= 2^{n-2j-1}\left(2\binom{n-j}{j} - \frac{n-2j}{n-j}\binom{n-j}{j}\right) = 2^{n-2j-1}\binom{n-j}{j}\frac{n}{n-j}$$

(3.121)

$$\boxed{\sum_{k=j}^{n/2} \binom{n+1}{2k+1}\binom{k}{j} = 2^{n-2j}\binom{n-j}{j}}$$

$$\sum_{k=j}^{n/2} \binom{n+1}{2k+1}\binom{k}{j} \stackrel{A}{=} [t^{n+1}]\frac{t}{(1-t)^2}\left[\frac{u^j}{(1-u)^{j+1}} \;\Big|\; u = \frac{t^2}{(1-t)^2}\right] =$$

$$= [t^n]\frac{1}{(1-t)^2} \cdot \frac{t^{2j}}{(1-t)^{2j}} \cdot \frac{(1-t)^{2j+2}}{(1-2t)^{j+1}} = [t^{n-2j}]\frac{1}{(1-2t)^{j+1}} = \binom{-j-1}{n-2j}(-2)^{n-2j} = 2^{n-2j}\binom{n-j}{j}.$$

***(3.122)**

$$\boxed{\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\binom{n}{k}\binom{n+k}{k} = 2H_n}$$

***(3.123)**

$$\boxed{\sum_{k=1}^{n} (-1)^k \binom{n}{k}\binom{n+k-1}{k} H_k = \frac{(-1)^n}{n}}$$

***(3.124)** - R. R. Goldberg:

$$\boxed{\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k-1}{k} H_{n+k-1} = \frac{(-1)^n}{n}}$$

***(3.125)**

$$\sum_{k=1}^{n} \binom{n}{k}^2 H_k = \binom{2n}{n}(2H_n - H_{2n})$$

***(3.126)**

$$\sum_{k=0}^{\infty} \binom{1/2}{k}^2 \frac{1}{2n+2k} = \sum_{k=0}^{\infty} \binom{1/2}{k}^2 \frac{1}{2n-2k+1} \qquad (n \geq 1)$$

***(3.127)**

$$\sum_{k=0}^{\infty} \binom{2k}{k}^2 \left(\frac{z}{4}\right)^{2k} = \frac{2}{\pi} \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{1 - z^2 \sin(x)^2}}$$

***(3.128)**

$$\sum_{k=0}^{\infty} \binom{2k}{k}^2 \left(\frac{z}{4}\right)^{2k} \frac{1}{1-2k} = \frac{2}{\pi} \int_0^{\pi/2} \sqrt{1 - z^2 \sin(x)^2}\ \mathrm{d}z$$

***(3.129)** - Bessel polynomials:

$$y_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} k! \left(\frac{x}{2}\right)^k = \frac{\exp(2/x)}{2^n} D_x^n \left(\frac{x^{2n}}{\exp(2/x)}\right)$$

***(3.130)** - Hermite polynomials:

$$H_n(x) = \sum_{k=0}^{n/2} (-1)^k \binom{n}{k}\binom{n-k}{k} k!(2x)^{n-2k} = (-1)^n \exp(x^2) D_x^n \exp(-x^2)$$

***(3.131)** - Jacobi polynomials:

$$P_n^{(a,b)}(x) = \sum_{k=0}^{n} \binom{n+a}{k}\binom{n+b}{n-k} \left(\frac{x-1}{2}\right)^{n-k} \left(\frac{x+1}{2}\right)^k =$$
$$\frac{(-1)^n}{2^n n!}(1-x)^{-a}(1+x)^{-b} D_x^n \left((1-x)^{a+n}(1+x)^{b+n}\right)$$

***(3.132)** - Legendre polynomials. Definition by formula of Rodriguez:

$$P_n(x) = \frac{1}{2^n n!} D_x^n (x^2-1)^n$$

In terms of generating functions:

$$\sum_{n=0}^{\infty} t^n P_n(x) = \frac{1}{\sqrt{1-2xt+t^2}}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} P_n(x) = \exp(tx) J_0\left(t\sqrt{1-x^2}\right).$$

In terms of hypergeometric function:

$$P_n(x) = F\left(\begin{matrix} n+1, \; n \\ 1 \end{matrix} \;\middle|\; \frac{1-x}{2}\right).$$

Many summation of the form S:2/0 in the literature involve $P_n(x)$, and the following table of equivalent forms of $P_n(x)$ may be found of use.

***(3.133)**

$$\boxed{P_n(x) = \frac{1}{2^n} \sum_{k=0}^{n/2} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}}$$

***(3.134)**

$$\boxed{P_n(x) = \left(\frac{x-1}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{x+1}{x-1}\right)^k}$$

***(3.135)**

$$\boxed{P_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \frac{(x^2-1)^{k/2}}{2^k} \left(x - \sqrt{x^2-1}\right)^{n-k}}$$

***(3.136)**

$$\boxed{P_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \frac{(x^2-1)^{k/2}}{2^k} \left(x - \sqrt{x^2-1}\right)^{n-k}}$$

***(3.137)**

$$\boxed{P_n(x) = \sum_{k=0}^{n/2} \binom{n}{2k} \binom{2k}{k} \frac{x^{n-2k}(x^2-1)^k}{4^k}}$$

***(3.138)** - R. P. Kelisky:

$$\boxed{\sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} x^{2k} = (4x)^n P_n\left(\frac{x+x^{-1}}{2}\right) = \frac{2 \cdot 4^n}{\pi} \int_0^{\pi/2} \left(x^2 \sin(t)^2 + \cos(t)^2\right)^n \mathrm{d}t}$$

***(3.139)**

$$\sum_{k=0}^{n} \binom{-1/2}{k} \binom{-1/2}{n-k} x^{2k} = (-x)^n P_n\left(\frac{x+x^{-1}}{2}\right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k-1/2}{n} x^{2k}$$

Integral of Laplace:

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left(x + \sqrt{x^2-1} \cdot \cos(t)\right)^n \mathrm{d}t.$$

***(3.140)** - Finite series of I. J. Good, valid for integral $t > n$:

$$P_n(x) = \frac{1}{t} \sum_{k=0}^{t-1} \left(x + \sqrt{x^2-1} \cdot \cos\frac{2\pi k}{t}\right)^n$$

Integral form of Schläfli:

$$P_n(x) = \frac{1}{2\pi i} \int_C \frac{(t^2-1)^n}{2^n (t-x)^{n+1}} \mathrm{d}t$$

where C is a circle counterclockwise about the point $t = x$.

***(3.141)** - In general one may define $P_n(x)$ for all real values of n, and relations such as the following are found:

$$(1-x)^{n+1} \sum_{k=0}^{\infty} \binom{-n-1}{k}^2 x^k = \frac{1}{(1-x)^n} \sum_{k=0}^{\infty} \binom{n}{k}^2 x^k$$

or if one defines:

$$f(z) = P_z\left(\frac{1+x}{1-x}\right) = \frac{1}{(1-x)^z} \sum_{k=0}^{\infty} \binom{z}{k}^2 x^k,$$

then the above reads: $f(n) = f(-n-1)$.

(3.142)

$$\sum_{k=0}^{n} \frac{x}{x+kz} \binom{x+kz}{k} \frac{y}{y+(n-k)z} \binom{y+(n-k)z}{n-k} = \frac{x+y}{x+y+nz} \binom{x+y+nz}{n}$$

$$\sum_{k=0}^{n} \frac{x}{x+kz} \binom{x+kz}{k} \frac{y}{y+(n-k)z} \binom{y+(n-k)z}{n-k} \overset{conv,GO1}{=} [t^n] \left[(1+w)^{x+y} \;\middle|\; w = t(1+w)^z\right] \overset{LIF}{=}$$

$$= \frac{1}{n}[t^{n-1}](x+y)(1+t)^{x+y-1}(1+t)^{zn} = \frac{x+y}{n} \binom{x+y+zn-1}{n-1} = \frac{x+y}{x+y+nz} \binom{x+y+nz}{n}$$

(3.143) - *It is possible to prove, in the same way, analogous identities relative to the first generating function of Gould and to the generating functions of Abel.*

$$\sum_{k=0}^{n} \binom{x+kz}{k} \binom{y+(n-k)z}{n-k} = \sum_{k=0}^{n} \binom{x+p+kz}{k} \binom{y-p+(n-k)z}{n-k}$$

$$(1) \quad S \overset{conv,GO2}{=} \left[\frac{(1+w)^{x+1}}{1-(z-1)w} \;\middle|\; w = t(1+w)^z\right] \cdot \left[\frac{(1+w)^{y+1}}{1-(z-1)w} \;\middle|\; w = t(1+w)^z\right] =$$

$$= \left[\frac{(1+w)^{x+y+2}}{1-(z-1)w} \;\Big|\; w = t(1+w)^z \right].$$

$$(2) \quad S \overset{conv,GO2}{=} \left[\frac{(1+w)^{x+p+1}}{1-(z-1)w} \;\Big|\; w = t(1+w)^z \right] \cdot \left[\frac{(1+w)^{y-p+1}}{1-(z-1)w} \;\Big|\; w = t(1+w)^z \right] =$$

$$= \left[\frac{(1+w)^{x+y+2}}{1-(z-1)w} \;\Big|\; w = t(1+w)^z \right].$$

(3.144) - Jensen:

$$\boxed{\sum_{k=0}^{n} \binom{x+kz}{k} \binom{y-kz}{n-k} = \sum_{k=0}^{n} \binom{x+y-k}{n-k} z^k}$$

$$\binom{y-kz}{n-k} = [t^{n-k}](1+t)^{y-kz} \rightsquigarrow \mathcal{R}((1+t)^y, (1+t)^{-z}).$$

$$(1) \quad \sum_{k=0}^{n} \binom{x+kz}{k} \binom{y-kz}{n-k} \overset{GO2}{=} [t^n](1+t)^y \left[\left[\frac{(1+w)^{x+1}}{1-(z-1)w} \;\Big|\; w = u(1+w)^z \right] \;\Big|\; u = \frac{t}{(1+t)^z} \right] =$$

$$= [t^n] \frac{(1+t)^{x+y+1}}{1-(z-1)t}.$$

$$(2) \quad \sum_{k=0}^{n} \binom{x+y-k}{n-k} z^k \overset{B}{=} [t^n](1+t)^{x+y} \left[\frac{1}{1-zu} \;\Big|\; u = \frac{t}{1+t} \right] = [t^n] \frac{(1+t)^{x+y+1}}{1-(z-1)t}.$$

(3.145) - Jensen:

$$\boxed{\sum_{k=0}^{n} \binom{x+kz}{k} \binom{p-x-kz}{n-k} = \begin{cases} z^{p+1}(z-1)^{n-p-1} & 0 \le p \le n-1 \\ \dfrac{z^{n+1}-1}{z-1} & p = n \end{cases}}$$

$$\sum_{k=0}^{n} \binom{x+kz}{k} \binom{p-x-kz}{n-k} \overset{GO2}{=} [t^n](1+t)^{p-x} \left[\left[\frac{(1+w)^{x+1}}{1-(z-1)w} \;\Big|\; w = u(1+w)^z \right] \;\Big|\; u = \frac{t}{(1+t)^z} \right] =$$

$$= [t^n](1+t)^{p-x} \frac{(1+t)^{x+1}}{1-(z-1)t} = [t^n] \frac{(1+t)^{p+1}}{1-(z-1)t} = \sum_{k=0}^{n} \binom{p+1}{n-k} (z-1)^k.$$

$$(\mathbf{p} = \mathbf{n}) \quad \sum_{k=0}^{n} \binom{n+1}{n-k} (z-1)^k = [t^n] \frac{(1+t)^{n+1}}{1-(z-1)t} = [t^n] \left[\frac{(1+w)^2}{1-(z-1)w} \;\Big|\; w = t(1+w) \right] =$$

$$= [t^n] \frac{1}{(1-t)^2} \cdot \frac{1-t}{1-zt} = \frac{z^{n+1}-1}{z-1}.$$

$$(\mathbf{p} < \mathbf{n}) \quad \sum_{k=0}^{n} \binom{p+1}{k} (z-1)^{n-k} = (z-1)^n \sum_{k=0}^{p+1} \binom{p+1}{k} \frac{1}{(z-1)^k} \overset{E}{=}$$

$$= (z-1)^n [t^{p+1}] \frac{1}{1-t} \left[\frac{1}{1-u/(z-1)} \;\Big|\; u = \frac{t}{1-t} \right] = (z-1)^n [t^{p+1}] \frac{1}{1-zt/(z-1)} =$$

$$= (z-1)^n \frac{z^{p+1}}{(z-1)^{p+1}} = z^{p+1}(z-1)^{n-p-1}.$$

***(3.146)** - Hegen / Rothe:

$$\sum_{k=0}^{n} \binom{x+kz}{k} \binom{y-kz}{n-k} \frac{p+qk}{(x+kz)(y-kz)} = \frac{p(x+y-nz)+nxq}{x(x+y)(y-nz)} \binom{x+y}{n}$$

***(3.147)** - Van der Corput:

$$\sum_{k=1}^{n-1} \binom{kz}{k} \binom{nz-kz}{n-k} \frac{1}{kz(nz-kz)} = \frac{2}{nz} \binom{nz}{n} \sum_{k=1}^{n-1} \frac{1}{nz-n+k}$$

***(3.148)** - Chung (cf. (7.18):

$$\binom{nd}{n} = d(d-1) \sum_{k=1}^{n} \frac{(dk-2)!}{(k-1)!(dk-k)!} \binom{nd-kd}{n-k}$$

***(3.149)** - cf. (3.27):

$$\sum_{k=0}^{n} \binom{2x+1}{2k} \binom{x-k}{n-k} = 4^n \binom{x+n}{2n}$$

(3.150) - *The case $j > n$ has not been closed.*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x+kz}{j} = \begin{cases} 0, & 0 \le j < n \\ (-z)^n, & j = n \\ (-z)^n \frac{2x+(z-1)n}{2}, & j = n+1 \end{cases}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x+kz}{j} \stackrel{B}{=} [t^j](1+t)^x \left[(1-u)^n \;\middle|\; u = (1+t)^z \right] = [t^j](1+t)^x (1-(1+t)^z)^n =$$

$$= [t^j](1+t)^x (-z)^n t^n (1 + \frac{n(z-1)}{2} t + \cdots) = \begin{cases} 0, & 0 \le j < n \\ (-z)^n, & j = n \\ (-z)^n \frac{2x+(z-1)n}{2}, & j = n+1 \end{cases} .$$

(3.151)** - ***NN. Knuth:

$$\sum_{k=0}^{n} \binom{x+k}{m} \binom{x-k}{m} k = \frac{m+1}{2} \left(\binom{x+1}{m+1} \binom{x}{m+1} - \binom{x+n+1}{m+1} \binom{x-n}{m+1} \right)$$

(3.152)** - ***NN. Knuth (extends (3.14)):

$$\sum_{k=0}^{m} \binom{x}{k} \binom{y}{n-k} (nx-(x+y)k) = (m+1)(n-m) \binom{x}{m+1} \binom{y}{n-m}$$

(3.153)** - ***NN. Knuth:

$$\sum_{k=0}^{m}\binom{-x}{k}\binom{x}{n+k}=\frac{x}{x-n}\binom{-x-1}{m}\binom{x-1}{n+m}$$

(3.154)** - ***NN. Gould (special case of Stanley's (6.52)):

$$\sum_{k=0}^{m}\binom{-x}{a-k}\binom{x}{b-k}=\binom{x+a}{b}\binom{b-x}{a}\qquad m=\min(a,b)$$

(3.155)** - ***NN. Knuth:

$$\sum_{k=0}^{s-1}\binom{k}{n}\binom{k+m}{m}=\frac{s-n}{m+n+1}\binom{s}{n}\binom{s+m}{m}$$

(3.156) Variation of (3.1):

$$\sum_{k=a}^{n}\binom{x}{k-a}\binom{y}{n-k}=\binom{x+y}{n-a}$$

$$\sum_{k=a}^{n}\binom{x}{k-a}\binom{y}{n-k}\overset{B}{=}[t^n](1+t)^y\left[u^a(1+u)^x\;\middle|\;u=t\right]=[t^{n-a}](1+t)^{x+y}=\binom{x+y}{n-a}.$$

***(3.157)**

$$\sum_{k=0}^{n-1}\binom{2x}{2k+1}\binom{x-k-1}{n-k-1}=\frac{n4^n}{x+n}\binom{x+n}{2n}$$

***(3.158)**

$$\sum_{k=0}^{n}\binom{2x}{2k+1}\binom{x-k-1}{n-k}=\frac{2\cdot 4^n(x+n)}{2n+1}\binom{x-1+n}{2n}$$

***(3.159)**

$$\sum_{k=0}^{n}\binom{2x}{2k+1}\binom{x-k}{n-k}=\frac{4^n(2x^2+n)}{(2n+1)(x+n)}\binom{x+n}{2n}$$

