Ramanujan J (2006) 11: 249–261
DOI 10.1007/s11139-006-6511-4

Quantitative irrationality for sums of reciprocals of Fibonacci and Lucas numbers

Tapani Matala-Aho · Marc Prévost

Received: 1 May 2003 / Accepted: 5 November 2003

Abstract Irrationality measures are given for the values of the series $\sum_{n=0}^{\infty} t^n / W_{an+b}$, where $a, b \in \mathbb{Z}^+$, $1 \leq b \leq a$, $(a, b) = 1$ and W_n is a rational valued Fibonacci or Lucas form, satisfying a second order linear recurrence. In particular, we prove irrationality of all the numbers

$$\sum_{n=0}^{\infty} \frac{1}{f_{an+b}}, \quad \sum_{n=0}^{\infty} \frac{1}{l_{an+b}},$$

where f_n and l_n are the Fibonacci and Lucas numbers, respectively.

Keywords Irrationality measure · Padé approximation · Cyclotomic polynomial · q-series

2000 Mathematics Subject Classification Primary—11J82, 11B39

1. Introduction and results

Let

$$W_n = W_n(r, s) = \gamma\alpha^n + \delta\beta^n, \quad n \in \mathbb{N}, \tag{1}$$

be a Fibonacci $F_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ or Lucas $L_n = \alpha^n + \beta^n$ form.

T. Matala-Aho
Matemaattisten tieteiden laitos, Linnanmaa, PL 3000, 90014 Oulun Yliopisto, Finland
e-mail: tma@cc.oulu.fi

M. Prévost
Laboratoire de Mathématiques Pures et Appliquées, Université du Littoral,
50 Rue F. Buisson, B.P. 699, 62228 Calais Cédex, France
e-mail: prevost@lmpa.univ-littoral.fr

In the sequel we will investigate some arithmetical properties of the values of the meromorphic function

$$W(t)=\sum_{n=0}^{\infty}\frac{t^n}{W_{an+b}},\quad t\in\mathbb{C}\setminus\left\{\alpha^{a(n+1)}/\beta^{an}\,\middle|\, n\in\mathbb{N}\right\}, \tag{2}$$

where $\alpha=(r+\sqrt{r^2+4s})/2$, $\beta=(r-\sqrt{r^2+4s})/2$, $a,b\in\mathbb{Z}^+$, $r,s\in\mathbb{Q}^*$ and $r^2+4s>0$. Here we may suppose without a loss of generality that $|\alpha|>|\beta|$ and hence $F_nL_n\neq 0$ for all $n\in\mathbb{Z}^+$.

André-Jeannin [1] proved the irrationality of the series (2) in the Fibonacci case, where $a=b=1$, and soon after followed some irrationality measure considerations of the series (2) in the case $a=1$, see [4, 9, 12, 13]. However, not much is known about the arithmetic character of the series (2) with arbitrary parameters a, b except the transcendence coming from Nesterenko's method [6, 10] for the numbers

$$\sum_{n=0}^{\infty}\frac{1}{L_n},\quad \sum_{n=0}^{\infty}\frac{1}{F_{2n+1}},\quad \sum_{n=0}^{\infty}\frac{1}{L_{2n}},$$

where α and β are algebraic numbers satisfying $\alpha\beta=1$ in the first case and $\alpha\beta=-1$ for the other two cases. When the indices grow at geometrical rate or more, the situation is dramatically different because Mahler's method applies. Namely, let $r,s\in\mathbb{Z}\setminus\{0\}$ and let $W_0,W_1\in\mathbb{Z}$ be not both zero. Then Nishioka's general arguments [11] lead even to algebraic independence results, for example of the numbers

$$\theta(d,l)=\sum_{n=0}^{\infty}\frac{1}{W_{d^n+l}},\quad d,l\in\mathbb{Z}^+, d\geq 3.$$

