ON THE REPRESENTATION OF INTEGERS AS SUMS OF TRIANGULAR NUMBERS

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Abstract. In this survey article we discuss the problem of determining the number of representations of an integer as sums of triangular numbers. This study reveals several interesting results. If $n \geq 0$ is a non-negative integer, then the n^{th} triangular number is $T_n = \frac{n(n+1)}{2}$. Let k be a positive integer. We denote by $\delta_k(n)$ the number of representations of n as a sum of k triangular numbers. Here we use the theory of modular forms to calculate $\delta_k(n)$. The case where $k = 24$ is particularly interesting. It turns out that if $n \geq 3$ is odd, then the number of points on the 24 dimensional Leech lattice of norm $2n$ is $2^{12}(2^{12} - 1)\delta_{24}(n-3)$. Furthermore the formula for $\delta_{24}(n)$ involves the Ramanujan $\tau(n)-$function. As a consequence, we get elementary congruences for $\tau(n)$. In a similar vein, when p is a prime we demonstrate $\delta_{24}(p^k - 3)$ as a Dirichlet convolution of $\sigma_{11}(n)$ and $\tau(n)$. It is also of interest to know that this study produces formulas for the number of lattice points inside $k-$dimensional spheres.

1. Introduction

Representations of non-negative integers by quadratic forms and as sums of squares have a long history [10]. For example, if $k \geq 1$ is a positive integer, then the number of representations of n as a sum of k squares, denoted by $r_k(n)$, has received considerable attention from Rankin [20]. To study $r_k(n)$, he used the classical theta function defined by

$$\Theta(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \dots .$$

Consequently, the values of $r_k(n)$ are the formal coefficients of the power series

$$\Theta^k(q) = \sum_{n \geq 0} r_k(n) q^n .$$

Fortunately, it is well known that $\Theta(q)$ is a modular form of weight $\frac{1}{2}$ on $\Gamma_0(4)$. It now follows that the modular form theory of $\Theta^k(q)$ defines $r_k(n)$. When k is odd, these calculations can be troublesome since $\Theta^k(q)$ is a modular form of half integral weight. Here we apply these classical methods to the representations of integers as

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sums of triangular numbers. Much of what follows is a special case of the problem of representations of numbers as sums of figurate numbers. We will show that the general case follows from the study of generalized Dedekind $\eta-$functions with little complication [22].

First we define the triangular numbers.

Definition. If n is a non-negative integer, then the triangular number T_n is defined by

$$T_n = \frac{n(n+1)}{2}.$$

Note that geometrically T_n is equal to the number of nodes that complete an equilateral triangle with sidelength n. Here are the first few triangular numbers:

$$T_0 = 0 \quad T_1 = 1 \quad T_2 = 3 \quad T_3 = 6 \quad T_4 = 10.$$

If $k \geq 1$ is a positive integer, then let $\delta_k(n)$ denote the number of representations of n as a sum of k triangular numbers. We calculate $\delta_k(n)$ for several values of k using modular form theory.

Incidentally, Gauss' famous *Eureka* theorem asserts that every non-negative integer is represented as a sum of three triangular numbers. In our notation this says that if $n \geq 0$, then $\delta_3(n) > 0$. The reader may consult [1] for a discussion of this theorem.

Now we give some basic preliminaries in the theory of modular forms. Let $N \geq 1$ be a rational integer. Then we define the following congruence subgroups of $SL_2(\mathbb{Z})$. Let A denote the matrix below with integer entries in $SL_2(\mathbb{Z})$:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Definition. The most common *congruence subgroups of level* N are defined below:
(i) $\Gamma_0(N) = \{A \in SL_2(\mathbb{Z}) \mid c \equiv 0 \mod N\}$
(ii) $\Gamma_1(N) = \{A \in SL_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \mod N \text{ and } c \equiv 0 \mod N\}$
(iii) $\Gamma(N) = \{A \in SL_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \mod N \text{ and } b \equiv c \equiv 0 \mod N\}$.