(3.160) - *Egorychev (2.1.4) Ex. 11.iii. This is one of the* Moriarty identities *by H. T. Davis:*

$$\sum_{k=a}^{n}(-1)^k\binom{k}{a}\binom{n+k}{2k}4^k\frac{n}{n+k}=(-1)^n\binom{n+a}{2a}\frac{n}{n+a}4^a$$

$$\sum_{k=a}^{n}\binom{n+k}{2k}\frac{n}{n+k}(-4)^k\binom{k}{a}=\sum_{k=a}^{n}\left(\binom{n+k}{2k}-\frac{1}{2}\binom{n+k-1}{2k-1}\right)\binom{k}{a}(-4)^k\overset{A}{=}$$

$$=\left([t^n]\frac{1}{1-t}+\frac{1}{2}[t^{n-1}]\frac{1}{t}\right)\left[\frac{(-4u)^a}{(1+4u)^{a+1}}\;\middle|\;u=\frac{t}{(1-t)^2}\right]=$$

$$=(-4)^a[t^n]\frac{1+t}{2(1-t)}\cdot\frac{t^a}{(1-t)^{2a}}\cdot\frac{(1-t)^{2a+2}}{(1+t)^{2a+2}}=\frac{(-4)^a}{2}[t^{n-a}]\frac{1-t}{(1+t)^{2a+1}}=$$

$$=\left(\binom{-2a-1}{n-a}-\binom{-2a-1}{n-a-1}\right)\frac{(-4)^a}{2}=\left(\binom{n+a}{2n}+\binom{n+a}{2n}\frac{n-a}{n+a}\right)\frac{(-1)^n4^a}{2}=$$

$$=(-1)^n\binom{n+a}{2a}\frac{n}{n+a}4^a.$$

(3.161) - *Egorychev (2.1.4) Ex. 11.iv. This is one of the* Moriarty identities *by H. T. Davis:*

$$\boxed{\sum_{k=a}^{n}(-1)^k\binom{k}{a}\binom{n+k}{2k}4^k\frac{2n+1}{2k+1}=(-1)^n\binom{n+a}{2a}4^a}$$

$$\sum_{k=a}^{n}\binom{n+k}{2k}\frac{2n+1}{2k+1}(-4)^k\binom{k}{a}=\sum_{k=a}^{n}\left(2\binom{n+k+1}{2k+1}-\binom{n+k}{2k}\right)\binom{k}{a}(-4)^k\overset{A}{=}$$

$$=\left([t^{n+1}]\frac{2t}{(1-t)^2}-[t^n]\frac{1}{1-t}\right)\left[\frac{(-4u)^a}{(1+4u)^{a+1}}\;\middle|\;u=\frac{t}{(1-t)^2}\right]=$$

$$=(-4)^a[t^n]\frac{1+t}{(1-t)^2}\cdot\frac{t^a}{(1-t)^{2a}}\cdot\frac{(1-t)^{2a+2}}{(1+t)^{2a+2}}=(-4)^a[t^{n-a}]\frac{1}{(1+t)^{2a+1}}=$$

$$=\binom{-2a-1}{n-a}(-4)^a=\binom{n+a}{2a}(-1)^n4^a.$$

(3.162) - *Egorychev (2.1.4) Ex. 11.i. This is one of the* Moriarty identities *by H. T. Davis:*

$$\boxed{\sum_{k=a}^{n}(-1)^k\binom{k}{a}\binom{n+k}{2k}4^k=(-1)^n\binom{n+a}{2a}4^a\frac{2n+1}{2a+1}}$$

$$\sum_{k=a}^{n}\binom{n+k}{2k}\binom{k}{a}(-4)^k\overset{A}{=}[t^n]\frac{1}{1-t}\left[\frac{(-4u)^a}{(1+4u)^{a+1}}\;\middle|\;u=\frac{t}{(1-t)^2}\right]=$$

$$=(-4)^a[t^n]\frac{1}{1-t}\cdot\frac{t^a}{(1-t)^{2a}}\cdot\frac{(1-t)^{2a+2}}{(1+t)^{2a+2}}=(-4)^a[t^{n-a}]\frac{1-t}{(1+t)^{2a+2}}=$$

$$=(-4)^a\left(\binom{-2a-2}{n-a}-\binom{-2a-2}{n-a-1}\right)=4^a(-1)^n\binom{a+n}{2a}\left(\frac{a+n+1}{2a+1}+\frac{n-a}{2a+1}\right)=$$

$$=(-1)^n\binom{n+a}{2a}4^a\frac{2n+1}{2a+1}.$$

(3.163) - Carlitz:

$$\boxed{\sum_{k=0}^{n}\binom{n}{k}\binom{k/2}{m}=\frac{n}{m}\binom{n-m-1}{m-1}2^{n-2m}}$$

$$S=\sum_{k=0}^{n}\binom{n}{k}\binom{k/2}{m}=[t^m]\sum_{k=0}^{n}\binom{n}{k}\sqrt{1+t}^k=[t^m](1+\sqrt{1+t})^n=2^n[t^m]\left(\frac{1+\sqrt{1+t}}{2}\right)^n.$$

By setting $1+y=(1+\sqrt{1+t})/2$ we find $y=t/(4+4y)$.

$$S=2^n[t^m]\left[(1+y)^n\ \Big|\ y=\frac{t}{4(1+y)}\right]=\frac{2^n}{m}[t^{m-1}]n(1+t)^{n-1}\frac{1}{4^m(1+t)^m}=$$

$$=\frac{n2^n}{m4^m}[t^{m-1}](1+t)^{n-m-1}=\frac{n2^n}{m4^m}\binom{n-m-1}{m-1}.$$

(3.164) - Rosenstock, Gray, Riordan:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k/2}{m}=(-1)^m2^{n-2m}\left(\binom{2m-n-1}{m-1}-\binom{2m-n-1}{m}\right)$$

$$S=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k/2}{m}=[t^m]\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sqrt{1+t}^k=[t^m](1-\sqrt{1+t})^n.$$

By setting $1-\sqrt{1+t}=2y$ *we find* $y=t/(4y-4)$ *and so:*

$$S=[t^m]\left[2^ny^n\ \Big|\ y=\frac{t}{-4(1-y)}\right]=\frac{2^n}{m}[t^{m-1}]nt^{n-1}\frac{(-1)^m}{4^m(1-t)^m}=$$

$$=\frac{2^n}{4^m}\cdot\frac{(-1)^mn}{m}[t^{m-n}]\frac{1}{(1-t)^m}=\frac{(-1)^m2^nn}{4^mm}\binom{-m}{m-n}(-1)^{m-n}=$$

$$=\frac{(-1)^m2^nn}{4^mm}\binom{2m-n-1}{m-1}=(-1)^m2^{n-2m}\left(\binom{2m-n-1}{m-1}-\binom{2m-n-1}{m}\right).$$

(3.165) - ***NN***. E. Catalan (1842):

$$\sum_{k=0}^{n}\binom{p-k}{p-n}\binom{q+k+1}{m}=\sum_{k=0}^{m}\binom{q-k}{q-m}\binom{p+k+1}{n}=\binom{p+q+2}{n-m+q+1}$$

$$(1)\quad\sum_{k=0}^{n}\binom{p-k}{n-k}\binom{q+k+1}{m}\stackrel{B}{=}[t^n](1+t)^p\left[\frac{u^{m-q-1}}{(1-u)^{m+1}}\ \Big|\ u=\frac{t}{1+t}\right]=$$

$$=[t^n](1+t)^p\frac{t^{m-q-1}}{(1+t)^{m-q-1}}(1+t)^{m+1}=[t^{n-m+q+1}](1+t)^{p+q+2}=\binom{p+q+2}{n-m+q+1}.$$

$$(2)\quad\sum_{k=0}^{n}\binom{q-k}{m-k}\binom{p+k+1}{n}\stackrel{B}{=}[t^m](1+t)^q\left[\frac{u^{n-p-1}}{(1-u)^{n+1}}\ \Big|\ u=\frac{t}{1+t}\right]=$$

$$=[t^m](1+t)^q\frac{t^{n-p-1}}{(1+t)^{n-p-1}}(1+t)^{n+1}=[t^{m-n+p+1}](1+t)^{p+q+2}=\binom{p+q+2}{m-n+p+1}.$$

(3.166) - Joel L. Brenner.

$$\sum_{k=0}^{n}\binom{2n+1}{2k+1}\binom{2n-2k}{n-k}(x^2+y^2)^{2k+1}(xy)^{2n-2k}=\sum_{k=0}^{2n+1}\binom{2n+1}{k}^2x^{2k}y^{4n+2-2k}$$

$$\sum_{k=0}^{n}\binom{2n+1}{2k+1}\binom{2n-2k}{n-k}(x^2+y^2)^{2k+1}(xy)^{2n-2k}\overset{B}{=}$$
$$=[t^n](1+x^2y^2t)^{2n}\left[\frac{(1+(x^2+y^2)\sqrt{u})^{2n+1}-(1-(x^2+y^2)\sqrt{u})^{2n+1}}{2\sqrt{u}}\;\Big|\; u=\frac{t}{(1+x^2y^2t)^2}\right]=$$
$$=[t^n]\frac{(1+x^2y^2t+(x^2+y^2)\sqrt{t})^{2n+1}-(1+x^2y^2t-(x^2+y^2)\sqrt{t})^{2n+1}}{2\sqrt{t}}=$$
$$=[t^{2n+1}](1+x^2t)^{2n+1}(1+y^2t)^{2n+1}=\sum_{k=0}^{2n+1}\binom{2n+1}{k}(x^2)^k\binom{2n+1}{2n+1-k}(y^2)^{2n+1-k}.$$

(3.167) - Joel L. Brenner.

$$\boxed{\sum_{k=0}^{n}\binom{2n}{2k}\binom{2n-2k}{n-k}(x^2+y^2)^{2k}(xy)^{2n-2k}=\sum_{k=0}^{2n}\binom{2n}{k}^2x^{2k}y^{4n-2k}}$$

$$\sum_{k=0}^{n}\binom{2n}{2k}\binom{2n-2k}{n-k}(x^2+y^2)^{2k}(xy)^{2n-2k}\overset{B}{=}$$
$$=[t^n]\frac{(1+x^2y^2t)^{2n}}{2}\left[(1+(x^2+y^2)\sqrt{u})^{2n}+(1-(x^2+y^2)\sqrt{u})^{2n}\;\Big|\; u=\frac{t}{(1+x^2y^2t)^2}\right]=$$
$$=[t^n]\frac{1}{2}\left((1+x^2y^2t+(x^2+y^2)\sqrt{t})^{2n}+(1+x^2y^2t-(x^2+y^2)\sqrt{t})^{2n}\right)=$$
$$=[t^{2n}](1+x^2t)^{2n}(1+y^2t)^{2n}=\sum_{k=0}^{2n}\binom{2n}{k}(x^2)^k\binom{2n}{2n-k}(y^2)^{2n-k}.$$

***(3.168)** - Equivalent to result of Graham and Riordan:

$$\boxed{\sum_{k=0}^{n}\frac{2k+1}{n+k+1}\binom{x-1-k}{n-k}\binom{x+k}{n+k}=\binom{x}{n}^2}$$

(3.169)

$$\boxed{\sum_{k=0}^{n/2}\binom{n+1}{2k+1}\binom{x+k}{n}=\binom{2x}{n}}$$

$$\sum_{k=0}^{n/2}\binom{n+1}{n-2k}\binom{x+k}{n}\overset{B}{=}[t^n](1+t)^{n+1}\left[\frac{u^{n-x}}{(1-u)^{n+1}}\;\Big|\; u=t^2\right]=[t^{2x-n}]\frac{(1+t)^{n+1}}{(1-t^2)^{n+1}}=$$
$$=[t^{2x-n}]\frac{1}{(1-t)^{n+1}}=\binom{-n-1}{2x-n}(-1)^{2x-n}=\binom{n+1+2x-n-1}{2x-n}=\binom{2x}{n}.$$

(3.170)

$$\boxed{\sum_{k=0}^{(n+1)/2}\binom{n+1}{2k}\binom{x+k}{n}=\binom{2x+1}{n}}$$

$$\sum_{k=0}^{(n+1)/2}\binom{n+1}{2k}\binom{x+k}{n} \stackrel{A}{=} [t^{n+1}]\frac{1}{1-t}\left[\frac{u^{n-x}}{(1-u)^{n+1}} \;\middle|\; u=\frac{t^2}{(1-t)^2}\right] =$$

$$= [t^{n+1}]\frac{1}{1-t}\cdot\frac{t^{2n-2x}}{(1-t)^{2n-2x}}\cdot\frac{(1-t)^{2n+2}}{(1-2t)^{n+1}} = [t^n]t^{2n-2x-1}\frac{(1-t)^{2x+1}}{(1-2t)^{n+1}} =$$

$$= [t^n]\left[\frac{(1-w)^{2x+1}(1-2w)}{w^{2x+1}(1-2w)} \;\middle|\; w=\frac{tw^2}{1-2w}\right] = [t^n]\left[\left(\frac{1-w}{w}\right)^{2x+1} \;\middle|\; w=\frac{1}{2+t}\right] =$$

$$= [t^n](1+t)^{2x+1} = \binom{2x+1}{n}.$$

(3.171)

$$\boxed{\sum_{k=0}^{n}\binom{2n+1}{2k+1}\binom{x+k}{2n} = \binom{2x}{2n}}$$

$$\sum_{k=0}^{n}\binom{2n+1}{2k+1}\binom{x+k}{2n} \stackrel{A}{=} [t^{2n+1}]\frac{t}{(1-t)^2}\left[\frac{u^{2n-x}}{(1-u)^{2n+1}} \;\middle|\; u=\frac{t^2}{(1-t)^2}\right] =$$

$$= [t^{2n}]\frac{1}{(1-t)^2}\cdot\frac{t^{4n-2x}}{(1-t)^{4n-2x}}\cdot\frac{(1-t)^{4n+2}}{(1-2t)^{2n+1}} = [t^{2n}]t^{4n-2x}\frac{(1-t)^{2x}}{(1-2t)^{2n+1}} =$$

$$= [t^m]t^{2m-2x}\frac{(1-t)^{2x}}{(1-2t)^{m+1}} = [t^m]\left[\frac{(1-w)^{2x}(1-2w)}{w^{2x}(1-2w)} \;\middle|\; w=\frac{tw^2}{1-2w}\right] =$$

$$= [t^m]\left[\left(\frac{1-w}{w}\right)^{2x} \;\middle|\; w=\frac{1}{2+t}\right] = [t^m](1+t)^{2x} = \binom{2x}{m} = \binom{2x}{2n}.$$

(3.172)

$$\boxed{\sum_{k=0}^{n}\binom{2n+2}{2k+1}\binom{x+k}{2n+1} = \binom{2x}{2n+1}}$$

$$\sum_{k=0}^{n}\binom{2n+2}{2n+1-2k}\binom{x+k}{2n+1} \stackrel{B}{=} [t^{2n+1}](1+t)^{2n+2}\left[\frac{u^{2n-x+1}}{(1-u)^{2n+2}} \;\middle|\; u=t^2\right] =$$

$$= [t^{2x-2n-1}]\frac{(1+t)^{2n+2}}{(1-t^2)^{2n+2}} = [t^{2x-2n-1}]\frac{1}{(1-t)^{2n+2}} = \binom{-2n-2}{2x-2n-1}(-1)^{2x-2n-1} = \binom{2x}{2n+1}.$$

(3.173) - Graham and Riordan:

$$\boxed{\sum_{k=0}^{n}\binom{2n+1}{2k}\binom{x+k}{2n} = \binom{2x+1}{2n}}$$

$$\sum_{k=0}^{n}\binom{2n+1}{2n+1-2k}\binom{x+k}{2n} \stackrel{B}{=} [t^{2n+1}](1+t)^{2n+1}\left[\frac{u^{2n-x}}{(1-u)^{2n+1}} \;\middle|\; u=t^2\right] =$$

$$= [t^{2x-2n+1}]\frac{(1+t)^{2n+1}}{(1-t^2)^{2n+1}} = [t^{2x-2n+1}]\frac{1}{(1-t)^{2n+1}} = \binom{-2n-1}{2x-2n+1}(-1)^{2x-2n+1} = \binom{2x+1}{2n}.$$

(3.174) *To check and re-do.*

$$\sum_{k=0}^{n+1} \binom{2n+2}{2k}\binom{x+k}{2n+1} = \binom{2x+1}{2n+1}$$

$$\sum_{k=0}^{n+1} \binom{2n+2}{2k}\binom{x+k}{2n+1} \overset{A}{=} [t^{2n}]\frac{t^{4n}}{t^{2x}} \cdot \frac{(1-t)^{2x+1}}{(1-2t)^{2n+2}} =$$

$$= [t^m]\left[\left(\frac{1-w}{w}\right)^{2x+1} \frac{w}{1-2w} \;\Big|\; w = \frac{1}{2+t}\right] = [t^m]\frac{(1+t)^{2x+1}}{t} = \binom{2x+1}{2n+1}.$$

(3.175) - Machover and Gould.

$$\sum_{k=0}^{n} \binom{x}{2k}\binom{x-2k}{n-k} 2^{2k} = \binom{2x}{2n}$$

$$\sum_{k=0}^{n} \binom{x}{2k}\binom{x-2k}{n-k} 4^k \overset{B}{=} [t^n](1+t)^x \left[\frac{(1+2\sqrt{u})^x + (1-2\sqrt{u})^x}{2} \;\Big|\; u = \frac{t}{(1+t)^2}\right] =$$

$$= [t^n](1+t)^x \left(\frac{(1+t+2\sqrt{t})^x}{2(1+t)^x} - \frac{(1+t-2\sqrt{t})^x}{2(1+t)^x}\right) = [t^{2n}](1+t)^{2x} = \binom{2x}{2n}.$$