In order to present our Theorem 1 we first define $c=c(W)$ by

$$c(F)^{-1}=\frac{1}{2}+\left(\frac{\phi(a)}{2a^2}-\frac{3}{\pi^2}\sum_{l=1,(l,a)=1}^{a-1}\frac{1}{l^2}\right)\prod_{p|a}\frac{p^2}{p^2-1}$$

and respectively

$$c(L)^{-1}=\frac{1}{2}+\left(\frac{\phi(a)\kappa^2}{6a^2}-\frac{4}{\pi^2}\sum_{l=1,(l,2a)=1}^{a-1}\frac{1}{l^2}\right)\prod_{p|a,p\geq 3}\frac{p^2}{p^2-1},\quad \kappa=\gcd(a,2).$$

Also we set $r=R/d$ and $s=S/d$, where $d,R\in\mathbb{Z}^+$ and $S\in\mathbb{Z}\setminus\{0\}$. In the following let $\mathbb{I}$ be an imaginary quadratic field and let $\mathbb{Z}_{\mathbb{I}}$ be it's ring of integers.

Theorem 1. *Let* $a,b\in\mathbb{Z}^+$, $1\leq b\leq a$, $\gcd(a,b)=1$ *and let* W_n *be a Fibonacci or Lucas form, where d, R and S satisfy*

$$R>|dS|^{c(W)}-\frac{dS}{|dS|^{c(W)}}. \tag{3}$$

Take $t \in \mathbb{I}^* \setminus \{\alpha^{a(n+1)}/(d\beta^{an})|\ n \in \mathbb{N}\}$. *Then for every* $\epsilon > 0$ *there exists a positive constant* $N_0 = N_0(\epsilon)$ *such that for all* $M, N \in \mathbb{Z}_{\mathbb{I}}$ *with* $|N| \geq N_0$

$$\left| \sum_{n=0}^{\infty} \frac{t^n}{W_{an+b}} - \frac{M}{N} \right| > |N|^{-m_W(a)-\epsilon}, \tag{4}$$

with $m_W(a) = \log(|\alpha|^2/|dS|)/\log(|\alpha|^{1/c(W)}/|dS|)$.

Corollary 1. *Let* $r \in \mathbb{Z} \setminus \{0\}$, *if* $s = 1$, *and* $r \in \mathbb{Z} \setminus \{0, \pm 1, \pm 2\}$, *if* $s = -1$; *take* $t \in \mathbb{I}^* \setminus \{\alpha^{a(n+1)}/\beta^{an}|\ n \in \mathbb{N}\}$. *Then* $m_F(2) = \frac{3\pi^2}{\pi^2-6} = 7.65163\ldots$, $m_F(3) = \frac{64\pi^2}{20\pi^2-135} = 10.12395\ldots$, $m_F(6) = \frac{1200\pi^2}{325\pi^2-2808} = 29.63686\ldots$, $m_L(2) = \frac{3\pi^2}{\pi^2-6} = 7.65163\ldots$, $m_L(3) = \frac{48\pi^2}{13\pi^2-108} = 23.3314\ldots$, $m_L(6) = \frac{1200\pi^2}{13(25\pi^2-216)} = 29.6368\ldots$.

2. Common multiples and factors

The following results are essential for our method even in proving just the irrationality of the numbers

$$\sum_{n=0}^{\infty} \frac{1}{f_{an+b}}, \quad \sum_{n=0}^{\infty} \frac{1}{l_{an+b}}.$$

Moreover, Lemmas 2 and 3 imply almost optimal common multiples and big common factors for the coefficients in the Padé approximation formulae (61) thus yielding the sharp irrationality measures listed in Corollary 1.

