Arithmetic of Weil numbers and Hecke fields.

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Abstract: Analyzing prime factorization of Weil numbers in the union of algebraic extensions with bounded degree of the cyclotomic field K of all p-power roots of unity, we show that there are only finitely many Weil p-numbers of a given weight for a prime p (upto roots of unity). Applying this fact to Hecke eigenvalues of cusp forms in p-adic analytic families of cusp forms of p-power level, we show that the field generated by the eigenvalues over the family has unbounded degree over K.

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§0.0. Hecke fields.

To me at least, one of the most mysterious number fields is the Hecke field. For example, in the space of modular forms on $SL_2(\mathbb{Z})$, for each even weight $2k \geq 4$, we have Eisenstein series

$$E_k = -\frac{B_{2k}}{2k} + \sum_{n=1}^{\infty} \Big(\sum_{0<d|n} d^{k-1} \Big) \exp(2\pi i n z).$$

Its Hecke eigenvalue for the hecke operator $T(n)$ is the sum of divisors $\sum_{0<d|n} d^{k-1}$. So its Hecke field $\mathbb{Q}(E_k)$ generated by Hecke eigenvalues is $\mathbb{Q}$ nothing imressive. However cusp forms are different. For example, $S_{24}(SL_2(\mathbb{Z}))$ has two eigenforms f and f^σ (a Galois conjugate, and $\mathbb{Q}(f) = \mathbb{Q}(\sqrt{144169})$.

Hecke eigenvalues of $T(l)$ for a prime l is the sum of a Weil l-number of weight $2k-1$ and its complex conjugate; so, arithmetic of Weil number is very important to analyze Hecke fields.

§0.1. What Hecke field means?

What is the meaning of this field $\mathbb{Q}(\sqrt{144169})$. Perhaps, the coefficients field of the unique rank 2 motive over $\mathbb{Q}$ of Hodge weight $(25, 0)$ "unramified" everywhere? After this, alway up to weight thousands, $\mathbb{Q}(f_{2k})$ for a Hecke eigenform f_{2k} of weight $2k$ has degree $\dim S_{2k}(SL_2(\mathbb{Z}))$ and conjugates of f_k span $S_{2k}(SL_2(\mathbb{Z}))$. This is conjectured to be true by Maeda for all $2k$.

Perhaps, if we have the moduli of motives of given weight, raising weight and making coefficient field bigger, it becomes more and more complicated, and it would not have many $\mathbb{Z}$-points? Thus the coefficient field (= the Hecke field) getting bigger?

§0.2. What we study?

Instead of raising weight, fixing the weight, I want to study by growing level considering a prime power level p. If Hecke eigenforms f are p-adic analytically deformed over the (spectrum) of the Iwasawa algebra, we describe how the degree $[K(f) : K]$ grows for the cyclotomic fields K of all p-powers root of unity.

Indeed, if the p-slope of the family is zero (an ordinary family), we prove $[K(f) : K]$ grows indefinitely if f runs over infinite set of fixed weight in the family.

We will come back to Maeda's conjecture at the end.

§1. What are Weil numbers.

A Weil l-number of weight k is an algebraic integer α with $|\alpha^\sigma| = l^{k/2}$ for all Galois conjugates α^σ of α. Here l is a prime.

If $E_{/\mathbb{F}_l}$ is an elliptic curve and define $a_l \in \mathbb{Z}$ by $1 + l - a_l = |\mathbf{P}^1(\mathbb{F}_l)| - a_l = |E(\mathbb{F}_l)|$, then by Hasse,

$$a_l = \alpha + \overline{\alpha} \quad \text{and} \quad \alpha\overline{\alpha} = l$$

for a Weil l-number α of weight 1. In other words, $X^2 - a_l X + l = (X - \alpha)(X - \overline{\alpha})$. Moreover

$$1 + l^n - (\alpha^n + \overline{\alpha}^n) = |E(F_{l^n})| \text{ for all } n.$$

Or α and $\overline{\alpha}$ is the Frobenius eigenvalue acting on $T_pE = \varprojlim_n E[p^n](\overline{\mathbb{F}}_l)$ for another prime $p \neq l$.

§2. Tate modules.

If E is defined by $y^2 = 4x^3 - g_2x - g_3$ with $g_j \in \mathbb{Z}$, by reducing its equation modulo a prime l, we get an elliptic curve $E_{l/\mathbb{F}_l}$ as long as l is prime to the discriminant Δ_E of $4x^3 - g_2x - g_3$.