(3.176) - Machover and Gould.

$$\sum_{k=0}^{n} \binom{x+1}{2k+1}\binom{x-2k}{n-k} 2^{2k+1} = \binom{2x+2}{2n+1}$$

$$\sum_{k=0}^{n} \binom{x+1}{2k+1}\binom{x-2k}{n-k} 2^{2k+1} \overset{B}{=} [t^n](1+t)^x \left[\frac{(1+2\sqrt{u})^{x+1} - (1-2\sqrt{u})^{x+1}}{2\sqrt{u}} \;\Big|\; u = \frac{t}{(1-t)^2}\right] =$$

$$= [t^n](1+t)^x \left(\frac{(1+t+2\sqrt{t})^{x+1}}{(1+t)^{x+1}} - \frac{(1+t-2\sqrt{t})^{x+1}}{(1+t)^{x+1}}\right) \frac{1+t}{2\sqrt{t}} = [t^n]\frac{(1+\sqrt{t})^{2x+2} - (1-\sqrt{t})^{2x+2}}{2\sqrt{t}} =$$

$$= [t^{2n+1}](1+t)^{2x+2} = \binom{2x+2}{2n+1}.$$

(3.177) - A *Moriarty identity* by H. T. Davis, et al.. *Egorychev (2.3.3) Ex. 5 (p. 74).*

$$\sum_{k=0}^{n-p} \binom{2n+1}{2p+2k+1}\binom{p+k}{k} = \binom{2n-p}{p} 4^{n-p}$$

$$\sum_{k=0}^{n-p} \binom{2n+1}{2p+2k+1}\binom{p+k}{k} \overset{A}{=} [t^{2n+1}]\frac{t^{2p+1}}{(1-t)^{2p+2}} \left[\frac{1}{(1-u)^{p+1}} \;\Big|\; u = \frac{t^2}{(1-t)^2}\right] =$$

$$= [t^{2n-2p}]\frac{1}{(1-t)^{2p+2}} \cdot \frac{(1-t)^{2p+2}}{(1-2t)^{p+1}} = \binom{-p-1}{2n-2p}(-2)^{2n-2p} = \binom{2n-p}{p} 4^{n-p}$$

(3.178) - (3.177) - (3.178) are equivalent to (3.120) - (3.121). A *Moriarty identity* by H. T. Davis, et al.. *Egorychev (2.3.3) Ex. 5 (p. 74).*

$$\boxed{\sum_{k=0}^{n-p}\binom{2n}{2p+2k}\binom{p+k}{k} = \frac{n}{2n-p}\binom{2n-p}{p}4^{n-p}}$$

$$\sum_{k=0}^{n-p}\binom{2n}{2p+2k}\binom{p+k}{k} \stackrel{A}{=} [t^{2n}]\frac{t^{2p}}{(1-t)^{2p+1}}\left[\frac{1}{(1-u)^{p+1}}\;\Big|\; u=\frac{t^2}{(1-t)^2}\right] =$$

$$= [t^{2n-2p}]\frac{1-t}{(1-2t)^{p+1}} = \binom{2n-p}{p}4^{n-p} - \frac{1}{2}\binom{2n-p-1}{p}4^{n-p} = \frac{n}{2n-p}\binom{2n-p}{p}4^{n-p}$$

(3.179) - Marcia Ascher.

$$\boxed{\sum_{k=r}^{n/2}(-1)^k\binom{n-k}{k}\binom{k}{r}2^{n-2k} = (-1)^r\binom{n+1}{2r+1}}$$

$$2^n\sum_{k=r}^{n/2}\binom{n-k}{k}\binom{k}{r}\left(-\frac{1}{4}\right)^k \stackrel{A}{=} 2^n[t^n]\frac{1}{1-t}\left[\frac{(-u/4)^r}{(1+u/4)^{r+1}}\;\Big|\; u=\frac{t^2}{1-t}\right] =$$

$$= 2^{n-2r}(-1)^r[t^n]\frac{1}{1-t}\cdot\frac{t^{2r}}{(1-t)^r}\cdot\frac{(1-t)^{r+1}}{(1-t/2)^{2r+2}} = 2^{n-2r}(-1)^r[t^{n-2r}]\frac{1}{(1-t/2)^{2r+2}} =$$

$$= \frac{2^{n-2r}(-1)^r}{2^{n-2r}}\binom{-2r-2}{n-2r}(-1)^{n-2r} = (-1)^r\binom{n+1}{2r+1}$$

(3.180) Companion piece to (3.179). These are inverse Moriarty formulas.

$$\boxed{\sum_{k=r}^{n/2}\frac{(-1)^k n}{n-k}\binom{n-k}{k}\binom{k}{r}2^{n-2k-1} = (-1)^r\binom{n}{2r}}$$

$$2^{n-1}\sum_k\left(\binom{n-k}{k}+\binom{n-k-1}{k-1}\right)\binom{k}{r}\left(-\frac{1}{4}\right)^k = 2^{n-1}[t^n]\frac{2-t}{1-t}\left[\frac{(-u/4)^r}{(1+u/4)^{r+1}}\;\Big|\; u=\frac{t^2}{1-t}\right] =$$

$$= 2^{n-1}[t^n]\frac{2(1-t/2)}{1-t}\cdot\frac{(-1)^r t^{2r}}{(1-t)^r}\cdot\frac{(1-t)^{r+1}}{(1-t/2)^{2r+2}} = (-1)^r 2^{n-2r}[t^{n-2r}]\frac{1}{(1-t/2)^{2r+1}} =$$

$$= 2^{n-2r}\binom{-2r-1}{n-2r}\frac{(-1)^{n-2r}}{2^{n-2r}} = (-1)^r\binom{n}{2r}.$$

(3.181) - ***(NN)*** *This is not a natural sum, because the term with $k=-1$ is missing. We add and subtract such term.* Brill:

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{x+k}{k+1}\binom{x}{n-k} = \binom{x}{n+1}}$$

$$\sum_{k=-1}^{n}(-1)^k\binom{x+k}{k+1}\binom{x}{n-k} + \binom{x}{n+1} \stackrel{B}{=} [t^n](1+t)^x\left[\frac{-1}{u(1+u)^x}\;\Big|\; x=t\right] + \binom{x}{n+1} =$$

$$= \left([t^{n+1}](-1)\right) + \binom{x}{n+1} = \binom{x}{n+1}$$

***(3.182)** - A. Brill, Math. Annalen, 36 (1890), 361 – 370. Discussions, proofs, extensions by Gould: Mathematica Monongaliae, N. 3, August 1961:

$$\sum_{k=0}^{t}(-1)^k\binom{2a+2b+1}{k}\binom{3a+2b+1-2k}{2a} = (-1)^b\frac{(2a+2b+1)!(2b)!}{(a+2b+1)!b!(a+b)!}$$
$$= (-1)^b\binom{2a+2b+1}{a}\binom{2b}{b}\binom{a+b}{b}^{-1} \qquad 2t = 3a+2b+1$$

(3.183) - Here we have $m = 2n+1-a$. This is equivalent to Brill's sum (3.182).

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\binom{m-1}{a+m-k} = (-1)^{m+n+a}\binom{2n-a}{n}$$

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\binom{m-1}{a+m-k} \stackrel{B}{=} [t^{a+m}](1+t)^{m-1}\left[(1-u)^m \;\middle|\; u=t\right] =$$
$$= [t^{a+m}](1-t)(1-t^2)^{m-1} = [t^{a+m}](1-t^2)^{m-1} - [t^{a+m-1}](1-t^2)^{m-1} = S;$$
$$a+m = 2n \qquad (\text{that is}: m-1 = 2n-a-1)$$
$$S = [t^n](1-t)^{m-1} = \binom{m-1}{n}(-1)^n = \binom{2n-a-1}{n}(-1)^n;$$
$$a+m = 2n+1 \qquad (\text{that is}: m-1 = 2n-a)$$
$$S = [t^n](1-t)^{m-1} = \binom{m-1}{n}(-1)^n = \binom{2n-a}{n}(-1)^n.$$

2.4 Table 4: summations of the form S:1/1

This table contains 30 identities.

(4.1) - *Egorychev, p. 70 – 71:*

$$\sum_{k=j}^{n}\binom{z}{k}\binom{x}{k}^{-1} = \frac{x+1}{x-z+1}\left(\binom{z}{j}\binom{x+1}{j}^{-1} - \binom{z}{n+1}\binom{x+1}{n+1}^{-1}\right)$$

$$\sum_{k=j}^{n}\binom{z}{k}\binom{x}{k}^{-1} = \sum_{k=j}^{n}\frac{z!k!(x-k)!}{k!(z-k)!x!} = \frac{z!(x-z)!}{x!}\sum_{k=j}^{n}\binom{x-k}{x-z} \stackrel{A}{=}$$
$$= \binom{x}{z}^{-1}[t^x]\frac{t^{x-z}}{(1-t)^{x-z+1}}\left[\frac{u^j-u^{n+1}}{1-u} \;\middle|\; u=t\right] = \binom{x}{z}^{-1}[t^z]\frac{t^j-t^{n+1}}{(1-t)^{x-z+2}} =$$

$$= \binom{x}{z}^{-1}\left(\binom{x-j+1}{z-j} - \binom{x-n}{z-n-1}\right) =$$

$$= \frac{z!(x-z)!}{x!}\left(\frac{(x-j+1)!j!}{(z-j)!(x-z+1)!j!} - \frac{(x-n)!(n+1)!}{(z-n-1)!(x-z+1)!(n+1)!}\right) =$$

$$= \binom{z}{j}\binom{x+1}{j}^{-1}\frac{x+1}{x-z+1} - \binom{z}{n+1}\binom{x+1}{n+1}^{-1}\frac{x+1}{x-z+1}.$$

(4.2) - R. Frisch.

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{b+k}{c}^{-1} = \frac{c}{n+c}\binom{n+b}{b-c}^{-1} \qquad b \ge c > 0}$$

$$(I) \qquad \sum_{k=0}^{n}\binom{n}{k}\binom{b+k}{c}^{-1}(-1)^k = \sum_{k=0}^{n}(-1)^k\frac{n!c!(b-c+k)!}{k!(n-k)!(b+k)!}\cdot\frac{(b-c)!(b+n)!}{(b-c)!(b+n)!} =$$

$$= \frac{n!c!(b-c)!}{(b+n)!}\sum_{k=0}^{n}\binom{b+n}{n-k}\binom{b-c+k}{k}(-1)^k \stackrel{B}{=} \frac{n!c!(b-c)!}{(b+n)!}[t^n](1+t)^{b+n}\left[\frac{1}{(1+u)^{b-c+1}} \;\middle|\; u=t\right] =$$

$$= \frac{n!c!(b-c)!}{(b+n)!}\cdot\frac{(n+c-1)!}{n!(c-1)!} = \frac{c}{c+n}\binom{b+n}{c+n}^{-1}.$$

$$(II) \qquad f_k = (-1)^k\binom{b+k}{c}^{-1} \qquad \frac{f_{k+1}}{f_k} = -\frac{b+k+1-c}{b+k+1} \qquad f_0 = \binom{b}{c}^{-1}.$$

$$S = \binom{b}{c}^{-1}[t^n]{}_2F_1\left({1,\ b-c+1 \atop b+1} \;\middle|\; -\frac{t}{1-t}\right) = \binom{b}{c}^{-1}[t^n]{}_2F_1\left({1,\ c \atop b+1} \;\middle|\; t\right) =$$

$$= \frac{(1)_n(c)_n}{(b+1)_n n!}\binom{b}{c}^{-1} = \frac{c(c+1)\cdots(c+n-1)}{(b+1)(b+2)\cdots(b+n)}\frac{c!(b-c)!}{b!} =$$

$$= \frac{c}{c+n}\frac{(c+n)!(b-c)!}{(b+n)!} = \frac{c}{c+n}\binom{b+n}{b-c}.$$

***(4.3)**

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x+y)^k\binom{y+k}{k}^{-1} = \sum_{k=0}^{n}\binom{n}{k}(x-k)^k(1-x+k)^{n-k}\frac{y}{y+k}}$$

(4.4)

$$\boxed{S_r(x) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}^{-1}k^r = \binom{-x-1}{n}^{-1}\sum_{k=0}^{r}\binom{-x}{n-k}\left\{ r \atop k \right\} k!}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}^{-1}k^r = \sum_{k=0}^{n}\frac{n!k!x!}{k!(n-k)!(x+k)!}(-1)^k k^r =$$

$$= \binom{x+n}{n}^{-1}\sum_{k=0}^{n}\binom{x+n}{n-k}(-1)^k k^r \stackrel{B}{=} \binom{x+n}{n}^{-1}[t^n](1+t)^{x+n}\sum_{k=0}^{r}\left\{ r \atop k \right\}\frac{k!t^k(-1)^k}{(1+t)^{k+1}} =$$

$$= \binom{x+n}{n}^{-1}\sum_{k=0}^{r}\left\{ r \atop k \right\}k!(-1)^k[t^{n-k}](1+t)^{x+n-k-1} =$$

$$= \binom{x+n}{n}^{-1} \sum_{k=0}^{r} \left\{ {r \atop k} \right\} k!(-1)^k \binom{x+n-k-1}{n-k} = \binom{-x-1}{n}^{-1} \sum_{k=0}^{r} \binom{-x}{n-k} \left\{ {r \atop k} \right\} k!.$$

(4.5) The identity for $S_0(x)$ is formula (1.42).

$$\boxed{S_1(n) = \frac{-n}{2(2n-1)} \qquad S_0(x) = \frac{x}{x+n}}$$

$$(1) \quad S_1(n) = -\binom{2n}{n}^{-1} \binom{2n-2}{n-1} = -\frac{n^2}{2n(2n-1)} = -\frac{n}{2(2n-1)}.$$

$$(2) \quad S_0(x) = \binom{x+n}{n}^{-1} \binom{x+n-1}{n} = \frac{x!n!}{(x+n)!} \cdot \frac{(x+n-1)!}{n!(x-1)!} = \frac{x}{x+n}.$$

(4.6)

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} \binom{n+2x}{k+x}^{-1} = \frac{2x+n+1}{2x+1} \binom{2x}{x}^{-1}}$$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+2x}{k+x}^{-1} = \sum_{k=0}^{n} \frac{n!(k+x)!(n+x-k)!}{k!(n-k)!(n+2x)!} = \frac{n!x!^2}{(n+2x)!} \sum_{k=0}^{n} \binom{k+x}{k} \binom{n+x-k}{n-k} \stackrel{B}{=}$$

$$= \frac{n!x!^2}{(n+2x)!} [t^n](1+t)^{n+x} \left[\frac{1}{(1-u)^{x+1}} \;\middle|\; u = \frac{t}{1+t} \right] = \frac{n!x!^2}{(n+2x)!} [t^n](1+t)^{n+2x+1} =$$

$$= \frac{n!x!^2}{(n+2x)!} \cdot \frac{(n+2x+1)!}{n!(2x+1)!} = \frac{n+2x+1}{2x+1} \binom{2x}{x}^{-1}.$$

(4.7)

$$\boxed{\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \binom{2n+1+2x}{k+x}^{-1} = 0}$$

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} \binom{2n+1+2x}{k+x}^{-1} (-1)^k = \sum_{k=0}^{2n+1} \frac{(2n+1)!(k+x)!(2n+x+1-k)!}{k!(2n+1-k)!(2n+2x+1)!} (-1)^k =$$

$$= \frac{(2n+1)!x!^2}{(2n+2x+1)!} \sum_{k=0}^{2n+1} \binom{k+x}{k} (-1)^k \binom{2n+x+1-k}{x} \stackrel{A}{=}$$

$$= \frac{(2n+1)!x!^2}{(2n+2x+1)!} [t^{2n+x+1}] \frac{t^x}{(1-t)^{x+1}} \left[\frac{1}{(1+u)^{x+1}} \;\middle|\; u = t \right] =$$

$$= \frac{(2n+1)!x!^2}{(2n+2x+1)!} [t^{2n+1}] \frac{1}{(1-t^2)^{x+1}} = 0.$$

(4.8)

$$\boxed{\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2n+2x}{k+x}^{-1} = \binom{x+n}{n} \binom{2x}{x}^{-1} \binom{2x+2n}{2n}^{-1} = \binom{2n}{n} \binom{x+n}{n}^{-1} \binom{2x+2n}{x+n}^{-1}}$$

$$\sum_{k=0}^{2n} \binom{2n}{k} \binom{2n+2x}{k+x}^{-1} (-1)^k = \sum_{k=0}^{2n} \frac{(2n)!(k+x)!(2n+x-k)!}{k!(2n-k)!(2n+2x)!} (-1)^k =$$

$$= \frac{(2n)!x!^2}{(2n+2x)!}\sum_{k=0}^{2n}\binom{2n+x-k}{x}\binom{k+x}{k}(-1)^k \stackrel{A}{=} \frac{(2n)!x!^2}{(2n+2x)!}[t^{2n+x}]\frac{t^x}{(1-t)^{x+1}}\left[\frac{1}{(1+u)^{x+1}}\;\Big|\; u=t\right] =$$

$$= \frac{(2n)!x!^2}{(2n+2x)!}[t^{2n}]\frac{1}{(1-t^2)^{x+1}} = \frac{(2n)!x!^2}{(2n+2x)!}\cdot\frac{(x+n)!}{x!n!} =$$

$$(1) \quad = \frac{(2n)!x!^2(2x)!(x+n)!}{(2n+2x)!x!(2x)!n!} = \binom{x+n}{n}\binom{2x}{x}^{-1}\binom{2x+2n}{2n}^{-1}.$$

$$(2) \quad = \frac{(2n)!x!(x+n)!^2n!}{(2n+2x)!(x+n)!n!^2} == \binom{2n}{n}\binom{x+n}{n}^{-1}\binom{2x+2n}{x+n}^{-1}.$$