Lemma 1. *Let* $(a, b) = 1$, $\kappa = (a, 2)$, *let* p *be a prime and let* $\phi(n)$ *and* $\mu(d)$ *denote the Euler and the Möbius functions, respectively. Then*

$$\sum_{d=1,(d,a)=1}^{\infty} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} \prod_{p|a} \frac{p^2}{p^2-1}, \tag{5}$$

$$\sum_{j=0}^{n} \phi(aj) = n^2 \frac{3a}{\pi^2} \prod_{p|a} \frac{p}{p+1} + O(n \log n) \tag{6}$$

$$\sum_{j=0}^{n} \phi(aj+b) = n^2 \frac{3a}{\pi^2} \prod_{p|a} \frac{p^2}{p^2-1} + O(n \log n), \tag{7}$$

$$\sum_{j=0}^{n} \phi(2(aj+b)) = n^2 \frac{4a}{\pi^2} \prod_{p|a, p\geq 3} \frac{p^2}{p^2-1} + O(n \log n) \tag{8}$$

and

$$\sum_{d=1, d\equiv l \pmod a}^{n} \left[\frac{n}{d}\right]\phi(d) = \frac{n^2}{2a}\prod_{p|a}\frac{p^2}{p^2-1} + O(n\log^2 n) \tag{9}$$

$$\sum_{d=1, d\equiv l \pmod a}^{n} \left[\frac{n\kappa}{2d}\right]\phi(2d) = \frac{n^2\kappa^2}{6a}\prod_{p|a, p\geq 3}\frac{p^2}{p^2-1} + O(n\log^2 n) \tag{10}$$

for all $(l, a) = 1$.

The result (5) is well known and (6–7) are proved in Bavencoffe [2], see also Bézivin [3]. Accordingly, we consider just the claims (8), (9), and (10).

Proof: First we consider (8). Let

$$S = \sum_{j=0}^{n}\phi(2(aj+b))$$

If $2|a$, then $(2, aj+b) = 1$ for all $j \geq 0$. Thus

$$S = \sum_{j=0}^{n}\phi(aj+b) = n^2\frac{3a}{\pi^2}\prod_{p|a}\frac{p^2}{p^2-1} + O(n\log n)$$

$$= n^2\frac{4a}{\pi^2}\prod_{p|a, p\geq 3}\frac{p^2}{p^2-1} + O(n\log n) \tag{11}$$

by (7). If $2 \nmid a$, then

$$S = \sum_{j=0}^{n}\sum_{d|2(aj+b)}\mu(d)\frac{2(aj+b)}{d} = \sum_{dl=2(aj+b)}\mu(d)l = S_1 + S_2 + S_3, \tag{12}$$

where

$$S_1 = \sum_{dl=2(aj+b), 2\nmid d}\mu(d)l, \quad S_2 = \sum_{dl=2(aj+b), 2||d}\mu(d)l$$

and

$$S_3 = \sum_{dl=2(aj+b), 4|d}\mu(d)l.$$

Here

$$S_1 = \sum_{d=1, (d,2a)=1}^{2(an+b)}\mu(d)\sum_{l\equiv 2b/d \pmod{2a}}^{[\frac{2(an+b)}{d}]} l = an^2\sum_{(d,2a)=1}\frac{\mu(d)}{d^2} + O(n\log n), \tag{13}$$

where we use (5) to get

$$S_1 = n^2 \frac{6a}{\pi^2} \prod_{p|2a} \frac{p^2}{p^2-1} + O(n\log n) = n^2 \frac{8a}{\pi^2} \prod_{p|a,\, p\geq 3} \frac{p^2}{p^2-1} + O(n\log n). \qquad (14)$$

Similarly

$$S_2 = \sum_{d'=1,(d',2a)=1}^{an+b} \mu(2d') \sum_{l\equiv b/d' \pmod a}^{\left[\frac{an+b}{d'}\right]} l = -\frac{an^2}{2} \sum_{(d',2a)=1} \frac{\mu(d')}{d'^2} + O(n\log n)$$
$$= -n^2 \frac{3a}{\pi^2} \prod_{p|2a} \frac{p^2}{p^2-1} + O(n\log n) = -n^2 \frac{4a}{\pi^2} \prod_{p|a,\, p\geq 3} \frac{p^2}{p^2-1} + O(n\log n). \quad (15)$$

Because $S_3 = 0$ we get (8).