Since $T_pE \cong T_pE_l$ as $\mathbb{Z}_p$-modules, the action of Frobebius ($Frob_l$) on T_pE has eigenvalues α_l for a Weil l-number α of weight 1, and $\alpha_l + \overline{\alpha}_l = a_l \in \mathbb{Z}$ as long as $l \nmid \Delta_E$.

Define

$$L(s,E) = \prod_l (1 + a_l l^{-s} + l^{1-2s})^{-1}$$
$$= \prod_l [(1 - \alpha_l l^{-s})(1 - \overline{\alpha}_l l^{-s})]^{-1},$$

and write $L(s,E) = \sum_n a_n n^{-s}$ as a Dirichlet series.

Weil numbers appear also related to cusp forms as we will see later.

§3. Prime decomposition in a cyclotomic field.

Consider the group μ_{p^n} of p^n-th roots of unity. The roots in μ_{p^n} generates the cyclotomic field $\mathbb{Q}(\mu_{p^n})$ whose Galois group $\mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^\times = \mathrm{Aut}(\mu_{p^n})$. For $\mu_{p^\infty} = \bigcup_n \mu_{p^n}$, $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \cong \mathbb{Z}_p^\times = \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})^\times$such that

$$\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \ni Frob_l \; (= \text{Frobenius at prime } l) \mapsto l \in \mathbb{Z}_p^\times$$

as $\zeta^{Frob_l} = \zeta^l$ for all $l \in \mu_{p^\infty}$. Thus the decomposition group D_l at l is generated by l in $\mathbb{Z}_p^\times$; so, D_l is an open subgroup. Thus

There are finitely many primes in $\mathbb{Q}(\mu_{p^\infty})$ above l.

§4. Not many Weil numbers.

Theorem 1 (finiteness). *Let $\overline{\mathbb{Q}}$ be the field of all algebraic numbers in $\mathbb{C}$. In*

$$\bigcup_{K;[K:\mathbb{Q}(\mu_{p^\infty})]\le d} K \subset \overline{\mathbb{Q}},$$

there are finitely many Weil l-numbers of a given weight k up to roots of unity.

Assume first $d = 1$. Consider a Weil number as above. Suppose $l \neq p$. Then $(\alpha) = \prod_{\mathfrak{l}} \mathfrak{l}^{e(\mathfrak{l})}$ for primes $\mathfrak{l}$ in $\mathbb{Q}(\mu_{p^\infty})$. Since $\alpha\overline{\alpha} = l^k$, we have $e(\mathfrak{l}) + e(\bar{\mathfrak{l}})$ is bounded by k. Thus there are finitely many possibilities of factorization. If $(\alpha) = (\beta)$, writing $\alpha = \zeta\beta$, ζ is a unit of the integer ring of $\mathbb{Q}(\mu_{p^\infty})$ with $|\zeta^\sigma| = 1$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. By Kronecker, ζ has to be a root of unity. If $l = p$, all equivalent to $\sqrt{\pm p}^k$.

General case can be similarly proven as $(\alpha^{d!})$ is basically factorized into a product of primes in a tamely ramified extension $\mathbb{Q}_l(\mu_{p^\infty})$ of degree $\le d$, and such extensions are finitely many if we bound the degree over $\mathbb{Q}_l(\mu_{p^\infty})$.

§5. Power series hitting roots of unity over roots of unity.

Pick $\Phi(T) \in W[[T]]$ for a DVR finite over $\mathbb{Z}_p$. Regard $W[[T]] = \varprojlim_n W[t,t^{-1}]/(t^{p^n}-1)$ for $t = 1+T$; so, $\mathrm{Spf}(\Lambda) = \widehat{\mathbb{G}}_{m/\mathbb{Z}_p}$. Regard Φ as a function of t (so, $\Phi(t) = \Phi|_{T=t-1}$).