(4.9)

$$\boxed{\sum_{k=0}^{n}\binom{n}{k}\binom{2n-1}{k}^{-1} = 2}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2n-1}{k}^{-1} = \sum_{k=0}^{n}\frac{n!k!(2n-1-k)!}{k!(n-k)!(2n-1)!} = \frac{n!(n-1)!}{(2n-1)!}\sum_{k=0}^{n}\binom{2n-1-k}{n-k} =$$

$$= \frac{n!(n-1)!}{(2n-1)!}[t^n](1+t)^{2n-1}\left[\frac{1}{1-u}\;\Big|\; u=\frac{t}{1+t}\right] = \frac{n!(n-1)!}{(2n-1)!}\binom{2n}{n} = \frac{2n}{n} = 2.$$

(4.10)

$$\boxed{\sum_{k=0}^{n}k\binom{n}{k}\binom{2n-1}{k}^{-1} = \frac{2n}{n+1}}$$

$$\sum_{k=0}^{n}k\binom{n}{k}\binom{2n-1}{k}^{-1} = \sum_{k=0}^{n}k\frac{n!k!(2n-1-k)!}{k!(n-k)!(2n-1)!} = \binom{2n-1}{n}^{-1}\sum_{k=0}^{n}\binom{2n-1-k}{n-k}k \stackrel{B}{=}$$

$$= \binom{2n-1}{n}^{-1}[t^n](1+t)^{2n-1}\left[\frac{u}{(1-u)^2}\;\Big|\; u=\frac{t}{1+t}\right] = \binom{2n-1}{n}^{-1}[t^n](1+t)^{2n-1}\frac{t}{1+t}(1+t)^2 =$$

$$= \binom{2n-1}{n}^{-1}[t^{n-1}](1+t)^{2n} = \binom{2n-1}{n}^{-1}\binom{2n}{n-1} = \frac{n!(n-1)!(2n)!}{(2n-1)!(n-1)!(n+1)!} = \frac{2n}{n+1}.$$

(4.11) See identity (2.9).

$$\boxed{\sum_{k=1}^{n}\binom{n}{k}\binom{2k}{k}^{-1}\frac{(-4)^k}{k} = H_n - 2H_{2n}}$$

$$\sum_{k=1}^{n}\binom{n}{k}\binom{2k}{k}^{-1}\frac{(-4)^k}{k} = (-1)^n[t^n]\frac{1}{1+t}\left[2\sqrt{\frac{y}{1-y}}\arctan\sqrt{\frac{y}{1-y}}\;\Big|\; y=\frac{t}{1+t}\right] =$$

$$= (-1)^n 2[t^n]\frac{1}{1+t}\sqrt{t}\arctan\sqrt{t} = (-1)^n 2\sum_{k=1}^{n}(-1)^{n-k}[t^k]\sqrt{t}\arctan\sqrt{t} =$$

$$= 2\sum_{k=1}^{n}(-1)^k[t^{2k-1}]\arctan(t) = 2\sum_{k=1}^{n}\frac{(-1)^k}{2k-1}[t^{2k-1}]\frac{1}{1+t^2} = 2\sum_{k=1}^{n}\frac{(-1)^k}{2k-1}(-1)^{k-1} =$$

$$= -2\sum_{k=1}^{n}\frac{1}{2k-1} = -2\left(H_{2n} - \frac{1}{2}H_n\right) = H_n - 2H_{2n}.$$

(4.12)

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}^{-1}(-4)^k=\frac{1}{1-2n}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}^{-1}(-4)^k=(-1)^n\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\binom{2k}{k}^{-1}4^k\stackrel{A}{=}$$

$$=(-1)^n[t^n]\frac{1}{1+t}\left[\sqrt{\frac{u}{(1-u)^3}}\arctan\sqrt{\frac{u}{1-u}}+\frac{1}{1-u}\;\Big|\;u=\frac{t}{1+t}\right]=$$

$$=(-1)^n[t^n]\frac{1}{1+t}\left(\sqrt{t(1+t)^2}\arctan\sqrt{t}+(1+t)\right)=(-1)^n[t^n](\sqrt{t}\arctan\sqrt{t}+1)=$$

$$=(-1)^n[t^{2n-1}]\arctan t+\delta_{n,0}=(-1)^n\frac{(-1)^{n-1}}{2n-1}=\frac{1}{1-2n}\quad\text{(valid also for } n=0\text{)}.$$

***(4.13)**

$$\sum_{k=0}^{n}\binom{n}{k}x^k\binom{k+r}{k}^{-1}=1+\left(\frac{(x+1)^{n+r}}{x^r}-\sum_{k=0}^{r}\binom{n+r}{k}x^{k-r}\right)\binom{n+r}{n}^{-1}$$

***(4.14)**

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(2k+1)(1+(-1)^{n+k})P_k(x)}{(n-k+1)2^{k+1}}\binom{(n+k+1)/2}{k}^{-1}=x^n$$

(4.15)

$$\sum_{k=0}^{2n+1}(-4)^k\binom{3n+2}{n+k+1}\binom{2n+2k+1}{n+k}^{-1}=\frac{1}{3}\binom{3n+2}{n+1}\binom{2n+1}{n}^{-1}$$

$$f_k=\binom{2n+2k+1}{n+k}^{-1}(-4)^k\qquad f_0=\binom{2n+1}{n}^{-1}.$$

$$\frac{f_{k+1}}{f_k}=-4\frac{(n+k+1)!(n+k+2)!(2n+2k+1)!}{(2n+2k+3)!(n+k)!(n+k+1)!}=-\frac{n+k+2}{n+k+3/2}\Rightarrow$$

$$\Rightarrow\mathcal{G}(f_k)=\binom{2n+1}{n}^{-1}{}_2F_1\left(\begin{matrix}n+2,\ 1\\ n+3/2\end{matrix}\;\Big|\;-t\right).$$

$$S\stackrel{A}{=}[t^{3n+2}]\frac{t^{n+1}}{(1-t)^{n+2}}\left[\binom{2n+1}{n}^{-1}{}_2F_1\left(\begin{matrix}n+2,\ 1\\ n+3/2\end{matrix}\;\Big|\;-u\right)\;\Big|\;u=\frac{t}{1-t}\right]=$$

$$=\binom{2n+1}{n}^{-1}[t^{2n+1}]\frac{1}{(1-t)^{n+2}}\;{}_2F_1\left(\begin{matrix}n+2,\ 1\\ n+3/2\end{matrix}\;\Big|\;\frac{-t}{1-t}\right)\stackrel{Pf}{=}$$

$$\stackrel{Pf}{=}\binom{2n+1}{n}^{-1}[t^{2n+1}]\;{}_2F_1\left(\begin{matrix}n+2,\ n+1/2\\ n+3/2\end{matrix}\;\Big|\;t\right)=\binom{2n+1}{n}^{-1}\frac{(n+2)_{2n+1}(n+1/2)_{2n+1}}{(n+3/2)_{2n+1}(2n+1)!}=$$

$$=\binom{2n+1}{n}^{-1}\frac{(n+2)(n+3)\cdots(3n+2)\cdot(n+1/2)}{(3n+3/2)(2n+1)!}=$$

$$= \binom{2n+1}{n}^{-1} \frac{(3n+2)!}{(n+1)!} \cdot \frac{2n+1}{6n+3} \cdot \frac{1}{(2n+1)!} = \frac{1}{3}\binom{3n+2}{n+1}\binom{2n+1}{n}^{-1}.$$

***(4.16)**

$$\sum_{k=0}^{n} \binom{2n}{n+k}\binom{2n+2k}{n+k}^{-1} \frac{(4k+1)4^k}{2n+2k+1} = 1$$

***(4.17)**

$$\sum_{k=0}^{n} \frac{(-1)^k}{2k+1}\binom{n}{k}\binom{k+n+1}{k}^{-1} = (n+1)\binom{n+1/2}{n}^{-2}$$

***(4.18)** - The preceding identity may be rewritten as an infinite series (H. F. Sandham); actually:

$$1 - \frac{1}{3} \cdot \frac{1-n}{1+n} + \frac{1}{5} \cdot \frac{(1-n)(2-n)}{(1+n)(2+n)} - \cdots = \frac{1}{4n} \cdot \left(\frac{2 \cdot 4 \cdot \cdots \cdot 2n}{1 \cdot 3 \cdot \cdots \cdot (2n-1)} \right)^2$$

***(4.19)**

$$1 + 2\sum_{k=1}^{\infty} \binom{x}{k}\binom{x+k}{k}^{-1} = \binom{x-1/2}{x}^{-1} = 4^x \binom{2x}{x}^{-1} \qquad \Re(x) \geq 0$$

***(4.20)**

$$\sum_{k=0}^{n} (-1)^k k^{2p} \binom{n}{k}\binom{k+n}{k}^{-1} = 0 \qquad \text{provided } 1 < 2p < 2n-1,\ p \in \mathbb{Z}$$

(4.21) - René Lagrange. Special case of general form when $y = n$. See (7.48):

$$\sum_{k=0}^{n} \binom{x}{k}\binom{x+n-k}{n}^{-1} \frac{x+n+1-2k}{x+n+1-k} = 1$$

$$\sum_{k=0}^{n} \binom{x}{k}\binom{x+n-k}{n}^{-1} \frac{x+n+1-2k}{x+n+1-k} = \sum_{k=0}^{n} \frac{x!n!(x-k)!}{k!(x-k)!(x+n-k)!} \cdot \frac{(x+n)!}{(x+n)!} \left(1 - \frac{k}{x+n-1-k}\right) =$$

$$= \binom{x+n}{n}^{-1} \sum_{k=0}^{n} \left(\binom{x+n}{k} - \binom{x+n}{k} \frac{k}{x+n-k+1} \right) = \binom{x+n}{n}^{-1} \sum_{k=0}^{n} \left(\binom{x+n}{k} - \binom{x+n}{k-1} \right) =$$

$$= \binom{x+n}{n}^{-1} \binom{x+n}{n} = 1$$

the last sum being telescoping.

(4.22) - This is the first of eight sums arising naturally in a statistical problem; it amounts to the evaluation of the moments of a certain distribution. Gould:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n}{2k}^{-1} = \frac{2n+1}{n+1}\cdot\frac{1+(-1)^n}{2} = S_n$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n}{2k}^{-1} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(2n+1)\int_0^1 y^{2k}(1-y)^{2n-2k}\mathrm{d}y =$$

$$= (2n+1)\int_0^1(1-y)^{2n}\left(\sum_k\binom{n}{k}\left(\frac{-y^2}{(1-y)^2}\right)^k\right)\mathrm{d}y =$$

$$= (2n+1)\int_0^1(1-2y)^{2n}[t^n]\frac{1}{1-t}\left[\frac{1}{1+y^2z/(1-y)^2}\;\Big|\; z=\frac{t}{1-t}\right]\mathrm{d}y =$$

$$= (2n+1)\int_0^1(1-y)^{2n}\left([t^n]\frac{(1-y)^2}{(1-y)^2-(1-2y)t}\right)\mathrm{d}y =$$

$$= (2n+1)\int_0^1(1-y)^{2n}\frac{(1-2y)^n}{(1-y)^{2n}}\mathrm{d}y = (2n+1)\int_0^1(1-2y)^n\mathrm{d}y = -\frac{2n+1}{2}\left[\frac{(1-2y)^{n+1}}{n+1}\right]_0^1 =$$

$$= -\frac{2n+1}{2(n+1)}\left((-1)^{n+1}-1\right) = \frac{2n+1}{n+1}\left(\frac{1+(-1)^n}{2}\right).$$

(4.23)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n+1}{2k+1}^{-1} = \frac{1}{n+2}\cdot\frac{1+(-1)^n}{2} + (-1)^n = T_n$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n+1}{2k+1}^{-1} = (2n+2)\int_0^1 y(1-y)^{2n}\left(\sum_k\binom{n}{k}\left(\frac{-y^2}{(1-y)^2}\right)^k\right)dy =$$

$$= 2(n+1)\int_0^1 y(1-y)^{2n}\frac{(1-2y)^n}{(1-y)^{2n}}\mathrm{d}y = 2(n+1)\int_0^1 y(1-2y)^n\mathrm{d}y =$$

$$= 2(n+1)\left(\left[-y\frac{(1-2y)^{n+1}}{2(n+1)}\right]_0^1 + \frac{1}{2}\int_0^1\frac{(1-2y)^{n+1}}{n+1}\mathrm{d}y\right) =$$

$$= -(-1)^{n+1} + \int_0^1(1-2y)^{n+1}\mathrm{d}y = (-1)^n - \left[\frac{(1-2y)^{n+2}}{2(n+2)}\right]_0^1 =$$

$$= (-1)^n - \frac{(-1)^{n+2}}{2(n+2)} + \frac{1}{2(n+2)} = (-1)^n + \frac{1-(-1)^n}{2(n+2)}.$$

***(4.24)**

$$\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\binom{2n}{2k+1}^{-1} = U_n = ?$$

***(4.25)**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n+1}{2k}^{-1} = V_n = (-1)^nT_n = \begin{cases} 1 & n \text{ even} \\ \frac{n+1}{n+2} & n \text{ odd}\end{cases}$$

***(4.26)**

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{2k}\binom{n}{k}^{-1} = \frac{1}{1-n}\cdot\frac{1+(-1)^n}{2} = A_n$$

***(4.27)**

$$\sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k+1}\binom{n}{k}^{-1} = 1+\frac{1}{n}\cdot\frac{1-(-1)^n}{2} = B_n$$

***(4.28)**

$$\sum_{k=0}^{n-1}(-1)^k\binom{2n}{2k+1}\binom{n}{k}^{-1} = C_n = B_n - A_n = \begin{cases} \frac{n}{n-1} & n \text{ even} \\ \frac{n+1}{n} & n \text{ odd} \end{cases}$$

***(4.29)**

$$\sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k}\binom{n}{k}^{-1} = D_n = (-1)^n B_n$$

(4.30) - This is also listed as (1.42) (because of the intimate relation with (1.41)). This is also $r = 0$ in (4.4).

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}^{-1} = \frac{x}{x+n}$$

2.5 Table 5: summations of the form S:0/2

This table contains 2 identities.

***(5.1)**

$$\sum_{k=1}^{\infty}\frac{1}{k^2}\binom{k+n}{k}^{-2} = \binom{2n}{n}\left(\frac{\pi^2}{6} - 3\sum_{k=1}^{n}\frac{1}{k^2}\binom{2k}{k}^{-1}\right)$$

***(5.2)** - Tor B. Staver:

$$\sum_{k=0}^{n}\binom{n}{k}^{-2} = \frac{3(n+1)^2}{2n+3}\binom{2n+2}{n+1}^{-1}\cdot\sum_{k=1}^{n+1}\frac{1}{k}\binom{2k}{k}$$

2.6 Table 6: summations of the form S:3/0

This table contains 52 identities.