We next consider (9) by writing

$$H_l = \sum_{d=1, d\equiv l \pmod a}^{n} \left[\frac{n}{d}\right] \phi(d) = \sum_{k=1}^{n} k \sum_{n/(k+1) < ia+l \leq n/k} \phi(ia+l) \qquad (16)$$

and using (7) to get

$$H_l = \frac{3a}{\pi^2} \prod_{p|a} \frac{p^2}{p^2-1} \sum_{k=1}^{n} k \left(\left(\frac{n}{ka}\right)^2 - \left(\frac{n}{(k+1)a}\right)^2 \right)$$
$$+O\left(\sum_{k=1}^{n} k \left(\frac{n}{ka} \log \frac{n}{ka} - \frac{n}{(k+1)a} \log \frac{n}{(k+1)a} \right) \right)$$
$$= \frac{3n^2}{\pi^2 a} \prod_{p|a} \frac{p^2}{p^2-1} \left(\sum_{k=1}^{n} \frac{1}{k^2} - \frac{n}{(n+1)^2} \right) + O\left(\frac{n}{a} \sum_{k=1}^{n+1} \frac{1}{k} \log \frac{n}{ka} \right)$$
$$= \frac{n^2}{2a} \prod_{p|a} \frac{p^2}{p^2-1} + O(n \log^2 n). \qquad (17)$$

The proof of (10) follows by an argument similar to that just given. □

Now set

$$E_k = E_k(\alpha, \beta) = \beta^{\phi(k)} \Phi_k(\alpha/\beta), \qquad (18)$$

where $\Phi_d = \Phi_d(x)$ is the dth cyclotomic polynomial. If $r, s \in \mathbb{Z}$, then $E_k(\alpha, \beta) \in \mathbb{Z}$ for all $k \in \mathbb{N}$, $(k \geq 2)$, see Carmichael [5]. We also note that Fibonacci and Lucas forms are given by

$$F_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} = \beta^k \frac{x^k - 1}{\alpha - \beta} = \prod_{d|k} E_d(\alpha, \beta) \qquad (19)$$

and

$$L_k = \alpha^k + \beta^k = \frac{F_{2k}}{F_k} = \prod_{d\nmid k, d|2k} E_d(\alpha, \beta). \tag{20}$$

Lemma 2. *Let $r \in \mathbb{Z}^+, s \in \mathbb{Z} \setminus \{0\}$ and let W_n be a Fibonacci or Lucas form. Then there exists $M_n \in \mathbb{Z}^+$ such that*

$$lcm[W_b, W_{a+b}, \ldots, W_{an+b}] \,\Big|\, M_n \quad \forall n \in \mathbb{Z}^+ , \tag{21}$$

and

$$M_n \leq |\alpha^a|^{M_W(a)n^2 + O(n \log n)} \tag{22}$$

with

$$M_F(a) = \frac{3}{\pi^2} \prod_{p|a} \frac{p^2}{p^2-1} \sum_{l=1,(l,a)=1}^{a-1} \frac{1}{l^2} < \frac{1}{2} \tag{23}$$

and

$$M_L(a) = \frac{4}{\pi^2} \prod_{p|a, p\geq 3} \frac{p^2}{p^2-1} \sum_{l=1,(l,2a)=1}^{a-1} \frac{1}{l^2} < \frac{1}{2}. \tag{24}$$

Proof: If $W = F$, then we may choose a common multiple

$$M_n = \prod_{\substack{d=1 \\ d|ak+b \ \text{ for some } k\leq n}}^{an+b} E_d. \tag{25}$$

Suppose the numbers $1 \leq b_l \leq a-1$, $(l, a) = 1$ satisfy

$$b_l \equiv b/l \pmod{a} \quad \text{for all } l,\ 1 \leq l \leq a-1. \tag{26}$$