Lemma 1 (Binomial lemma)**.** *If* $\Phi(1) = 1$ *and* $\Phi(\zeta) \in \mu_{p^\infty}$ *for infinitely many* $\zeta \in \mu_{p^\infty}$*, then there exists* $s \in \mathbb{Z}_p$ *such that* $\Phi(t) = t^s = \sum_{n=0}^{\infty} \binom{s}{n} T^n$*, where* $\binom{s}{n} = \frac{s(s-1)\cdots(s-n+1)}{n!}$ *if* $n > 0$ *and* $\binom{s}{0} = 1$*.*

Proof. For simplicity, assume $W = \mathbb{Z}_p$. Note $\Phi(\zeta^\sigma) = \Phi(\zeta)^\sigma$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ as long as $\zeta, \Phi(\zeta) \in \mu_{p^\infty}$. Since $\zeta^\sigma = \zeta^z$ for some $z \in \mathbb{Z}_p^\times$, we have $\Phi(t^z) = \Phi(t)^z$ for all $z \in \mathbb{Z}_p^\times$. The graph $\Gamma_\Phi \subset \widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m$ of Φ contains $(t_0^n, \Phi(t_0)^n)$ for all n prime to p for an infinite order $t_0 \in \widehat{\mathbb{G}}_m$. From this, taking the logarithm, it is not hard to conclude, Γ_Φ is a formal subgroup as in the lemma. $\square$

§6. Power series hitting Weil numbers over roots of unity.

Lemma 2 (Degree lemma)**.** *Let* $\Phi \in \Lambda^\times$*. Suppose* $\Phi(\zeta\gamma^k)$ *for* $\gamma = 1+p \in 1+p\mathbb{Z}_p$ *and* $\zeta \in \Omega$ *is a Weil number of weight* k *for an infinite subset* $\Omega \subset \mu_{p^\infty}$*. Then* $\limsup_{\zeta\in\Omega}[\mathbb{Q}(\Phi(\zeta\gamma^k)) : \mathbb{Q}] < \infty$ *if and only if* $\Phi(t) = ct^s$ *for some* $s \in \mathbb{Z}_p$ *and a constant* $0 \neq c \in \overline{\mathbb{Q}}$*.*

Proof. Suppose $\limsup_{\zeta\in\Omega}[\mathbb{Q}(\Phi(\zeta\gamma^k)) : \mathbb{Q}] \leq d$. By the finiteness lemma, for an infinite subset $\Omega_1 \subset \Omega$, $\Phi(\zeta\gamma^k) = \zeta'\alpha$ for a Weil number $\alpha \in \bigcup_{\zeta\in\Omega}\mathbb{Q}(\Phi(\zeta\gamma^k))$. Note that $\alpha \in W$ is invertible as $\Phi \in \Lambda^\times$. Replacing Φ by $\alpha^{-1}\Phi$ and making a varibale change, $t \mapsto t\gamma^{-k}$ on $\widehat{\mathbb{G}}_m$, by Binomial formula lemma, we have $\Phi(T) = \alpha(\gamma^k t)^s = ct^s$. □

To give a typical example of a power series hitting Weil numbers over roots of unity, we look into Hecke eigen modular forms.

§7. Cusp forms.

What are cusp forms? Define a group of nteger matirces

$$\Gamma_0(N) = \Big\{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \\ \Big| c \equiv 0 \mod N, \det(\gamma) = ad - bc = 1\Big\}.$$

Here N is a fixed positive integer. A cusp form $f \in S_{k+1}(N, \chi)$ is a holomorphic function on $\mathfrak{H} = \{z \in \mathbb{C} | 2\,\mathrm{Im}(z) = -i(z - \overline{z}) > 0\}$ satisfies $f(\frac{az+b}{cz+d}) = \chi(a)^{-1} f(z)(cz+d)^{k+1}$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $|f(z)\,\mathrm{Im}(z)^{(k+1)/2}|$ bounded over $\mathfrak{H}$. Here χ is a Dirichlet character modulo N.

Such f has Fourier expansion for $a(n, f) \in \mathbb{C}$ and $q = \exp(2\pi i z)$

$$f(z) = \sum_{n=1}^{\infty} a(n, f) \exp(2\pi i n z) = \sum_{n=1}^{\infty} a(n, f) q^n.$$

Put $S_{k+1}(N, \chi; A) = S_{k+1}(N, \chi) \cap A[[q]]$ for $A \subset \mathbb{C}$.

§8. Hecke operators.

Put $T(n) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \middle| ad - bc = n, (a, N) = 1, N|c \right\}$, Decompose $T(n) = \bigsqcup_\alpha \Gamma_0(N)\alpha$ for finitely many cosets $\Gamma_0(N)\alpha$. Starting from a cusp form f on $\Gamma_0(N)$, define

$f|T(n) = \sum_\alpha f|\alpha$ (an average over the set $T(n)$).