***(6.1)** - Fjeldstad:

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k}\binom{2x}{x-n+k}\binom{2z}{z-n+k}=(-1)^n\frac{(n+x+z)!(2n)!(2x)!(2z)!}{(n+x)!(n+z)!(x+z)!n!x!z!}$$

***(6.2)** - The previous identity may be restated as:

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2x+2n}{x+k}\binom{2z+2n}{z+k}=$$

$$=(-1)^n\binom{2n}{n}\binom{2x+2n}{x+n}\binom{2z+2n}{z+n}\binom{x+z+3n}{n}\binom{x+2n}{n}^{-1}\binom{z+2n}{n}^{-1}$$

***(6.3)** - The previous identities may also be restated in symmetrical form. Th. Bang:

$$\sum_{k=-m}^{m}(-1)^k\binom{m+n}{m+k}\binom{n+p}{n+k}\binom{p+m}{p+k}=\frac{(m+n+p)!}{m!n!p!}$$

***(6.4)**

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x}{k}\binom{x}{2n-k}=\binom{x}{n}\binom{-x-1}{n}=(-1)^n\binom{2n}{n}\binom{x+n}{2n}$$

***(6.5)**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x}{k}\binom{x}{n-k}=\binom{x}{n/2}\binom{-x-1}{n/2}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}\binom{x+n-k}{n-k}$$

***(6.6)** - A. C. Dixon:

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3=(-1)^n\binom{2n}{n}\binom{3n}{n}=(-1)^n\frac{(3n)!}{n!^3}$$

***(6.7)** - MacMahon:

$$\sum_{k=0}^{n}\binom{n}{k}^3 x^k y^{n-k} = \sum_{k=0}^{n/2}\binom{n}{2k}\binom{2k}{k}\binom{n+k}{k}x^k y^k (x+y)^{n-2k}$$

***(6.8)** - Dougall:

$$\sum_{k=0}^{\infty}\frac{x+2k}{x}\binom{-x}{k}^3 = \frac{\sin(\pi x)}{\pi x} \qquad (x \le 1/3)$$

***(6.9)** - Dougall:

$$\sum_{k=0}^{\infty}(-1)^k\binom{-x}{k}^3 = \cos\frac{\pi x}{2}\left(-\frac{3x}{2}\right)!\left(-\frac{x}{2}\right)!^{-3} \qquad \Re(x) < 2/3$$

***(6.10)** - Dougall:

$$\sum_{k=0}^{\infty}(-1)^k\binom{-x}{k}^3\frac{x+2k}{x} = \frac{\sin(\pi x)}{\pi x}\frac{((x-1)/2)!((-3x-1)/2)!}{((-x-1)/2)!^2}$$

***(6.11)**

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2k}{k}\binom{4n-2k}{2n-k} = \binom{2n}{n}^2$$

***(6.12)**

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k}\binom{2k}{k}\binom{4n-2k}{2n-k} = \binom{n}{n/2}^2$$

***(6.13)**

$$\sum_{k=0}^{n/2}(-1)^k\binom{n}{k}\binom{2n-2k}{n}\binom{n-2k}{r} = 2^{n-r}\binom{n}{r}\binom{n+r}{r}$$

(6.14)

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k}\binom{k}{j} = \binom{x}{j}\binom{y+x-j}{n-j}$$

$$\sum_{k=0}^{n}\binom{x}{k}\binom{k}{j}\binom{y}{n-k}=\sum_{k=0}^{n}\binom{x}{j}\binom{x-j}{k-j}\binom{y}{n-k}\overset{B}{=}$$

$$=\binom{x}{j}[t^n](1+t)^y\left[u^j(1+u)^{x-j}\;\middle|\;u=t\right]=\binom{x}{j}[t^{n-j}](1+t)^{x+y-j}=\binom{x}{j}\binom{y+x-j}{n-j}.$$

(6.15)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{-x+j-1}{k+r}\binom{x}{j-r-k}=(-1)^r\binom{j+n}{j-r}\binom{x}{j}}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{-x+j-1}{k+r}\binom{x}{j-r-k}=(-1)^r\sum_{k=0}^{n}\binom{n}{k}\binom{x-j+r+k}{x-j}\binom{x}{x-j+r+k}=$$

$$=(-1)^r\binom{x}{j}\sum_{k=0}^{n}\binom{n}{k}\binom{j}{r+k}\overset{E}{=}(-1)^r\binom{x}{j}[t^n]\frac{1}{1-t}\left[\frac{(1+u)^j}{u^r}\;\middle|\;u=\frac{t}{1-t}\right]=$$

$$=(-1)^r\binom{x}{j}[t^{n+r}]\frac{1}{(1-t)^{j+1-r}}=(-1)^r\binom{x}{j}\binom{-j-1+r}{n+r}(-1)^{n+r}=$$

$$=(-1)^r\binom{x}{j}\binom{j+n}{n+r}=(-1)^r\binom{j+n}{j-r}\binom{x}{j}.$$

(6.16)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{r-x-1}{r-k}\binom{x-r-n}{j-r-k}=(-1)^r\binom{x}{j}\binom{j+n}{j-r}}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^r\binom{x-r+1+r+k-1}{r+k}\binom{x-r-n}{j-r-k}=(-1)^r\sum_{k}\binom{n}{k}\binom{x-r-n}{j-r-k}\binom{x+k}{x-r}\overset{T_+}{=}$$

$$=(-1)^r\binom{x}{x-r-j+r}\binom{x+j-r-x+r+n}{j-r}=(-1)^r\binom{x}{j}\binom{j+n}{j-r}.$$

(6.17)

$$\boxed{\sum_{k=0}^{n}\binom{m-x+y}{k}\binom{n+x-y}{n-k}\binom{x+k}{m+n}=\binom{x}{m}\binom{y}{n}}$$

$$\sum_{k=0}^{n}\binom{m-x+y}{k}\binom{n+x-y}{n-k}\binom{x+k}{m+n}\overset{T_+}{=}\binom{x}{m+n-n}\binom{x+n-n-x+y}{n}=\binom{x}{m}\binom{y}{n}.$$

(6.18)

$$\boxed{\sum_{k=0}^{n}\binom{x+1}{k}\binom{x+a-1}{x-k}\binom{x-k}{x-n}=\binom{x+a-1}{n+a-1}\binom{x+n+a}{n}}$$

$$\sum_{k=0}^{n}\binom{x+1}{k}\binom{x+a-1}{x-k}\binom{x-k}{x-n}=\binom{x+a-1}{x-n}\sum_{k=0}^{n}\binom{x+1}{k}\binom{n+a-1}{n-k}\overset{conv}{=}$$

$$=\binom{x+a-1}{n+a-1}\binom{x+n+a}{n}.$$

(6.19)

$$\sum_{k=0}^{n}\binom{n}{k}\binom{r}{k}\binom{x+n+r-k}{n+r}=\binom{x+r}{r}\binom{x+n}{n}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{r}{r-k}\binom{x+n+r-k}{n+r}\overset{T-}{=}\binom{x+n+r-n}{r}\binom{x+n+r-r}{n+r-r}=\binom{x+r}{r}\binom{x+n}{n}.$$

***(6.20)**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n}\binom{n+k}{n}=\sum_{k=0}^{n}(-1)^k\binom{n+k}{n-k}\binom{2k}{k}\binom{2n-k}{n}$$

(6.21)

$$\sum_{k=j}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n}\binom{k}{j}=(-1)^j\binom{n}{j}^2$$

$$\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j}(-1)^k\binom{2n-k}{n}=(-1)^j\binom{n}{j}\sum_{k=j}^{n}\binom{n-j}{k-j}(-1)^{k-j}\binom{2n-k}{n}\overset{A}{=}$$

$$=(-1)^j\binom{n}{j}[t^{2n}]\frac{t^n}{(1-t)^{n+1}}\left[u^j(1-u)^{n-j}\;\middle|\;u=t\right]=(-1)^j\binom{n}{j}[t^{n-j}]\frac{1}{(1-t)^{j+1}}=$$

$$=(-1)^j\binom{n}{j}\binom{-j-1}{n-j}(-1)^{n-j}=(-1)^j\binom{n}{j}^2.$$

***(6.22)**

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k}\binom{2n-k}{n}^2\frac{x}{x+k}=\binom{2n}{n}\binom{2n+x}{n}\binom{x+n}{n}^{-1}$$

***(6.23)**

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k}\binom{2n-k}{n}^2\frac{2n+1}{2n+k+1}=\binom{2n}{n}\binom{4n+1}{n}\binom{3n+1}{n}^{-1}$$

***(6.24)**

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k}\binom{2n-k}{n}^2\frac{2n+1}{2n-k+1}=1$$

***(6.25)**

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k}^2\binom{2n}{n-k}=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{4n-k}{2n}\binom{2n+k}{n}=\sum_{k=0}^{n}(-1)^k\binom{2n}{k}\binom{4n-k}{3n}\binom{2n+k}{2n}$$

***(6.26)** - Carlitz:

$$\sum_{k=0}^{n}(-1)^k\binom{2r}{k}\binom{2n-2r}{n-k}^2=(-1)^r\binom{2r}{r}\binom{2n-r}{n}^2\binom{2n-r}{r}^{-1}$$

***(6.27)** - The previous identity can be restated as:

$$\sum_{k=-n}^{n}(-1)^k\binom{2r}{r-k}^2\binom{2n}{n-k}=\binom{2r+n}{r}\frac{(2n)!(2r)!}{(n+r)!n!r!}$$

(6.28)

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{k}{n-j}=\binom{n}{j}\binom{n+j}{j}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n}{k}\binom{k}{n-j}=\binom{n}{n-j}\sum_{k=0}^{n}\binom{n}{k}\binom{j}{k-n+j}\stackrel{E}{=}$$

$$=\binom{n}{j}[t^n]\frac{1}{1-t}\left[\frac{(1+u)^j}{u^{j-n}}\;\middle|\;u=\frac{t}{1-t}\right]=\binom{n}{j}[t^n]\frac{1}{(1-t)^{j+1}}\cdot\frac{(1-t)^{j-n}}{t^{j-n}}=$$

$$=\binom{n}{j}[t^j]\frac{1}{(1-t)^{n+1}}=\binom{n}{j}\binom{-n-1}{j}(-1)^j=\binom{n}{j}\binom{n+j}{j}.$$

***(6.29)**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2\binom{2x-n}{x-k}=(-1)^{n/2}\binom{n}{n/2}\binom{x+n/2}{n}\binom{2x}{x}\binom{2x}{n}^{-1}$$

(6.30) This and the next two formulas are essentially the one given in Math Review, V. 17 (1956), p. 653 - 654.

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{x+k}{2n}=\binom{x}{n}^2$$

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{x+k}{2n}\stackrel{T_+}{=}\binom{x}{2n-n}\binom{x+n-n}{n}=\binom{x}{n}^2.$$

(6.31)

$$\sum_{k=0}^{n}\binom{x-n}{k}^2\binom{2x-n-k}{n-k}=\binom{x}{n}^2$$

$$\sum_{k=0}^{n}\binom{x-n}{k}\binom{x-n}{x-n-k}\binom{2x-n-k}{2x-2n}\stackrel{T_-}{=}\binom{2x-n-x+n}{x-n}\binom{2x-n-x+n}{2x-2n-x+n}=\binom{x}{n}^2.$$

(6.32) Math. Review, V. 18 (1957), p. 4. Known to Le Jen-Shoo, 1867. *Egorychev (2.1.4) (p. 51).*

$$\boxed{\sum_{k=0}^{n} \binom{n}{k}^2 \binom{x+2n-k}{2n} = \binom{x+n}{n}^2}$$

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}\binom{x+2n-k}{2n} \overset{T_-}{=} \binom{x+2n-n}{n}\binom{x+2n-n}{2n-n} = \binom{x+n}{n}^2.$$

(6.33)

$$\boxed{\sum_{k=j}^{n/2} \binom{n}{2k}\binom{2k}{k}\binom{k}{j} 2^{n-2k} = \binom{n}{j}\binom{2n-2j}{n}}$$

$$\sum_{k=j}^{n/2} \binom{n}{2k}\binom{2k}{k}\binom{k}{j} 2^{n-2k} = \sum_{k=j}^{n/2} \binom{n}{k}\binom{k}{j}\binom{n-k}{k} 2^{n-2k} =$$

$$= \binom{n}{j} \sum_{k=j}^{n/2} \binom{n-j}{k-j}\binom{n-k}{n-2k} 2^{n-2k} \overset{B}{=} \binom{n}{j} [t^n](1+2t)^n \left[u^j (1+u)^{n-j} \;\middle|\; u = \frac{t^2}{1+2t} \right] =$$

$$= \binom{n}{j} [t^n](1+2t)^n \frac{t^j}{(1+2t)^j} \cdot \frac{(1+2t+t^2)^{n-j}}{(1+2t)^{n-j}} = \binom{n}{j} [t^{n-2j}](1+t)^{2n-2j} =$$

$$= \binom{n}{j}\binom{2n-2j}{n-2j} = \binom{n}{j}\binom{2n-2j}{n}.$$

***(6.34)** - Generalization of (6.35):

$$\boxed{\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k}\binom{2k}{k+j} 4^{n-k} = \binom{n+j}{(n+j)/2}\binom{n-j}{(n-j)/2} \frac{(-1)^n + (-1)^j}{2}}$$

***(6.35)** - E. T. Bell:

$$\boxed{\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k}\binom{2k}{k} 4^{n-k} = \binom{n}{n/2}^2 = 4^n \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{-n-1}{k}\binom{-1/2}{k}}$$

(6.36)

$$\boxed{\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{n+k}{k}^2 = \binom{2n}{n}}$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{n+k}{k}^2 = \sum_{k=0}^{2n} [t^k](1-t)^{2n} [v^n](1+v)^{n+k} [w^n](1+w)^{n+k} =$$

$$= [v^n](1+v)^n [w^n](1+w)^n \sum_{k=0}^{\infty} [t^k](1-t)^{2n} (1+v)^k (1+w)^k =$$

$$= [v^n](1+v)^n[w^n](1+w)^n(1-(1+v)(1+w))^{2n} = [v^n](1+v)^n[w^n](1+w)^n(v+(1+v)w)^{2n} =$$

$$= [v^n](1+v)^{3n}[w^n](1+w)^n \sum_{k=0}^{2n} \binom{2n}{k} \frac{v^k}{(1+v)^k} w^{2n-k} =$$

$$= \sum_{k=0}^{2n} \binom{2n}{k} [v^{n-k}](1+v)^{3n-k}[w^{k-n}](1+w)^n = \sum_{k=0}^{2n} \binom{2n}{k} \binom{3n-k}{n-k} \binom{n}{k-n} =$$

$$= \binom{2n}{n}. \quad \text{In fact, for } \mathbf{k} < \mathbf{n}: \binom{n}{k-n} = 0; \text{ for } \mathbf{k} > \mathbf{n}: \binom{3n-k}{n-k} = 0.$$

(6.37)

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{3n+k}{2n} = \binom{3n}{n}^2$$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} \binom{3n+k}{2n} \overset{T_+}{=} \binom{3n}{2n-n} \binom{3n+n-n}{n} = \binom{3n}{n}^2.$$

***(6.38)**

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \frac{k}{k+2nx} \binom{k+2nx}{2n} \frac{2n-k}{2n-k+2nx} \binom{2n-k+2nx}{2n} =$$

$$= (-1)^n x \binom{2n}{n} \frac{n}{n+2nx} \binom{n+2nx}{2n} = \frac{(-1)^n}{2 \cdot n!^2} \prod_{k=0}^{n-1} \left((2nx)^2 - k^2\right) \qquad (n \geq 1)$$

***(6.39)** - Carlitz:

$$\sum_{k=0}^{n} \binom{n}{k}^3 = [x^n](1-x^2)^n P_n\left(\frac{1+x}{1-x}\right)$$

***(6.40)**

$$D_x^r P_n(x) = \frac{r!}{2^n} \sum_{k=0}^{n/2} (-1)^k \binom{n}{k} \binom{2n-2k}{n} \binom{n-2k}{r} x^{n-2k-r}$$

***(6.41)** - Gould; special case $a = 0$, $b = 2$, $x = 2n$ suggested by Samuel Karlin:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{a+bk}{j} \binom{x-a-bk}{n-j} = (-1)^j \binom{n}{j} b^n$$

(6.42) - M. T. L. Bizley

$$\sum_{k} \binom{b}{k} \binom{c}{k-d} \binom{a+k}{b+c} = \binom{a}{b-d} \binom{a+d}{c+d}$$

$$\sum_k \binom{b}{k}\binom{c}{c+d-k}\binom{a+k}{b+c} \overset{T_+}{=} \binom{a}{b+c-c-d}\binom{a+c+d-c}{c+d} = \binom{a}{b-d}\binom{a+d}{c+d}.$$

(6.43) - Bizley. *This is another form of* (T_+)*:*

$$\boxed{\sum_k \binom{b}{k}\binom{c}{d-k}\binom{a+k}{b+c} = \binom{a}{b+c-d}\binom{a-c+d}{d}}$$

(6.44) - Riordan:

$$\boxed{\sum_{k=0}^{n} \binom{n}{k}\binom{m+n-k}{m-k}\binom{x}{m+n-k} = \binom{x}{m}\binom{x}{n}}$$

$$\sum_{k=0}^{n} \binom{n}{k}\binom{x}{m+n-k}\binom{m+n-k}{n} = \binom{x}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{x-n}{m-k} \overset{B}{=}$$

$$= \binom{x}{n}[t^m](1+t)^{x-n}\left[(1+u)^n \;\middle|\; u=t\right] = \binom{x}{n}[t^m](1+t)^{x-n}(1+t)^n = \binom{x}{n}\binom{x}{m}.$$

(6.45) - Riordan. *We apply* (T_-)*:*

$$\boxed{\sum_{k=0}^{n} \binom{n}{k}\binom{m}{n-k}\binom{x+n-k}{m+n} = \binom{x}{m}\binom{x}{n}}$$

(6.46) - Equivalent to formula of Surányi:

$$\boxed{\sum_{k=0}^{n} \binom{x}{k}\binom{y}{k}\binom{x+y+n-k}{n-k} = \binom{x+n}{n}\binom{y+n}{n}}$$

$$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{y-k}\binom{x+y+n-k}{x+y} \overset{T_-}{=} \binom{x+n}{n}\binom{y+n}{n}.$$

(6.47) - Equivalent to formula posed by H. L. Krall:

$$\boxed{\sum_{k=0}^{n} \binom{y}{k}\binom{r+y+z-k}{n-k}\binom{r+z-n}{r-k} = \binom{r+z}{r}\binom{y+z}{n}}$$

$$\sum_{k=0}^{n} \binom{y}{k}\binom{r+z-n}{r-k}\binom{r+y+z-k}{r+y+z-n} \overset{T_-}{=} \binom{r+y+z-y}{r}\binom{r+y+z-r}{y+r+z-n-r} = \binom{r+z}{r}\binom{y+z}{n}.$$

(6.48) - Extension of a formula of David Zeitlin; equivalent to formula of Surányi:

$$\boxed{\sum_{k=0}^{n} (-1)^{n-k}\binom{x+y+1}{n-k}\binom{x+k}{k}\binom{y+k}{k} = \binom{x}{n}\binom{y}{n}}$$

If we are not afraid to introduce negative denominators in binomial coefficients:

$$\sum_{k=0}^{n}\binom{-x-y-1+n-k-1}{n-k}\binom{-x-k+k-1}{k}(-1)^k\binom{-y-k+k-1}{k}(-1)^k =$$

$$=\sum_k\binom{-x-1}{k}\binom{-y-1}{k}\binom{n-x-y-2-k}{-x-y-2} = \binom{n-x-y-2+x+1}{-y-1}\binom{n-x-y-2+y+1}{-x-y-2+y+1} =$$

$$=\binom{n-y-1}{-y-1}\binom{n-x-1}{-x-1} = \binom{n-y-1}{n}\binom{n-x-1}{n} = \binom{x}{n}\binom{y}{n}.$$

(6.49) is equivalent to (6.7).