Thus $d \equiv b_l \pmod{a}$ for every divisor d satisfying $dl = ak + b$. Hence

$$M_n = \prod_{l=1,(l,a)=1}^{a-1} \prod_{j=0}^{[n/l-(lb_l-b)/(la)]} E_{aj+b_l} \,\Bigg|\, \prod_{l=1,(l,a)=1}^{a-1} \prod_{j=0}^{[n/l]} E_{aj+b_l}, \tag{27}$$

where

$$\deg_\alpha \prod_{j=0}^{n} E_{aj+b} = \sum_{j=0}^{n} \phi(aj+b) = n^2 \frac{3a}{\pi^2} \prod_{p|a} \frac{p^2}{p^2-1} + O(n \log n), \tag{28}$$

yielding

$$\deg_\alpha M_n = n^2 \frac{3a}{\pi^2} \prod_{p|a} \frac{p^2}{p^2-1} \sum_{l=1,(l,a)=1}^{a-1} \frac{1}{l^2} + O(n \log n). \tag{29}$$

If $W = L$ we continue similarly to the case $W = F$. First we note that, if $2|l$ in $dl = ak + b$, then

$$ak + b = 2l'd, \quad l' \in \mathbb{Z}^+ \tag{30}$$

and if $d|ak' + b$, where $k' \geq k$, then

$$ak' + b = 2hl'd \tag{31}$$

for some $h \in \mathbb{Z}^+$. From the representation

$$L_{ak'+b} = \frac{\prod_{D|4hl'd} E_D}{\prod_{D|2hl'd} E_D} \tag{32}$$

we see that the term E_{2d} cancels. Thus we obtain a common multiple

$$M_n = \prod_{\substack{d=1 \\ dl=ak+b,\, 2\nmid l, \text{ for some } k\leq n}}^{an+b} E_{2d}. \tag{33}$$

We now choose

$$M_n = \prod_{l=1,(l,2a)=1}^{a-1} \prod_{j=0}^{[n/l+(b-lb_l)/(la)]} E_{2(aj+b_l)} \Bigg| \prod_{l=1,(l,2a)=1}^{a-1} \prod_{j=0}^{[n/l]} E_{2(aj+b_l)}, \tag{34}$$

where

$$\deg_\alpha \prod_{j=0}^{n} E_{2(aj+b)} = \sum_{j=0}^{n} \phi(2(aj+b)) = n^2 \frac{4a}{\pi^2} \prod_{p|a,\, p\geq 3} \frac{p^2}{p^2-1} + O(n \log n). \tag{35}$$

Thus

$$\deg_\alpha M_n = n^2 \frac{4a}{\pi^2} \prod_{p|a,\, p\geq 3} \frac{p^2}{p^2-1} \sum_{l=1,(l,2a)=1}^{a-1} \frac{1}{l^2} + O(n \log n). \tag{36}$$

It is now clear that

$$\Phi_n(x) = \prod_{d|n} (x^{n/d} - 1)^{\mu(d)}, \tag{37}$$

together with (29) and (36) implies (22), (23), and (24).

To complete our argument we note that

$$M_L(a) = \frac{3}{\pi^2}\prod_{p|2a}\frac{p^2}{p^2-1}\sum_{l=1,(l,2a)=1}^{a-1}\frac{1}{l^2}$$
$$= \frac{3}{\pi^2}\prod_{p|2a}\sum_{k=0}^{\infty}\frac{1}{(p^k)^2}\sum_{l=1,(l,2a)=1}^{a-1}\frac{1}{l^2} < \frac{3}{\pi^2}\sum_{L=1}^{\infty}\frac{1}{L^2} = \frac{1}{2} \tag{38}$$

and hence that $M_W(a) < 1/2$ in both cases. □

Lemma 3. *Set $r \in \mathbb{Z}^+$, $s \in \mathbb{Z}\setminus\{0\}$ and let W_n be a Fibonacci or Lucas form. Then there exists $G_n \in \mathbb{Z}^+$ such that*

$$G_n \Big| \prod_{j=1}^{n} W_{a(h+j)+b} \quad \text{for all } h \in \mathbb{N} \tag{39}$$

and

$$G_n \geq |\alpha^a|^{G_W(a)n^2+O(n\log^2 n)}, \tag{40}$$

with

$$G_F(a) = \frac{\phi(a)}{2a^2}\prod_{p|a}\frac{p^2}{p^2-1} \tag{41}$$

and

$$G_L(a) = \frac{\phi(a)\kappa^2}{6a^2}\prod_{p|a,\, p\geq 3}\frac{p^2}{p^2-1}. \tag{42}$$