Here $f|\alpha(z) = \det(\alpha)^k \chi(a) f(\frac{az+b}{cz+d})(cz + d)^{-k-1}$ for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$. Since $T(n)\Gamma_0(N) \subset \Gamma_0(N)$, $f|T(n) \in S_{k+1}(\Gamma_0(N), \chi)$ and $T(n) : S_{k+1}(N, \chi) \to S_{k+1}(N, \chi)$ is a linear operator. By a computation,

$$a(n, f|T(m)) = \sum_{0<d|(m,n),(d,N)=1} d^k \chi(d) a(\frac{mn}{d^2}, f).$$

So $T(n)T(m) = T(m)T(n)$,
and $T(n) \in \mathrm{End}(S_{k+1}(N, \chi; \mathbb{Z}[\chi]))$ for the ring $\mathbb{Z}[\chi]$ generated by the values of χ: the eigenvalues of $T(n)$ are algebraic **integers**.

§9. Hecke eigenvalues.

The algebra

$$h_{k+1}(N,\chi;A) = A[T(n)|n = 1,2,\dots]$$

inside $\mathrm{End}(S_{k+1}(N,\chi;A))$ is commutative; so, we can make $T(n)$ simultaneously upper triangular. Define $(\cdot,\cdot) : h_{k+1} \times S_{k+1} \to A$ by

$$(h,f) = a(1,f|h)(\Rightarrow (T(n),f) = a(n,f)).$$

By the above formula, this is a perfect duality if A is a field or a DVR. For simplicity, assume χ is **primitive**.

Suppose hereafter $f|T(n) = a_n f$ (for all n). Then $a_1 = 1$ as $T(1)$ is the identity. Since

$$a(n,f) = (T(n),f) = a(1,f|T(n)) = a_n a(1,f),$$

normalizing $a(1,f) = 1$, we have

$$a_n = a(n,f) \quad \text{for all } n > 0.$$

§10. Weil numbers and L-function.

If $p|N$ is a prime, a_p is a Weil p-number of weight k, and if $l \nmid N$, roots α_l and β_l of $X^2 - a_l X + \chi(l) l^k = 0$ are Weil l-numbers of weight k (Ramanujan-Petersson conjecture proven by Eichler-Shimura/Deligne/Deligne-Serre in different settings). Put

$$L(s, f) = \sum_n a_n n^{-s} = \prod_l (1 - a_l l^{-s} + l^{k-2s})^{-1}.$$

As conjectured by Shimura–Taniyama and proven Wiles–Taylor et al, for every elliptic curve $E_{/\mathbb{Q}}$, there exists $f = f_E$ as abobe such that

$$L(s, E) = L(s, f)$$

and $L(s, E)$ is analytically continued to the whole $\mathbb{C}$-plane. This fact is true for any simple abelian variety A over $\mathbb{Q}$ such that $\mathrm{End}(A_{/\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a field of degree $\dim A$.

§11. Galois representation

We fix an algebraic closure $\overline{\mathbb{Q}}_p$ which contains an algebraic closure $\overline{\mathbb{Q}} \subset \mathbb{C}$. For a subfield K of $\overline{\mathbb{Q}}_p$, define

$$K(f) = K(a_n | n = 1, 2, \dots),$$

which is called the Hecke field of f over K. By Deligne/Deligne–Serre/Eichler–Shimura, we have an irreducible representation

$$\rho = \rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Q}_p(f))$$

unramified outside Np characterized by

$$\det(1 - \rho(Frob_l)X) = 1 - a_l X + \chi(l) l^k X^2,$$

If $|a_p|_p = p^\alpha$ with $\alpha = 0$ (slope $\alpha = 0$), we have

$$(\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)})^{ss} \cong \epsilon \oplus \delta$$

with unramified character δ satisfying $\delta(Frob_p) = a_p$.