Let χ be a Dirichlet character $\mod N$ and $k \in \mathbb{Z}^+$ satisfying $\chi(-1) = (-1)^k$. Let $A \in SL_2(\mathbb{Z})$ act on H, the complex upper half plane, by the linear fractional transformation

$$A\tau = \frac{a\tau + b}{c\tau + d}.$$

Let $f(\tau)$ be a holomorphic function on H such that

$$f(A\tau) = \chi(d)(c\tau + d)^k f(\tau)$$

for all $A \in \Gamma_0(N)$ and all $\tau \in H$. We call such $f(\tau)$ a *modular form of weight* k and character χ on $\Gamma_0(N)$. If $f(\tau)$ is holomorphic (resp. vanishes) at the *cusps* of $\Gamma_0(N)$ then $f(\tau)$ is a holomorphic modular form (resp. cusp form). The holomorphic modular forms and cusp forms of weight k and character χ form finite dimensional $\mathbb{C}-$vector spaces. These spaces are denoted by $M_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$, respectively. It is well known that $M_k(\Gamma_0(N), \chi)$ is the direct sum of $S_k(\Gamma_0(N), \chi)$

and forms known as Eisenstein series. When $k = \lambda + \frac{1}{2}$ with $\lambda \in \mathbb{N}$, there is a similar theory of modular forms with half-integral weight [12].

Since the the transformation $\tau \to \tau + 1$ is in $\Gamma_0(N)$, a holomorphic modular form $f(\tau)$ admits a Fourier expansion at the point at infinity in the uniformizing variable $q = e^{2\pi i \tau}$

$$f(\tau) = \sum_{n=0}^{\infty} a(n) q^n.$$

Understanding the arithmetic nature of these coefficients $a(n)$ has been a major topic in number theory; their behavior is related to quadratic forms, elliptic curves, integral lattices, the splitting of prime ideals in number fields etc... It is of interest to know that the Fourier coefficients of Eisenstein series are determined by generalized divisor functions.

There are natural linear transformations, the Hecke operators, which act on Fourier expansions of modular forms preserving $M_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$. If p is prime, then the Hecke operator T_p is defined by

$$f \mid T_p = \sum_{n=0}^{\infty} a(pn) q^n + \chi(p) p^{k-1} \sum_{n=0}^{\infty} a(n) q^{pn}.$$

Note that if $p \mid N$, then $\chi(p) = 0$, so T_p reduces to the dissection operator U_p defined by

$$f \mid U_p = \sum_{n=0}^{\infty} a(pn) q^n.$$

For a thorough treatment of the theory of modular forms the reader should consult [11], [12],[15], [23] or [28].

2. The generating function

To carry out our study, we will use the generating function $\Psi(q)$

$$\Psi(q) = \sum_{n=0}^{\infty} q^{T_n} = 1 + q + q^3 + q^6 + q^{10} + \dots.$$

Consequently, it is easy to see that if $k \geq 1$ then

$$\Psi^k(q) = \sum_{n=0}^{\infty} \delta_k(n) q^n.$$

We will see that $\Psi(q)$ is essentially a quotient of Dedekind η- functions. The Dedekind $\eta-$function is a modular form of weight $\frac{1}{2}$ that is defined by the infinite product

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{where} \quad q = e^{2\pi i \tau}.$$

Products and quotients of this function have been studied extensively because of its applications to combinatorics and representations of the symmetric group [1],[2],[7], [8], [18], [21].

We are interested in $q\Psi(q^8)$; it is the Fourier expansion for the weight $\frac{1}{2}$ modular form

$$\frac{\eta^2(16\tau)}{\eta(8\tau)} = q + q^9 + q^{25} + q^{49} + \dots.$$

Using the Serre-Stark Basis theorem [27], it turns out that $\frac{\eta^2(16\tau)}{\eta(8\tau)}$ is the theta series

$$\frac{\eta^2(16\tau)}{\eta(8\tau)} = \sum_{\substack{n \geq 1 \\ n \text{ odd}}} q^{n^2}.$$

Here we use this result to derive an infinite product representation for $\Psi(q)$:

$$q^{\frac{-1}{8}} \frac{\eta^2(2\tau)}{\eta(\tau)} = \sum_{n \geq 0,\ n \text{ odd}} q^{\frac{n^2-1}{8}} = \sum_{n=0}^{\infty} q^{T_n} = \Psi(q).$$

This proves the following proposition.

Proposition 1. *If T_n is the n^{th} triangular number, then*

$$\Psi(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)} = \sum_{n=0}^{\infty} q^{T_n}.$$

Furthermore, we establish a simple relationship between square and triangular representations:

Proposition 2. *If $\delta_k(n)$ is the number of representations of n as a sum of k triangular numbers and $q_k(n)$ is the number of representations of n as a sum of k odd squares, then*

$$\delta_k(n) = q_k(8n+k).$$

Proof. This follows by rewriting a representation of n as a sum of k triangular numbers in the following way:

$$n = \sum_{i=1}^{k} \frac{x_i(x_i+1)}{2} \Leftrightarrow 8n = \sum_{i=1}^{k} (4x_i^2 + 4x_i)$$

$$\Leftrightarrow 8n + k = \sum_{i=1}^{k} (2x_i+1)^2.$$

$\square$

3. Formulae for some $\delta_k(n)$

In this section we compute formulae for $\delta_k(n)$ for various values of k. When $k = 2 or 3$, this reduces to calculating $r_2(8n+2)$ and $r_3(8n+3)$. For other values of k we apply classical modular form theory.