Proof: Let $W = F$. It is known that

$$E_d | F_{a(h+j)+b} \quad \text{if and only if } d \mid a(h+j)+b \tag{43}$$

and, if we suppose $(a, d) = 1$, then (43) holds exactly for every

$$j \equiv -h - b/a \pmod d \tag{44}$$

and thus

$$\#\{1 \leq j \leq n : E_d \mid F_{a(h+j)+b}\} \geq \left[\frac{n}{d}\right]. \tag{45}$$

So we may choose

$$G_n = \prod_{d=1,(a,d)=1}^{n} E_d^{[n/d]}, \tag{46}$$

which, by (9), has degree

$$\deg_\alpha G_n = \sum_{l=1,(l,a)=1}^{a-1} \sum_{d=1,d\equiv l \pmod a}^{n} \left[\frac{n}{d}\right]\phi(d)$$
$$= \phi(a)\frac{n^2}{2a}\prod_{p|a}\frac{p^2}{p^2-1} + O(n\log^2 n). \tag{47}$$

Now set $W = L$. By (20) we know that

$$E_D \mid L_k \quad \text{if and only if } D \mid 2k \text{ and } D \not| k. \tag{48}$$

We consider factors $D = 2d$, $d|k$ and $(d,a) = 1$. Let $k = a(h+j)+b$. First we suppose $2|a$, hence $2 \not| d$ and $D = 2d$. So

$$\#\left\{1 \le j \le n : E_D | L_{a(h+j)+b}\right\} \ge \left[\frac{n}{d}\right]. \tag{49}$$

If $2 \not| a$, then we look only for the instances

$$k = (2l+1)d = a(h+j)+b. \tag{50}$$

Now we take

$$j_0 = \min\{1 \le j \le n : a(h+j) \equiv b-d \pmod{2d}\}$$

and thus (50) holds exactly for every $j \equiv j_0 \pmod{2d}$ giving

$$\#\left\{1 \le j \le n : E_{2d} \mid L_{a(h+j)+b}\right\} \ge \left[\frac{n}{2d}\right]. \tag{51}$$

So

$$G_n = \prod_{d=1,(a,d)=1}^{n} E_{2d}^{[n\kappa/2d]}, \tag{52}$$

which by (10) has degree

$$\deg_\alpha G_n = \sum_{l=1,(l,a)=1}^{a-1} \sum_{d=1,d\equiv l \pmod a}^{n} \left[\frac{n\kappa}{2d}\right]\phi(2d)$$
$$= \phi(a)\frac{n^2\kappa^2}{6a}\prod_{p|a,p\ge 3}\frac{p^2}{p^2-1} + O(n\log^2 n). \tag{53}$$

□

3. Padé approximations

In the following Padé approximations the q-series factorials

$$(b,a)_0 = 1,\ (b,a)_n = (b-a)(b-aq)\dots(b-aq^{n-1}),\quad n \in \mathbb{Z}^+,$$

$(a)_n = (1,a)_n$ and the q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k(q)_{n-k}} \tag{54}$$

are used with the W-nomials (Fibonomials)

$$\begin{bmatrix} n \\ k \end{bmatrix}_W = \begin{bmatrix} n \\ k \end{bmatrix}_{W(\alpha,\beta)},\quad \deg_\alpha \begin{bmatrix} n \\ k \end{bmatrix}_W = k(n-k)$$

defined by

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_W = \begin{bmatrix} n \\ n \end{bmatrix}_W = 1,\quad \begin{bmatrix} n \\ k \end{bmatrix}_W = \frac{W_1\cdots W_n}{W_1\cdots W_k W_1\cdots W_{n-k}}$$

for all $k, n \in \mathbb{N}$ with $1 \le k \le n-1$ for any form W_n. More generally, for a given $m \in \mathbb{Z}^+$ we set

$$\begin{bmatrix} n \\ k \end{bmatrix}_{W_{m\times}} = \begin{bmatrix} n \\ k \end{bmatrix}_Z,\quad Z_h = W_{mh}\quad \text{for all } h \in \mathbb{Z}^+. \tag{55}$$