§**12.** p**-adic deformation** Now assume $N = N_r = Np^r$ for a fixed prime $p \nmid N$. Recall that $\widehat{\mathbb{G}}_m = \mathrm{Spf}(\Lambda)$ for $\Lambda = W[[T]]$ with $t = 1 + T$. Let $\gamma = 1 + p$. The evaluation at $t = \varepsilon(\gamma)\gamma^k$: $\Phi(t) \mapsto \Phi(\varepsilon(\gamma)\gamma^k)$ induces $P_{k,\varepsilon} \in \widehat{\mathbb{G}}_m(\overline{\mathbb{Q}}_p)$ called an arithmetic point of weight k, where $\varepsilon : \mathbb{Z}_p^\times \to \mu_{p^\infty}$ is a p-power order character.

A formal expansion

$$F(q) = \sum_{n=1}^{\infty} a(n, F)(t)q^n \in \Lambda[[q]]$$

is called a Λ-adic form if

$$F_{k,\varepsilon} = \sum_{n=1} a(n, F)(\varepsilon(\gamma)\gamma^k)q^n$$

is a slope 0 Hecke eigenform in $S_{k+1}(N, \chi\varepsilon\omega^{-k}; \overline{\mathbb{Q}})$ for all $k \geq 2$ and ε, where $\omega : \mathbb{Z}_p^\times \to \mu_{p-1}$ is the Teichmüller character. Write $S(Np, \chi; \Lambda)$ for the space of slope 0 Λ-adic forms, which is a Λ-module. Then $S(Np, \chi; \Lambda)$ is free of finite rank over Λ and by $F \mapsto F_{k,\varepsilon}$,

$$S(Np, \chi; \Lambda) \otimes_{\Lambda, P_{k,\varepsilon}} \overline{\mathbb{Q}}_p \cong S^0_{k+1}(Np^{r+1}, \chi\varepsilon\omega^{-k}; \overline{\mathbb{Q}}_p).$$

§13. Hecke eigenvalues of Λ-adic form.

Let $T(n)$ acts on $S(Np,\chi;\Lambda)$ by

$$a(n,F|T(m)) = \sum_{0<d|(m,n),(d,Np)=1} \kappa(d)\chi(d)a(\frac{mn}{d^2},F),$$

where $\kappa(d)(t) = t^{\log_p(d)/\log_p(\gamma)}$ for the p-adic logarithm $\log_p$. Then $\kappa(d)(\varepsilon(\gamma)\gamma^k) = d^k\omega^{-k}(d)$. Thus we have a commutative diagram

$$\begin{array}{ccc} S(Np,\chi;\Lambda) & \xrightarrow{T(n)} & S(Np,\chi;\Lambda) \\ {\scriptstyle P_{k,\varepsilon}}\downarrow & & \downarrow{\scriptstyle P_{k,\varepsilon}} \\ S_{k+1}(N_r,\chi\varepsilon\omega^{-k};\overline{\mathbb{Q}}_p) & \xrightarrow{T(n)} & S_{k+1}(N_r,\chi\varepsilon\omega^{-k};\overline{\mathbb{Q}}_p). \end{array}$$

Thus $T(n)$ is a well defined Λ-linear operator of $S(Np,\chi;\Lambda)$.

§14. Weil numbers and $a(l, F)$.

Hereafter we assume that F is a Hecke eigenform with $F|T(n) = a(n, F)F$ for $a(p, F) \in \Lambda$. Then

$$a(p, F)(\varepsilon(\gamma)\gamma^k) = P_{k,\varepsilon}(a(p, F)) \\ = a(p, F)(\varepsilon(\gamma)\gamma^k) = a_p$$

for Hecke eigenvalue a_p of the specialized form $F_{k,\varepsilon}$. In particular, a Weil p-number of weight k.

If we take a root A_l of $X^2 - a(l, F)X + \kappa(l)\chi(l)$, similarly, $A_l(\varepsilon(\gamma)\gamma^k)$ is a Weil l-number of weight k.

This is exactly the setting in the degree lemma.

§15. Λ-adic Galois representation.

We have a Galois representation

$$\rho_F : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\Lambda)$$

interpolating $F_{k,\varepsilon}$. It is unramified outside Np and is characterized by

$$\det(1 - \rho_F(Frob_l)) = 1 - a(l, F)X + \kappa(l)\chi(l)X^2$$

for all primes $l \nmid Np$.

By the above formula,

$$\begin{aligned} P_{k,\varepsilon}(\det(1 - \rho_F(Frob_l)) \\ = 1 - a(l, F_{k,\varepsilon})X - \chi\varepsilon\omega^{-k}(l)X^2 \\ = \det(1 - \rho_{F_{k,\varepsilon}}(Frob_l)). \end{aligned}$$

Thus

$$P_{k,\varepsilon} \circ \rho_F \cong \rho_{F_{k,\varepsilon}}.$$

§16. Degree of Hecke fields; Horizontal theorem.