(6.50) C. van Ebbenhorst Tengbergen, 1913.

$$\boxed{\sum_{k=a}^{n}\binom{k+a-1}{k}\binom{2n}{n-k}\binom{k}{a} = \binom{2n-1}{n}\binom{n}{a}}$$

$$S=\sum_{k=a}^{n}\binom{-a}{k}\binom{k}{a}\binom{2n}{n-k}(-1)^k = \binom{-a}{a}\sum_{k=a}^{n}\binom{-2a}{k-a}\binom{2n}{n-k}(-1)^k \overset{B}{=}$$

$$=\binom{-a}{a}[t^n](1+t)^{2n}\left[(-u)^a(1-u)^{-2a}\;\middle|\; u=t\right] = \binom{-a}{a}(-1)^a[t^n]t^a\frac{(1+t)^{2n}}{(1-t)^{2a}} =$$

$$=\binom{-a}{a}(-1)^a[t^n]\left[\frac{w^a}{(1-w)^{2a}}\cdot\frac{1+w}{1-w}\;\middle|\; w=t(1+w)^2\right].$$

$$w=\frac{1-2t-\sqrt{1-4t}}{2t};\qquad \frac{1+w}{1-w}=\frac{1}{\sqrt{1-4t}};\qquad \frac{w^a}{(1-w)^{2a}}=\frac{t^a}{(1-4t)^a}.$$

$$S=\binom{-a}{a}(-1)^a[t^n]\frac{t^a}{(1-4t)^a}\cdot\frac{1}{\sqrt{1-4t}} = \binom{2a-1}{a}\binom{-a-1/2}{n-a}(-4)^{n-a} =$$

$$=\binom{2a-1}{a}\binom{n-1/2}{n-a}4^{n-a} \overset{Z45}{=} \frac{1}{2}\binom{2a}{a}\binom{2n}{n}\binom{n}{n-a}\binom{2n-2n+2a}{n-n+a}^{-1}\frac{4^{n-a}}{4^{n-a}} =$$

$$=\frac{1}{2}\binom{2n}{n}\binom{n}{a} = \binom{2n-1}{n}\binom{n}{a}.$$

(6.51) Gould. Contrast with (6.19).

$$\boxed{\sum_{k=0}^{n}\binom{n}{k}\binom{r}{k}\binom{x+n+r+k}{n+r} = \binom{x+n+r}{n}\binom{x+n+r}{r}}$$

By (T_+).

(6.52) Case $q=1$ of identity found by R. P. Stanley. Gould, J. Combinatorial Theory.

$$\boxed{\sum_{k=0}^{m}\binom{x+y+k}{k}\binom{y}{a-k}\binom{x}{b-k} = \binom{x+a}{b}\binom{y+b}{a}\qquad m=\min(a,b)}$$

$$\sum_{k=0}^{m}\binom{x}{x-b+k}\binom{y}{a-k}\binom{x+y+k}{x+y} \overset{T_+}{=} \binom{x+y-x+b}{x+y-a-x+b}\binom{x+y+a-y}{a+x-b} = \binom{x+a}{b}\binom{y+b}{a}.$$

2.7 Table 7: summations of the form S:2/1

This table contains 49 identities.

(7.1)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{z}{k}\binom{y}{k}^{-1}=\binom{y-z}{n}\binom{y}{n}^{-1}$$

$$\binom{n}{k}\binom{y}{k}^{-1}=\frac{n!(y-k)!}{(n-k)!y!}=\binom{y}{n}^{-1}\binom{y-k}{n-k}.$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{z}{k}\binom{y}{k}^{-1}=\binom{y}{n}^{-1}\sum_{k=0}^{n}\binom{y-k}{n-k}\binom{z}{k}(-1)^k\overset{B}{=}$$

$$=\binom{y}{n}^{-1}[t^n](1+t)^y\left[(1-u)^z \;\middle|\; u=\frac{t}{1+t}\right]=\binom{y}{n}^{-1}[t^n](1+t)^{y-z}=\binom{y-z}{n}\binom{y}{n}^{-1}.$$

(7.2)

$$\sum_{k=0}^{n}\binom{n}{k}\binom{z}{k}\binom{x+k}{k}^{-1}=\binom{x+z+n}{n}\binom{x+n}{n}^{-1}$$

$$\binom{n}{k}\binom{x+k}{k}^{-1}=\frac{n!x!}{(n-k)!(x+k)!}=\binom{n+x}{n}^{-1}\binom{n+x}{n-k}.$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{z}{k}\binom{x+k}{k}^{-1}=\binom{n+x}{x}^{-1}\sum_{k=0}^{n}\binom{z}{k}\binom{n+x}{n-k}\overset{B}{=}$$

$$=\binom{n+x}{n}^{-1}[t^n](1+t)^{n+x}\left[(1+u)^z \;\middle|\; u=t\right]=$$

$$=\binom{n+x}{n}^{-1}[t^n](1+t)^{n+x+z}=\binom{x+z+n}{n}\binom{n+x}{n}^{-1}.$$

(7.3) The Riordan Array method allows to find a more compact formula.

$$\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}\binom{-x+n-1}{k}^{-1}=\binom{2x-n}{x}^{-1}\binom{(n-1)/2}{x}(-4)^n$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}\binom{x-n+k}{k}^{-1}(-1)^k=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x!k!(x-n)!}{k!(x-k)!(x-n+k)!}\cdot\frac{(2x-n)!}{(2x-n)!}=$$

$$=\binom{2x-n}{x}^{-1}\sum_{k=0}^{n}\binom{2x-n}{x-k}(-1)^k\binom{n}{k}\overset{B}{=}\binom{2x-n}{x}^{-1}[t^x](1+t)^{2x-n}\left[(1-u)^n \;\middle|\; u=t\right]=$$

$$=\binom{2x-n}{x}^{-1}[t^x](1+t)^{2x}\left(\frac{1-t}{1+t}\right)^n=\binom{2x-n}{x}^{-1}[t^x]\left[\left(\frac{1-w}{1+w}\right)^n\frac{1+w}{1-w} \;\middle|\; w=t(1+w)^2\right]=$$

$$=\binom{2x-n}{x}^{-1}[t^x]\left[\left(\frac{1-w}{1+w}\right)^{n-1} \;\middle|\; w=\frac{1-2t-\sqrt{1-4t}}{2t}\right]=\binom{2x-n}{x}^{-1}[t^x]\frac{1}{\sqrt{1-4t}^{n-1}}=$$

$$= \binom{2x-n}{x}^{-1} \binom{(n-1)/2}{x} (-4)^n.$$

(7.4) -** ***(NN) P. Terdy: Giornale di Matematiche, 3 (1865) p. 1 – 3:

$$\sum_{k=0}^{n} (-1)^k \binom{x}{k} \binom{x+1}{k} \binom{2x}{k}^{-1} = (-1)^n \binom{x-1}{n} \binom{x}{n} \binom{2x}{n}^{-1}$$

***(7.5)**

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2kx}{2j} \binom{kx}{j}^{-1} = \begin{cases} 0 & 0 \le j < n \\ (-4x)^n \binom{2n}{n}^{-1} & j = n \end{cases}$$

(7.6)

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{j+k}{k}^{-1} \frac{1}{4^k} = \binom{2n+2j}{n+j} \binom{2j}{j}^{-1} \frac{1}{4^n}$$

$$\binom{n}{k} \binom{j+k}{k}^{-1} = \frac{n!j!}{(n-k)!(j+k)!} = \binom{n+j}{n}^{-1} \binom{n+j}{n-k}.$$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{j+k}{k}^{-1} \frac{(-1)^k}{4^k} = \binom{n+j}{n}^{-1} \sum_{k=0}^{n} \binom{n+j}{j+k} \binom{2k}{k} \frac{(-1)^k}{4^k} \overset{A}{=}$$

$$= \binom{n+j}{n}^{-1} [t^{n+j}] \frac{t^j}{(1-t)^{j+1}} \left[\frac{1}{\sqrt{1+u}} \;\middle|\; u = \frac{t}{1-t} \right] = \binom{n+j}{n}^{-1} [t^n] \frac{1}{(1-t)^{j+1/2}} =$$

$$= \binom{n+j}{n}^{-1} \binom{-j-1/2}{n} (-1)^n = \binom{n+j}{n}^{-1} \binom{n+j-1/2}{n} \overset{Z45}{=}$$

$$= \binom{n+j}{n}^{-1} \binom{2n+2j}{n+j} \binom{n+j}{n} \binom{2j}{j}^{-1} \frac{1}{4^n} = \binom{2n+2j}{n+j} \binom{2j}{j}^{-1} \frac{1}{4^n}.$$

(7.7)

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2k}{k} \binom{n+k}{k}^{-1} \frac{1}{4^k} = \binom{6n}{3n} \binom{2n}{n}^{-1} \frac{1}{16^n}$$

$$\binom{2n}{k} \binom{n+k}{k}^{-1} = \frac{(2n)!n!}{(2n-k)!(n+k)!} = \binom{3n}{n}^{-1} \binom{3n}{n+k}$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2k}{k} \binom{n+k}{k}^{-1} \frac{1}{4^k} = \binom{3n}{n}^{-1} \sum_{k=0}^{2n} \binom{3n}{n+k} \binom{2k}{k} \frac{(-1)^k}{4^k} \overset{A}{=}$$

$$= \binom{3n}{n}^{-1} [t^{3n}] \frac{t^n}{(1-t)^{n+1}} \left[\frac{1}{\sqrt{1+u}} \;\middle|\; u = \frac{t}{1-t} \right] = \binom{3n}{n}^{-1} [t^{2n}] \frac{1}{(1-t)^{n+1/2}} =$$

$$= \binom{3n}{n}^{-1} \binom{-n-1/2}{2n} (-1)^{2n} = \binom{3n}{n}^{-1} \binom{3n-1/2}{2n} \overset{Z45}{=}$$

$$= \binom{3n}{n}^{-1}\binom{6n}{3n}\binom{3n}{2n}\binom{2n}{n}^{-1}\frac{1}{16^n} = \binom{6n}{3n}\binom{2n}{n}^{-1}\frac{1}{16^n}.$$

***(7.8)**

$$\boxed{\sum_{k=0}^{2n}\frac{(-1)^k(2k+1)}{n+k+1}\binom{2n}{k}\binom{2k}{k}\binom{n+k}{k}^{-1} = 1}$$

(7.9)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}\binom{2k}{k}^{-1}\frac{4^k}{2k+1} = \binom{n-x-1/2}{n}\frac{4^n}{2n+1}\binom{2n}{n}^{-1}}$$

$$S = \sum_k (-1)^k\binom{n}{k}f_k \qquad \text{where} \qquad f_k = \binom{x+k}{k}\binom{2k}{k}^{-1}\frac{(-4)^k}{2k+1} \qquad f_0 = 0.$$

$$\frac{f_{k+1}}{f_k} = -4\frac{(x+k+1)!k!x!}{(k+1)!x!(x+k)!}\cdot\frac{(k+1)(k+1)}{2(k+1)(2k+1)}\cdot\frac{2k+1}{2k+3} = \frac{x+k+1}{2k+3}$$

$$\Rightarrow \mathcal{G}(f_k) = {}_2F_1\left(\begin{matrix}1,\ x+1\\ 3/2\end{matrix}\ \Big|\ -t\right).$$

$$S = [t^n]\frac{1}{1-t}{}_2F_1\left(\begin{matrix}1,\ x+1\\ 3/2\end{matrix}\ \Big|\ \frac{-t}{1-t}\right) \stackrel{Pf}{=} [t^n]\ {}_2F_1\left(\begin{matrix}1,\ 1/2-x\\ 3/2\end{matrix}\ \Big|\ t\right) = \frac{(1)_n(1/2-x)_n}{(3/2)_n n!}.$$

$$(3/2)_n = \frac{(2n+1)!}{4^n n!} \qquad (1/2-x)_n = (x-n+1/2)^{\underline{n}}.$$

$$S = \frac{n!(x-n+1/2)^{\underline{n}}4^n n!}{(2n+1)!n!} = \frac{1}{2n+1}\binom{2n}{n}^{-1}\binom{x-n+1/2}{n}.$$

(7.10)

$$\boxed{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}^{-1}\frac{4^k}{2k+1} = \frac{(-1)^n}{2n+1}}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}^{-1}\frac{(-4)^k}{2k+1} = \sum_{k=0}^{n}\binom{n}{k}\frac{(n+k)!k!k!}{k!n!(2k)!}\cdot\frac{(n-k)!}{(n-k)!}\cdot\frac{(-4)^k}{2k+1} =$$

$$= \sum_{k=0}^{n}\binom{n}{k}\binom{n}{k}^{-1}\binom{n+k}{n-k}\frac{(-4)^k}{2k+1}.$$

$$\mathcal{G}\left(\frac{(-4)^k}{2k+1}\right) = \frac{1}{2\sqrt{t}}\int_0^t\frac{1}{1-4z}\cdot\frac{\mathrm{d}z}{\sqrt{z}} = \frac{1}{2\sqrt{t}}\arctan 2\sqrt{t}.$$

$$S = \sum_{k=0}^{n}\binom{n+k}{n-k}\frac{(-4)^k}{2k+1} \stackrel{B}{=} [t^n](1+t)^n\left[\frac{1}{2\sqrt{u}}\arctan 2\sqrt{u}\ \Big|\ u = t(1+t)\right] =$$

$$= [t^n](1+t)^n\frac{1}{2\sqrt{t(1+t)}}\arctan 2\sqrt{t(1+t)} = [t^n]\left[\frac{1+w}{2\sqrt{w(1+w)}}\arctan 2\sqrt{w(1+w)}\ \Big|\ w = t(1+w)\right] =$$

$$= [t^n]\frac{1}{2\sqrt{t}}\arctan\frac{2\sqrt{t}}{1-t} = \frac{1}{2}[t^{2n+1}]\arctan\frac{2t}{1-t^2} = \frac{1}{2}\cdot\frac{1}{2n+1}[t^{2n}]\frac{\mathrm{d}}{\mathrm{d}t}\arctan\frac{2t}{1-t^2} =$$

$$= \frac{1}{2n+1}[t^{2n}]\frac{1}{1+t^2} = \frac{(-1)^n}{2n+1}.$$

(7.11)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}\binom{2k}{k}^{-1}\frac{4^k}{x+k}=\frac{(-1)^n}{x}\binom{2x}{2n}\binom{x}{n}^{-1}$$

$$S=\sum_k\binom{n}{k}f_k \qquad \text{where} \qquad f_k=\binom{x+k}{k}\binom{2k}{k}^{-1}\frac{(-4)^k}{x+k} \qquad f_0=\frac{1}{x}.$$

$$\frac{f_{k+1}}{f_k}=-4\cdot\frac{x+k+1}{k+1}\cdot\frac{k+1}{2(2k+1)}\cdot\frac{x+k}{x+k+1}=-\frac{x+k}{k+1/2}\Rightarrow$$

$$\Rightarrow\mathcal{G}(f_k)=\frac{1}{x}\,{}_2F_1\left(\begin{matrix}1,\ x\\ 1/2\end{matrix}\,\middle|\,-t\right).$$

$$S\overset{Pf}{=}\frac{1}{x}[t^n]\frac{1}{1-t}\left[{}_2F_1\left(\begin{matrix}1,\ x\\ 1/2\end{matrix}\,\middle|\,-u\right)\,\middle|\,u=\frac{t}{1-t}\right]=\frac{1}{x}[t^n]\,{}_2F_1\left(\begin{matrix}1,\ 1/2-x\\ 1/2\end{matrix}\,\middle|\,-t\right).$$

$$(1/2)_n=\frac{(2n)!}{4^n n!} \qquad (1/2-x)_n=(1/2-x)(3/2-x)\cdots(n-1/2-x)=(-1)^n\frac{(2x-1)\cdots(2x-2n+1)}{2^n}=$$

$$=(-1)^n\frac{2x(2x-1)\cdots((2x-2n+1)}{2^n\cdot 2^n\cdot x(x-1)\cdots(x-n+1)}=\frac{(-1)^n(2x)^{\underline{2n}}}{4^n\cdot x^{\underline{n}}}.$$

$$S=\frac{(-1)^n}{x4^n}\cdot\frac{(2x)^{\underline{2n}}}{x^{\underline{n}}}\cdot\frac{4^n n!}{(2n)!}=\frac{1}{2n+1}\binom{2n}{n}\binom{x-n+1/2}{n}.$$