If $r, s \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$ then (see [9])

$$\begin{bmatrix} n \\ k \end{bmatrix}_{F_{m\times}} \in \mathbb{Z}\quad \text{for all } k, n \in \mathbb{Z}\quad (0 \le k \le n). \tag{56}$$

The series

$$f(z) = \sum_{n=0}^{\infty} \frac{(B)_n}{(C)_n} z^n$$

is a special case of Heine's q-series for which closed form (n, n) Padé approximations were constructed in [7].

Lemma 4. *Let*

$$Q_n^*(z) \;=\; \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} (Bq^{n-k+1})_k (Cq^n)_{n-k} (-z)^k, \tag{57}$$

and

$$R_n^*(z) \;=\; z^{2n+1} q^{n^2} \frac{(q)_n (B)_{n+1} (B,C)_n}{(C)_{2n+1}} \sum_{i=0}^{\infty} \frac{(q^{n+1})_i (Bq^{n+1})_i}{(q)_i (Cq^{2n+1})_i} z^i. \tag{58}$$

Then there exists a polynomial $P_n^*(z)$ of degree $\le n$ such that

$$Q_n^*(z)f(z) - P_n^*(z) = R_n^*(z). \tag{59}$$

In the equations above we set

$$B = -\frac{\delta}{\gamma}\left(\frac{\beta}{\alpha}\right)^b, \quad C = Bq, \quad q = \left(\frac{\beta}{\alpha}\right)^a, \quad z = \frac{t}{\alpha^a} \tag{60}$$

to get the approximation formula

$$Q_n(t)W(t) - P_n(t) = R_n(t), \tag{61}$$

where

$$Q_n(t) \;=\; \sum_{k=0}^{n} q_{n,k}t^k = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{F_{a\times}} W_{a(n-k+1)+b}\cdots W_{a(2n-k)+b}(\alpha\beta)^{a\binom{k}{2}}(-t)^k, \tag{62}$$

$$P_n(t) \;=\; \sum_{k=0}^{n} p_{n,k}t^k, \quad p_{n,k} = \sum_{i+j=k} q_{n,i}/W_{aj+b} \tag{63}$$

and

$$R_n(t) = (-\delta)^n t^{2n+1}\beta^{an^2+bn}\alpha^{a(n^2-3n-2)/2}S_n(t), \tag{64}$$

with

$$S_n(t) = \frac{(1-B)(q)_n^2}{(Bq^{n+1})_{n+1}} \sum_{i=0}^{\infty} \frac{(q^{n+1})_i(Bq^{n+1})_i}{(q)_i(Bq^{2n+2})_i}\left(\frac{t}{\alpha^a}\right)^i.$$

The following results follow similarly to the corresponding lemmas in [9].

Lemma 5. *Let $r, s \in \mathbb{Z}\setminus\{0\}$ and let W_n be a Fibonacci or Lucas form. Then*

$$q_{nk} \in \mathbb{Z}, \quad G_n | q_{nk} \quad \textit{for all } k, n \in \mathbb{N} \quad (0 \le k \le n) \tag{65}$$

and

$$M_n p_{nk} \in \mathbb{Z} \quad \text{for all } k, n \in \mathbb{N} \quad (0 \le k \le n). \tag{66}$$