Theorem 2. *Assume $N = 1$. Fix a weight k and put $K = \mathbb{Q}(\mu_{p^\infty})$. Then for any infinite set $\Omega \subset \mathrm{Hom}(\mathbb{Z}_p^\times, \mu_{p^\infty}(\overline{\mathbb{Q}}))$,*

$$\limsup_{\varepsilon \in \Omega} [K(F_{k,\varepsilon}) : K] = \infty.$$

If we use the filter made of complement of finite subsets of $\mu_{p^\infty}(\overline{\mathbb{Q}}_p)$, we can replace "lim sup" by "lim".

If the limit is finite, by Degree lemma, $A_l = ct^{s_l}$ for all l. Making slightly more effort, one can show that there exists a finite extension field $L/\mathbb{Q}$ such that $Frob_{\mathfrak{l}}$ in $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$ satisfies

$$\mathrm{Tr}(\rho_F(Frob_{\mathfrak{l}})) = t^{s_l} + t^{s'_l}$$

for $s_l, s'_l \in \mathbb{Z}_p$ and $\det \rho_F = \kappa\chi$.

§17. Abelian image.

The above formula implies

$$\mathrm{Tr}(\rho_F^{sym\otimes 2}(Frob_{\mathfrak{l}})) = t^{2s_l} + t^{2s'_l} + \kappa\chi(Frob_{\mathfrak{l}})$$
$$= \mathrm{Tr}(\rho_F^2(Frob_{\mathfrak{l}})) + \kappa\chi(Frob_{\mathfrak{l}})$$

regarding $\kappa\chi$ as a Galois character by the iden-tiy $\mathbb{Z}_p^\times = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$. Thus the trace of the square ρ_F^2 is equal to the trace of virtual representation

$$\rho_F^{sym\otimes 2} - \kappa\chi.$$

It is an exercise to show that this happen only when $\rho_F|_{\mathrm{Gal}(\overline{\mathbb{Q}}/L)}$ has abelian image.

By Ribet, if $N = 1$, the image of ρ_f contains (an open subgroup of) $SL_2(\mathbb{Z}_p)$ and hence $P_{k,\varepsilon} \circ \rho_F = \rho_{F_{k,\varepsilon}}$ can never be abelian over $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$ for any finite extension $L/\mathbb{Q}$.

§18. Maeda's conjecture again.

In the 1970s, Y. Maeda conjecured

Conjecture 1. *Any Hecke eigenform* f *in* $S_k := S_k(SL_2(\mathbb{Z}))$ *are Galois conjugate each other; so,* $d = \dim_{\mathbb{Q}} \mathbb{Q}(f) = \dim_{\mathbb{C}} S_k$ *for any weight* k*. Moreover the Galois closure* $\mathbb{Q}(f)^{gal}$ *has Galois group isomorphic to the symmetric group of* d *letters.*

This conjecture has been numerically checked up to a big weight in the order of thousand.

This would implies

Conjecture 2. *If* $N = 1$,

$$\limsup_k [\mathbb{Q}(F_{k,1}) : \mathbb{Q}] = \infty.$$

§19. Vertical theorem.

In the direction of Maeda's conjecture, reducing it to the horizontal theorem, we can prove

Theorem 3. *Let* $F_k = F_{k,\varepsilon}$ *with* $\varepsilon = 1$. *Suppose that* $F_{11} = \Delta$ *(Ramanujan's Delta function). Then* $[\mathbb{Q}(F_k|k \in [11,m]) : \mathbb{Q}] \to \infty$ *as* $m \to \infty$. *Here* $\mathbb{Q}(F_k|k \in [11,m])$ *is the compositum of* $\mathbb{Q}(F_k)$ *for all* $11 \leq k \leq m$.

If you have interested in the proof; see, my paper in JAMS **24** (2011), 51–80 or Section 3.3 of my new book: "Elliptic Curves and Arithmetic Invariants," Springer Monographs in Mathematics, 2013 or a new preprint: "Hecke fields of Hilbert modular analytic families" posted in my web page `http://www.math.ucla.edu/~hida/`.