(7.12)

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}^{-1}\frac{(-4)^k}{n+k}=\frac{(-1)^n}{n} \qquad n\geq 1$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}^{-1}\frac{(-4)^k}{n+k}=\sum_k\frac{n!(n+k)!k!k!}{k!(n-k)!n!k!(2k)!}\cdot\frac{(-4)^k}{n+k}=$$

$$=\sum_k\frac{(n+k-1)!}{(n-k)!(2k-1)!}\cdot\frac{(-4)^k}{2k}=\frac{1}{2}\sum_k\binom{n-1+k}{n-k}\cdot\frac{(-4)^k}{k}\overset{A}{=}\frac{1}{2}[t^{n-1}]\frac{1}{t}\left[\ln\frac{1}{1+4u}\,\middle|\,u=\frac{t}{(1-t)^2}\right]=$$

$$=\frac{1}{2}[t^n]\ln\frac{t(1-t)^2}{(1-t)^2(1+2t+t^2)}=[t^n]\ln\frac{t}{1+t}=\frac{1}{n}[t^{n-1}]\frac{1}{1+t}=\frac{(-1)^n}{n}.$$

***(7.13)**

$$\sum_{k=0}^{n}\binom{n}{k}\binom{-n-1}{n-k}\binom{k+r}{k}^{-1}=(-1)^n\sum_{k=0}^{n}\binom{n}{k}^2\frac{r}{r+k}$$

***(7.14)**

$$\sum_{k=0}^{n}\frac{1}{2n+2k+1}\binom{2n}{n+k}\binom{2k}{k}\binom{2n+2k}{n+k}^{-1}=\frac{(-16)^n}{4n+1}\binom{-3/4}{n}\binom{4n}{2n}^{-1}$$

***(7.15)**

$$\sum_{k=1}^{n}\binom{n}{k}\binom{z}{k}\binom{x+k}{k}^{-1}(H_{x+k}-H_x)=\binom{x+z+n}{n}\binom{x+n}{n}^{-1}(H_{x+n}-H_x-H_{x+z+n}+H_{x+z})$$

***(7.16)**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2\binom{rn+n}{k}^{-1}=\frac{(rn)!^2}{(rn-n)!(rn+n)!}$$

***(7.17)**

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k}^2\binom{m}{k}^{-1}=2H_n-H_m\qquad(m\geq n)$$

***(7.18)** - Equivalent to a formula of K. L. Chung (3.148):

$$\sum_{k=0}^{n}\frac{1}{jk-1}\binom{n}{k}\binom{jn-n}{jk-k}\binom{jn}{jk}^{-1}=\begin{cases}\frac{j-2}{1-j} & n\geq 1\\ -1 & n=0\end{cases}$$

***(7.19)**

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2n+2x}{k+x}\binom{2n+2y}{k+y}^{-1}=\frac{\binom{2n}{n}\binom{n+x-y-1}{n}\binom{2n+2x}{x}}{\binom{x+n}{n}\binom{y+n}{n}\binom{2n+2y}{n+y}}$$

***(7.20)** - Many closed expressions for sums of the form S:2/1 are known from the theory of hypergeometric function. Definition: if a, b, c, z are complex numbers:

$$F\left(\begin{matrix}a,\ b\\ c\end{matrix}\ \middle|\ z\right)=\sum_{k=0}^{\infty}(-1)^k\binom{-a}{k}\binom{-b}{k}\binom{-c}{k}^{-1}z^k$$

***(7.21)**

$$F\left(\begin{matrix}a,\ b\\ c\end{matrix}\ \middle|\ 1\right)=\frac{(c-1)!(c-a-b-1)!}{(c-a-1)!(c-b-1)!}\qquad\Re(c)>\Re(a+b)$$

***(7.22)**

$$\sum_{k=0}^{\infty}\binom{-n/2}{k}\binom{n/2}{k}\binom{2k}{k}^{-1}(2\sin(x))^{2k}=\cos(nx)\qquad n\in\mathbb{R},|x|\leq\frac{\pi}{2}$$

***(7.23)**

$$\sum_{k=0}^{\infty} \binom{-n/2-1}{k} \binom{n/2-1}{k} \binom{2k}{k}^{-1} \frac{4^k \sin(x)^{2k+1}}{2k+1} = \frac{\sin(nx)}{cos(x)} \qquad n \in \mathbb{R}, |x| < 1$$

***(7.24)**

$$\sum_{k=0}^{\infty} \binom{-(n+1)/2}{k} \binom{(n-1)/2}{k} \binom{2k}{k}^{-1} \frac{4^k \sin(x)^{2k+1}}{2k+1} = \sin(nx) \qquad n \in \mathbb{R}, |x| \le \frac{\pi}{2}$$

***(7.25)**

$$\sum_{k=0}^{\infty} \binom{-(n+1)/2}{k} \binom{(n-1)/2}{k} \binom{2k}{k}^{-1} (2\sin(x))^{2k} = \frac{\cos(nx)}{cos(x)} \qquad n \in \mathbb{R}, |x| < 1$$

***(7.26)**

$$\sum_{k=0}^{\infty} (-1)^k \binom{-2a}{k} \binom{-2b}{k} \binom{-a-b-1/2}{k}^{-1} 2^{-k} = \frac{(a+b-1/2)!(-1/2)!}{(a-1/2)!(b-1/2)!} \qquad a+b+1/2 \neq 0, -1, -2, \ldots$$

***(7.27)**

$$\sum_{k=0}^{\infty} (-1)^k \binom{-2a}{k} \binom{-2b}{k} \binom{-a-b-1}{k}^{-1} 2^{-k} = \frac{\sqrt{\pi}(a+b)!}{a-b} \left(\frac{1}{(b-1/2)!(a-1)!} - \frac{1}{(b-1)!(a-1/2)!} \right)$$

$$a+b+1 \neq 0, -1, -2, \ldots$$

***(7.28)**

$$\sum_{k=0}^{\infty} (-1)^k \binom{-a}{k} \binom{a-1}{k} \binom{-b}{k}^{-1} 2^{-k} = \frac{(b-1)!(-1/2)!2^{1-b}}{(a/2+b/2-1)!(b/2-a/2-1/2)!} \qquad b \neq 0, -1, -2, \ldots$$

***(7.29)**

$$S_j^n = F\left(\begin{matrix} -2n, \ 1/2 \\ n+j+1 \end{matrix} \,\middle|\, 4 \right) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2k}{k} \binom{n+j+k}{k}^{-1}$$

***(7.30)**

$$R_j^n = F\left(\begin{matrix}-2n-1,\ 1/2\\ n+j+1\end{matrix}\ \middle|\ 4\right) = \sum_{k=0}^{2n+1}(-1)^k\binom{2n+1}{k}\binom{2k}{k}\binom{n+j+k}{k}^{-1}$$

***(7.31)** - The two previous sums satisfy the recurrence relations:

$$2\frac{2n+2j+1}{n+j+1}S_{j+1}^n = 3S_j^n + R_j^n$$

***(7.32)** - and:

$$2\frac{2n+2j+1}{n+j+1}R_{j+1}^n = 3R_j^n + S_{j-1}^{n+1}$$

***(7.33)** - Some special values of these summations are as follows:

$$S_{-1}^n = 3 \qquad (n \geq 1)$$

***(7.34)**

$$S_0^n = 1 \qquad (n \geq 0)$$

***(7.35)**

$$S_1^n = \frac{n+1}{2n+1}$$

***(7.36)**

$$S_2^n = \frac{3(n+1)(n+2)}{2(2n+1)(2n+3)}$$

***(7.37)**

$$S_3^n = \frac{(n+2)(n+3)(11n+10)}{4(2n+1)(2n+3)(2n+5)}$$

***(7.38)**

$$S_4^n = \frac{(n+2)(n+3)(n+4)(43n+35)}{8(2n+1)(2n+3)(2n+5)(2n+7)}$$

***(7.39)**

$$R_{-1}^n = -\frac{5n+2}{n} \qquad (n \geq 1)$$

***(7.40)**

$$R_0^n = -1 \qquad (n \geq 0)$$

***(7.41)**

$$R_1^n = 0$$

***(7.42)**

$$R_2^n = \frac{n+2}{2(2n+3)}$$

***(7.43)**

$$R_3^n = \frac{5(n+2)(n+3)}{4(2n+3)(2n+5)}$$

***(7.44)**

$$R_4^n = \frac{21(n+2)(n+3)(n+4)}{8(2n+3)(2n+5)(2n+7)}$$

***(7.45)** - In general one has transformations of the following type:

$$F\left(\begin{matrix} -n, \ 1/2 \\ j+1 \end{matrix} \,\middle|\, 4z\right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2k}{k}\binom{j+k}{k}^{-1} z^k =$$
$$4^j \binom{2j}{j}^{-1} \int_0^1 (\cos(2\pi x))^{2j} (1 - 4z(\sin(2\pi x))^2)^n \mathrm{d}x$$

***(7.46)**

$$= 4^j \binom{2j}{j}^{-1} \frac{1}{t} \sum_{k=0}^{t-1} \left(\cos\frac{2\pi k}{t}\right)^{2j} \left(1 - 4z\left(\sin\frac{2\pi k}{t}\right)^2\right)^n \qquad t > 2(n+j)$$

***(7.47)** - These relations, (7.29) - (7.47) appear in a paper by the author [H. W. Gould] pending publication.

$$= \binom{2j}{j}^{-1} \sum_{k=0}^{n} \binom{n}{k}\binom{2j+2k}{j+k} \frac{(4z)^k (1-4z)^{n-k}}{4^k}$$

***(7.48)** - R. Lagrange:

$$\sum_{k=0}^{n} \binom{x}{k}\binom{y-k}{n-k}\binom{x+y-k}{n}^{-1} \frac{x+y+1-2k}{x+y+1-k} = 1$$

(7.49)

$$\sum_{k=1}^{n} \frac{1}{k}\binom{n}{k}^{-1}\binom{x}{k-1}\binom{y-k}{n-k} = \frac{1}{x+1}\binom{y}{n}\sum_{k=1}^{n}\binom{x+1}{k}\binom{y}{k}^{-1} = \frac{1}{x-y}\left(\binom{x}{n}-\binom{y}{n}\right)$$

$$(1) \quad \sum_{k=1}^{n} \frac{1}{k}\binom{n}{k}^{-1}\binom{x}{k-1}\binom{y-k}{n-k} = \sum_{k=1}^{n} \frac{x!(y-k)!k!(n-k)!}{k!(x-k+1)!(n-k)!(y-n)!n!} =$$

$$= \frac{x!(y-x-1)!}{n!(y-n)!}\sum_{k=1}^{n}\binom{y-k}{x-k+1} \stackrel{B}{=} \frac{x!(y-x-1)!}{n!(y-n)!}[t^{x+1}](1+t)^y\left[\frac{u-u^{n+1}}{1-u} \;\Big|\; u=\frac{t}{1+t}\right] =$$

$$= \frac{x!(y-x-1)!}{n!(y-n)!}[t^{x+1}](1+t)^y\left(\frac{t}{1-t}-\frac{t^{n+1}}{(1+t)^{n+1}}\right)\Big/\left(1-\frac{t}{1+t}\right) =$$

$$= \left([t^x](1+t)^y-[t^{x-n}](1+t)^{y-n}\right)\frac{x!(y-x-1)!}{n!(y-n)!} =$$

$$= \frac{x!(y-x-1)!}{n!(y-n)!}\left(\frac{y!}{x!(y-x)!}-\frac{(y-n)!}{(x-n)!(y-x)!}\right) = \frac{1}{y-x}\binom{y}{n}-\frac{1}{y-x}\binom{x}{n}.$$

$$(2) \quad \frac{1}{x+1}\binom{y}{n}\sum_{k=1}^{n}\binom{x+1}{k}\binom{y}{k}^{-1} = \sum_{k=1}^{n}\frac{1}{x+1}\frac{(x+1)!k!(y-k)!}{k!(x+1-k)!y!}\cdot\frac{y!}{n!(y-n)!} =$$

$$= \frac{x!(y-x-1)!}{n!(y-n)!}\sum_{k=1}^{n}\binom{y-k}{x-k+1} = \text{as above}\ldots$$

2.8 Table 8: summations of the form S:1/2

This table contains 1 identity.

***(8.1)**

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2n+2x}{k+x}^{-1}\binom{2n+2y}{k+y}^{-1} = \frac{\binom{2n}{n}\binom{2n+x+y+1}{n}}{\binom{n+x}{n}\binom{n+y}{n}\binom{2n+2x}{n+x}\binom{2n+2y}{n+y}}$$

2.9 Table 10: summations of the form S:4/0

This table contains 9 identities.

***(10.1)** - Dougall:

$$\sum_{k=0}^{\infty} \binom{-x}{k}^4 \frac{x+2k}{x} = \frac{\sin(\pi x)}{\pi x} \cdot \frac{(-2x)!}{(-x)!^2} \qquad x < \frac{1}{2}$$

***(10.2)** - Dougall / Staver:

$$\sum_{k=0}^{n} k \binom{n}{k}^4 = \frac{n}{2} \sum_{k=0}^{n} \binom{n}{k}^4$$

***(10.3)**

$$\sum_{k=0}^{n} \binom{n}{k}^4 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-k}{n} \sum_{j=0}^{k} (-1)^j \binom{n}{j}^2 \binom{k}{j}$$

***(10.4)**

$$\sum_{k=0}^{2n} (-1)^k \binom{3n}{k} \binom{3n-k}{n}^3 = \binom{3n}{2n} \binom{2n}{n}$$

***(10.5)**

$$\sum_{k=0}^{2n} (-1)^k \binom{3n+1}{k} \binom{3n-k}{n}^3 = 1$$

***(10.6)**

$$P_n(x) P_r(x) = \left(\frac{x+1}{2}\right)^{n+r} \sum_{k=0}^{n+r} \left(\frac{x+1}{x-1}\right)^k \sum_{j=0}^{k} \binom{n}{j}^2 \binom{r}{k-j}^2$$

***(10.7)**

$$\sum_{k=0}^{n+r} (-1)^k \binom{n+r}{k}^{-1} \sum_{j=0}^{k} \binom{n}{j}^2 \binom{r}{k-j}^2 = \delta_{n,r}$$

***(10.8)**

$$\sum_{k=0}^{n+r} (-1)^k \sum_{j=0}^{k} \binom{n}{j}^2 \binom{r}{k-j}^2 = (-1)^{(n+r)/2} \binom{n}{n/2} \binom{r}{r/2} \frac{1+(-1)^n}{2} \cdot \frac{1+(-1)^r}{2}$$

***(10.9)** - Carlitz:

$$\sum_{k=0}^{n} \binom{n}{k}^4 = [x^n](1-x)^{2n} P_n\left(\frac{1+x}{1-x}\right)^2$$

2.10 Table 11: summations of the form S:3/1

This table contains 5 identities.

***(11.1)**

$$\sum_{k=0}^{n} \binom{n}{k}\binom{x}{k}\binom{y}{k+r}\binom{x+y+n}{k}^{-1} = \binom{x+r+n}{n}\binom{y+n}{n+r}\binom{x+y+n}{n}^{-1}$$

***(11.2)**

$$\sum_{k=0}^{n} \frac{1}{2k+1}\binom{n}{k}^2\binom{4n+2k+1}{2k}\binom{2n+k}{k}^{-1} = \frac{1}{2n+1}\binom{4n+1}{2n}$$

***(11.3)**

$$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{k}\binom{z}{n-k}\binom{x+y+z}{k}^{-1} = \binom{x+z}{n}\binom{y+z}{n}\binom{x+y+z}{n}^{-1}$$

***(11.4)** - Harry Bateman: *Notes on Binomial Coefficients*. The limiting case of this, when we replace x by x/z and y by y/z and multiply through with z^{n+1}, letting $z \to 0$ is precisely (1.60). Bateman does a similar thing with a series of the form S:2/1. The result suggests analogies between power relations and binomial sums.

$$\sum_{k=0}^{n/2} \frac{(-1)^k}{n-k+1}\binom{x+y-2k-1}{n-2k}\binom{x}{k}\binom{y}{k}\binom{n+1}{k}^{-1} = \frac{1}{x-y}\left(\binom{x}{n+1} - \binom{y}{n+1}\right)$$

***(11.5)** - Harry Bateman: *Notes on Binomial Coefficients*.

$$\sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{y}{k}\binom{x+y-2k-1}{n-2k-1}\binom{n-1}{k}^{-1} = \frac{n}{x-y}\left(\binom{x}{n} - \binom{y}{n}\right)$$

2.11 Table 12: summations of the form S:2/2

This table contains 9 identities.