If we set $t = u/v$,

$$q_n = v^n M_n G_n^{-1} Q_n(u/v), \quad p_n = v^n M_n G_n^{-1} P_n(u/v),$$

$$r_n = v^n M_n G_n^{-1} R_n(u/v),$$

where $u, v \in \mathbb{Z}_{\mathbb{I}} \setminus \{0\}$, we obtain the numerical approximations

$$q_n W(u/v) - p_n = r_n, \quad q_n,\ p_n \in \mathbb{Z}_{\mathbb{I}} \quad \text{for all } n \in \mathbb{N}. \tag{67}$$

Lemma 6. *Let $r, s \in \mathbb{Z} \setminus \{0\}$ and let W_n be a Fibonacci or Lucas form. Then for every $\delta > 0$ there exists $n_0 \in \mathbb{Z}^+$ such that*

$$|q_n| \le |\alpha^a|^{(3/2+M_W(a)-G_W(a)+\delta)n^2} \tag{68}$$

and

$$|r_n| \le |\beta^a|^{n^2} |\alpha^a|^{(1/2+M_W(a)-G_W(a)+\delta)n^2} \tag{69}$$

for all $n \ge n_0$.

Lemma 7. *Let (W_n) be a series defined by* (1) *such that $W_n \neq 0$ for all $n \in \mathbb{Z}^+$ and $\gamma\delta rst \neq 0$. Then*

$$q_n p_{n+1} - p_n q_{n+1} \neq 0 \quad \forall n \in \mathbb{Z}^+. \tag{70}$$

4. Proof of Theorem 1

Lemma 8 below is standard and may be obtained as was Theorem 3.3 in [8].

Lemma 8. *Let $\Phi \in \mathbb{C}$ and $y > 1$. Let*

$$q_n \Phi - p_n = r_n, \quad q_n,\ p_n \in \mathbb{Z}_{\mathbb{I}} \quad \forall n \in \mathbb{N} \tag{71}$$

be numerical approximation forms satisfying

$$q_n p_{n+1} - p_n q_{n+1} \neq 0, \tag{72}$$

$$|q_n| \le y^{An^2}, \quad |r_n| \le y^{-Bn^2} \tag{73}$$

for all $n \ge n_0$ with some positive A and B. Then for every $\epsilon > 0$ there exists a positive constant $N_0 = N_0(\epsilon)$ such that

$$\left|\Phi - \frac{M}{N}\right| > |N|^{-(1+A/B)-\epsilon} \tag{74}$$

for all $M, N \in \mathbb{Z}_{\mathbb{I}}$ with $|N| \ge N_0$.

Thus we may call $\mu = (A+B)/B$ an irrationality measure for Φ. Our bounds in Lemma 6 are of the form

$$|q_n| \le y^{(A+\delta)n^2}, \quad |r_n| \le y^{-(B-\delta)n^2} \tag{75}$$

where we are free to choose $\delta > 0$. So we get an irrationality measure

$$\mu = \frac{A+\delta+B-\delta}{B-\delta} \leq \frac{A+B}{B} + \epsilon \tag{76}$$

for every $\epsilon > 0$ whenever $\delta \leq B^2\epsilon/(A+B+B\epsilon)$. Hence also in this case we call $(A+B)/B$ an irrationality measure of Φ.

Proof of Theorem 1: Let

$$c = c_W(a) = 1/(1/2 + G_W(a) - M_W(a)). \tag{77}$$

By Lemmas 2, 3, 5 and 6 we have

$$A = 3/2 + M_W(a) - G_W(a), \tag{78}$$

$$B = G_W(a) - M_W(a) - 1/2 - \log|\beta|/\log|\alpha| > 0, \tag{79}$$

and $c > 0$, which give

$$\mu = \frac{1 - \log|\beta|/\log|\alpha|}{G_W(a) - M_W(a) - 1/2 - \log|\beta|/\log|\alpha|} = \frac{\log|\alpha|/|\beta|}{\log|\alpha|^{1/c-1}/|\beta|}.$$

This completes the proof of the theorem.

Acknowledgments The authors are indebted to the anonymous referee for improving the English of the paper.

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