***(12.1)**

$$1+2\sum_{k=1}^{\infty}\binom{x}{k}\binom{y}{k}\binom{x+k}{k}^{-1}\binom{y+k}{k}^{-1}=\binom{2x+2y}{2x}\binom{x+y}{x}^{-2}\qquad \Re(x+y)>-\frac{1}{2}$$

***(12.2)**

$$1+2\sum_{k=1}^{\infty}(-1)^k\binom{x}{k}\binom{y}{k}\binom{x+k}{k}^{-1}\binom{y+k}{k}^{-1}=\binom{x+y}{x}^{-1}\qquad \Re(x+y)>-\frac{1}{2}$$

(12.3) - (**NN**)

$$\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}\binom{n+k}{k}^{-1}\binom{x+k}{k}^{-1}=\frac{1}{2}\left(1+\binom{2x+2n}{2n}\binom{x+n}{n}^{-2}\right)$$

$$\binom{n}{k}\binom{x}{k}\binom{n+k}{k}^{-1}\binom{x+k}{k}^{-1}=\binom{2n}{n}^{-1}\binom{2x}{x}^{-1}\binom{2n}{n-k}\binom{2x}{x+k}.$$

$$\sum_{k=-n}^{n}\binom{2n}{n-k}\binom{2x}{x+k}\stackrel{B}{=}[t^n](1+t)^{2n}\left[\frac{(1+u)^{2x}}{u^x}\;\middle|\;u=t\right]=[t^{n+x}](1+t)^{2n+2x}=\binom{2n+2x}{n+x}.$$

The natural sum is therefore:

$$\frac{n!^2x!^2(2n+2x)!}{(2n)!(2x)!(n+x)!}=\binom{2n+2x}{2n}\binom{n+x}{n}^{-2}.$$

If we call S this sum, the original sum is $S/2$ plus one half of the central element $k=0$, that is:

$$\binom{2n}{n}^{-1}\binom{2x}{x}^{-1}\binom{2n}{n}\binom{2x}{x}=1.$$

***(12.4)**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x}{k}\binom{n+k}{k}^{-1}\binom{x+k}{k}^{-1}=\frac{1}{2}\left(1+\binom{x+n}{n}^{-1}\right)$$

***(12.5)** - Equivalent to a problem of H. F. Sandham: N. 4519, in *American Mathematical Montly.*

$$\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{x-1}{k}\binom{-n-1}{k}^{-1}\binom{-x-1}{k}^{-1}=\frac{(-1)^n}{2}\binom{-x-1/2}{n-1}\binom{-x-1}{n-1}^{-1}\binom{-1/2}{n}^{-1}$$

***(12.6)**

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{(n+k+1)/2}{k}^{-2} \frac{2k+1}{4^k(n-k+1)^2} \cdot \frac{1+(-1)^{n+k}}{2} = \frac{1}{2n+1}$$

***(12.7)**

$$\sum_{k=0}^{n} \binom{2n}{2k}^2 \binom{n+k+1/2}{2k}^{-2} \frac{4k+1}{16^k(2n-2k+1)^2} = \frac{1}{4n+1}$$

***(12.8)** - The previous identity can be rewritten in the form:

$$\sum_{k=0}^{n} \binom{2n}{n+k}^2 \binom{2n+2k}{n+k}^{-2} \frac{16^k(4k+1)}{(2n+2k+1)^2} = \frac{1}{4n+1}$$

***(12.9)** - Harry Bateman: *Notes on Binomial Coefficients*; see Gould, *Duke Mathematical Journal*, 32 (1965), p. 706.

$$\sum_{k=0}^{n} \binom{n}{k}\binom{y+k}{k}\binom{x+k}{k}^{-1}\binom{x+y+k+n+1}{n}^{-1} \frac{x+y+2k+1}{x+y+k+1} = \binom{x+n}{n}^{-1}$$

2.12 Table 16: summations of the form S:4/1

This table contains 3 identities.

***(16.1)** - Equivalent to a result of Bailey:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^{-1}\binom{-x}{k}\binom{x-1}{k}\binom{-y}{n-k}\binom{y-1}{n-k} = (-4)^n \binom{(n-x-y)/2}{n}\binom{(n+x-y-1)/2}{n}$$

***(16.2)** - Equivalent to a formula of W. A. Al-Salam; see (22.1).

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2a+2n}{a+k}\binom{2b+2n}{b+k}\binom{2c+2n}{c+k}\binom{2d+2n}{d+k}^{-1}=$$

$$=\frac{(-1)^n}{n!}\frac{(2a+2n)!(2b+2n)!(2c+2n)!(n+d)!(2n)!d!}{(a+n)!(b+n)!(c+n)!(2d+2n)!(a+2n)!}\times$$

$$\times\frac{(a+b+3n)!(a+c+3n)!(b+c+3n)!}{(b+2n)!(c+2n)!(a+b+2n)!(a+c+2n)!(b+c+2n)!}$$

$$a,b,c\in\mathbb{R},\ d=a+b+c+3n$$

***(16.3)** - H. L. Krall:

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{k}\binom{z}{n-k}\binom{x+y+z-k}{n-k}\binom{n}{k}^{-1}=\binom{x+z}{n}\binom{y+z}{n}$$

2.13 Table 17: summations of the form S:3/2

This table contains 5 identities.

***(17.1)** - Dougall / Dixon:

$$\sum_{k=0}^{\infty}(-1)^k\binom{-z}{k}\binom{x}{k}\binom{y}{k}\binom{x+z+k}{k}^{1}\binom{y+z+k}{k}^{-1}=\frac{(x+z)!(y+z)!(z/2)!(x+y+z/2)!}{(x+z/2)!(y+z/2)!(x+y+z)!z!}$$

$$\Re(x+y+z/2)>-1$$

***(17.2)**Saalschütz:

$$\sum_{k=0}^{\infty}\binom{n}{k}\binom{x}{k}\binom{y}{k}\binom{x+y+z+n}{k}^{1}\binom{z+k}{k}^{-1}=\binom{x+z+n}{n}\binom{y+z+n}{n}\binom{z+n}{n}^{-1}\binom{x+y+z+n}{n}^{-1}$$

***(17.3)** - Watson:

$$\sum_{k=0}^{\infty}(-1)^k\binom{-x}{k}\binom{-y}{k}\binom{-z}{k}\binom{-(x+z+k)/2}{k}^{1}\binom{-2z}{k}^{-1}=$$

$$=\frac{(-1/2)!(z-1/2)!((x+y-1)/2)!(z-(x+y+1)/2)!}{((x-1)/2)!((y-1)/2)!(z-(x+1)/2)!(x-(y+1)/2)!}$$

***(17.4)** - Bailey. We have $f(x,y) = f(y,x)$ if $f(x,y)$ is defined as:

$$f(x,y) = (1-z)^{2x-1} \sum_{k=0}^{\infty} (-1)^k \binom{1-2x}{k} \binom{-x-1/2}{k} \binom{y-x+1/2}{k} \binom{1/2-x}{k}^{-1} \binom{-x-y-1/2}{k}^{-1} z^k$$

***(17.5)** - Th. Clausen:

$$\sum_{k=0}^{\infty} (-1)^k \binom{-2x}{k} \binom{-2y}{k} \binom{-x-y}{k} \binom{-2x-2y}{k}^{-1} \binom{-x-y-1/2}{k}^{-1} z^k = \left(\sum_{k=0}^{\infty} (-1)^k \binom{-x}{k} \binom{-y}{k} \binom{-x-y-1/2}{k}^{-1} z^k \right)^2$$

2.14 Table 21: summations of the form S:6/0

This table contains 1 identity.

***(21.1)**

$$\sum_{k=0}^{2n} (-1)^k \binom{3n}{k}^3 \binom{3n-k}{n}^3 = (-1)^n \binom{2n}{n} \binom{3n}{2n}^4$$

2.15 Table 22: summations of the form S:5/1

This table contains 2 identities.

***(22.1)**

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2a}{a-n+k} \binom{2b}{b-n+k} \binom{2c}{c-n+k} \binom{d+k}{k} \binom{d+2n}{k}^{-1} =$$

$$= \frac{(-1)^n}{n!} \cdot \frac{(2a)!(2b)!(2c)!(n+d)!(2n)!(n+a+b)!}{(n+a)!(n+b)!(n+c)!(d+2n)!a!b!c!} \cdot \frac{(n+a+c)!(n+b+c)!}{(a+b)!(a+c)!(b+c)!} \qquad d = a+b+c$$

By use of the identity:

$$\binom{x+k}{k}\binom{x+2n}{k}^{-1} = \binom{2x+2n}{x}\binom{2x+2n}{x+k}^{-1}$$

it is possible to transform the above S:5/1 into a series of the form S:4/1 and the result is tabulated under that heading.

***(22.2)** - Special case of the previous identity:

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^4\binom{3n+k}{k}\binom{5n}{k}^{-1} = (-1)^n\binom{3n}{n}^3\frac{(4n)!(2n)!}{(5n)!n!}$$

2.16 Table 23: summations of the form S:4/2

This table contains 1 identity.

***(23.1)** - This identity follows from (17.5):

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{k}\binom{x}{n-k}\binom{y}{n-k}\binom{x+y-1/2}{k}^{-1}\binom{x+y-1/2}{n-k}^{-1} =$$

$$= \binom{2x}{n}\binom{2y}{n}\binom{x+y}{n}\binom{2x+2y}{n}^{-1}\binom{x+y-1/2}{n}^{-1}$$

2.17 Table 24: summations of the form S:3/3

This table contains 1 identity.

***(24.1)** - Dougall. The author [Gould] shows in a paper awaiting publication, that this relation implies Fjeldstad's relation (6.1).:

$$1+2\sum_{k=1}^{\infty}(-1)^k\binom{x}{k}\binom{y}{k}\binom{z}{k}\binom{x+k}{k}^{-1}\binom{y+k}{k}^{-1}\binom{z+k}{k}^{-1} =$$

$$= \binom{x+y+z}{y}\binom{x+y}{y}^{-1}\binom{y+z}{y}^{-1} = \frac{x!y!z!(x+y+z)!}{(x+y)!(y+z)!(z+x)!} \qquad \Re(x+y+z) > -1$$

2.18 Table 31: summations of the form S:4/3

This table contains 2 identities.

***(31.1)** - Dougall:

$$\sum_{k=0}^{\infty} \frac{(-1)^k(c+2k)}{c} \binom{c+k-1}{k} \binom{x}{k} \binom{y}{k} \binom{z}{k} \binom{x+c+k}{k}^{-1} \binom{y+c+k}{k}^{-1} \binom{z+c+k}{k}^{-1} =$$

$$= \frac{(x+c)!(y+c)!(z+c)!(x+y+z+c)!}{(y+z+c)!(z+x+c)!(x+y+c)!c!} \qquad \Re(x+y+z+c) > -1$$

***(31.2)**

$$\sum_{k=0}^{\infty} \binom{-x}{k} \binom{-1-x/2}{k} \binom{-y}{k} \binom{-z}{k} \binom{-x/2}{k}^{-1} \binom{-1-x+y}{k}^{-1} \binom{-1-x+z}{k}^{-1} =$$

$$= \frac{(x-y)!(x-z)!}{x!(x-y-z)!}$$

2.19 Table 71: summations of the form S:6/5

This table contains 1 identity.

***(71.1)** - One of the most general identities known is that of Dougall, which may be expressed as an S:6/5 or as a ${}_7F_6$ with unit argument:

$$\sum_{k=0}^{n} \frac{(-1)^k(c+2k)}{c} \cdot \frac{\binom{c+k-1}{k}\binom{n}{k}\binom{x}{k}\binom{y}{k}\binom{z}{k}\binom{x+y+z+n+2c+k}{k}}{\binom{n+c+k}{k}\binom{x+c+k}{k}\binom{y+c+k}{k}\binom{z+c+k}{k}\binom{x+y+z+c+n}{k}} =$$

$$= \frac{\binom{x+y+c+n}{n}\binom{y+z+c+n}{n}\binom{z+x+c+n}{n}\binom{c+n}{n}}{\binom{x+c+n}{n}\binom{y+c+n}{n}\binom{z+c+n}{n}\binom{x+y+z+c+n}{n}}$$

2.20 Table 97: summations of the form S:7/6

This table contains 1 identity.

***(97.1)** - Expressed as a ${}_7F_6$ one has the equivalent form of Dougall's formula:

$$ {}_7F_6\left[\begin{matrix} a, 1+a/2, b, c, d, e, -n \\ a/2, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a+n \end{matrix}\right] = $$

$$ = \frac{(1+a)_n(1+a-b-c)_n(1+a-b-d)_n(1+a-c-d)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-b-c-d)_n} $$

$$ 1+2a = b+c+d+e-n,\ n \in \mathbb{N},\ (x)_n = x(x+1)\cdots(x+n-1) $$

2.21 Table X: summations of the form S:p/0

This table contains 18 identities.

***(X.1)** - An asymptotic formula. For $p \in \mathbb{N}_0$, Polya and Szegö.

$$ \sum_{k=0}^{n} \binom{n}{k}^p \sim \frac{2^{pn}}{\sqrt{p}} \left(\frac{2}{\pi n}\right)^{(p-1)/2} $$

***(X.2)** - Gould:

$$ \sum_{k=0}^{n} \binom{x}{k}^p \frac{k!^p}{x^{(k+1)p}} \left((x-k)^p - x^p\right) = \binom{x}{n+1}^p \frac{(n+1)!^p}{x^{(n+1)p}} - 1 $$

***(X.3)** - The three relations (X.3), (X.4) and (X.5) are connected with general expansions of Worpitzky, as shown by Carlitz, and were "rediscovered" by various modern writers, in particular by Shanks.

$$ \sum_{k=0}^{rp-r+1} \binom{x+k-1}{rp} \sum_{j=0}^{k-1} (-1)^j \binom{rp+1}{j} \binom{k+r-j-1}{r}^p = \binom{x}{r}^p \qquad r,p \in \mathbb{N}_0 1 $$

***(X.4)**

$$ \sum_{k=0}^{rp} (-1)^k \binom{rp+1}{k} \binom{rp-k+y+r-1}{r}^p = \binom{1-y}{r}^p $$

***(X.5)**

$$\sum_{k=0}^{rp-r+1} (-1)^k \binom{rp+1}{k} \binom{rp-k}{r}^p = 1$$

***(X.6)** - Staver. Definition:

$$S_n(q) = \sum_{k=0}^{n} \binom{n}{k}^q$$

***(X.7)** - Staver. Definition:

$$S_{n,p}(q) = \sum_{k=0}^{n} \binom{n}{k}^q k^p$$

***(X.8)** - Then the following formulas are known:

$$S_{n,1}(3) = \frac{n}{2} S_n(3)$$

***(X.9)**

$$S_{n,2}(3) = \frac{n^2}{6} S_n(3) + \frac{2n^2}{3} S_{n-1}(3)$$

***(X.10)**

$$S_{n,3}(3) = n^3 S_{n-1}(3)$$

***(X.11)**

$$S_{n,4}(3) = \frac{n^3(n+1)}{2} S_{n-1}(3)$$

***(X.12)** - Staver:

$$2S_{n,5}(3) = \frac{n^5}{6} S_n(3) - \frac{5n^4(n-3)}{6} S_{n-1}(3)$$

***(X.13)** - Franel. This relation was first proved by J. Franel, in answer to a question posed by C. A. Laisant in the first volume (1894) of the journal *L'Intermédiarie des Mathématiciens*.

$$n^2 S_n(3) = (7n^2 - 7n + 2) S_{n-1}(3) + 8(n-1)^2 S_{n-2}(3)$$

***(X.14)** - Frenel / Staver. Frenel also found the following formula as well as other interesting results:

$$n^3 S_n(4) = 2(2n-1)(3n^2-3n+1)S_{n-1}(4) + (4n-3)(4n-4)(4n-5)S_{n-2}(4)$$

***(X.15)** - Nanjundiah:

$$\sum_{k=0}^{n} \binom{n}{k}^q x^{n-k} y^k = \sum_{k=0}^{n/2} C_{n,k}^{(q)} \binom{n-k}{k} (x+y)^{n-2k} (xy)^k$$

$$C_{n,k}^{(q+1)} = \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} C_{n,j}^{(q)}$$

This general result includes several special cases in the preceding tables. Thus:

$$C_{n,k}^{(-1)} = (-1)^k \binom{n+1}{k} \binom{n}{k}^{-1}, \qquad C_{n,k}^{(0)} = (-1)^k,$$

$$C_{n,k}^{(1)} = \delta_{n,k}, \qquad C_{n,k}^{(2)} = \binom{n}{k}, \qquad C_{n,k}^{(3)} = \binom{n}{k}\binom{n+k}{k}$$

MacMahon [53] has given very general generating functions for $S_n(q)$; however, this is too extensive to list in the present tables.

***(X.16)** - This identity follows trivially from (Z.8):

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{a_1+b_1k}{m_1} \binom{a_2+b_2k}{m_2} \cdots \binom{a_r+b_rk}{m_r} = 0 \qquad 0 \le m_1+m_2+\cdots+m_r < n$$

***(X.17)** - Again, a trivial consequence of (Z.8):

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{\binom{a_1k}{b_1}}{c_1} \binom{\binom{a_2k}{b_2}}{c_2} \cdots \binom{\binom{a_rk}{b_r}}{c_r} = 0 \qquad 0 \le b_1c_1+b_2c_2+\cdots+b_rc_r < n$$

***(X.18)** - Another trivial consequence of (Z.8):

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{a_1k^{b_1}}{c_1} \binom{a_2k^{b_2}}{c_2} \cdots \binom{a_rk^{b_r}}{c_r} = 0 \qquad 0 \le b_1c_1+b_2c_2+\cdots+b_rc_r < n$$