The case $k = 2$ **:**

By Proposition 2 we know that $\delta_2(n) = q_2(8n+2)$. It is easy to see that if $\alpha, \beta \in \mathbb{Z}$ and are solutions to

$$\alpha^2 + \beta^2 \equiv 2 \mod 8,$$

then α and β are necessarily odd. Consequently, we obtain

$$\delta_2(n) = q_2(8n+2) = \frac{1}{4} r_2(8n+2).$$

The scalar $\frac{1}{4}$ compensates for the 4 possible choices of sign that are counted with multiplicity in $r_2(n)$. Here we state the well-known formula for $r_2(n)$ [10, p.15].

Theorem 1. *Denote the number of divisors of n by $d(n)$, and write $d_a(n)$ for the number of divisors d of n with $d \equiv a \mod 4$. Let $n = 2^f n_1 n_2$, where the prime factorizations of n_1 and n_2 are:*

$$n_1 = \prod_{p \equiv 1 \mod 4} p^r \quad \text{and} \quad n_2 = \prod_{q \equiv 3 \mod 4} q^s.$$

If any of the exponents s are odd, then $r_2(n) = 0$. If all s are even, then

$$r_2(n) = 4d(n_1) = 4(d_1(n) - d_3(n)).$$

As a simple corollary we get

Corollary 1. *Let $8n+2 = 2^f n_1 n_2$, where*

$$n_1 = \prod_{p \equiv 1 \mod 4} p^r \quad \text{and} \quad n_2 = \prod_{q \equiv 3 \mod 4} q^s.$$

If any of the exponents s are odd, then $\delta_2(n) = 0$. If all of the exponents s are even, then

$$\delta_2(n) = d(n_1) = d_1(8n+2) - d_3(8n+2).$$

The case $k = 3$ **:**

Here we make use of the identity

$$\delta_3(n) = q_3(8n+3).$$

It is easy to verify that if $\alpha, \beta, \gamma \in \mathbb{Z}$ and are solutions to

$$\alpha^2 + \beta^2 + \gamma^2 \equiv 3 \mod 8,$$

then α, β and γ are all odd. Consequently we obtain the following identity:

$$\delta_3(n) = q_3(8n+3) = \frac{1}{8} r_3(8n+3).$$

Here the scalar $\frac{1}{8}$ compensates for the 8 choices of sign that are counted with multiplicity in $r_3(n)$. Again we state a classical result [10,p.53] that gives a formula for $r_3(n)$.

Theorem 2. *Let $[x]$ be the greatest integer function and let $\left(\frac{r}{n}\right)$ be the usual Jacobi symbol.*
If $n \equiv 1 \mod 4$, then define $R_3(n)$ by

$$R_3(n) = 24 \sum_{r=1}^{[\frac{n}{4}]} \left(\frac{r}{n}\right).$$

If $n \equiv 3 \mod 4$, then define $R_3(n)$ by

$$R_3(n) = 8 \sum_{r=1}^{[\frac{n}{2}]} \left(\frac{r}{n}\right).$$

Given these definitions, we obtain

$$r_3(n) = \sum_{d^2|n} R_3\left(\frac{n}{d^2}\right).$$

As a corollary, we obtain

Corollary 2. *Given the notation above, we have:*

$$\delta_3(n) = \frac{1}{8} \sum_{d^2|(8n+3)} R_3\left(\frac{8n+3}{d^2}\right).$$

The case $k = 4$:

It is easy to see that $q\Psi^4(q^2)$ is the weight 2 modular form on $\Gamma_0(4)$ defined by

$$q\Psi^4(q^2) = \frac{\eta^8(4\tau)}{\eta^4(2\tau)} = q\left[\prod_{n=1}^{\infty} \frac{(1-q^{4n})^2}{(1-q^{2n})}\right]^4.$$

Its Fourier expansion is

$$q\Psi^4(q^2) = \sum_{n=0}^{\infty} \delta_4(n) q^{2n+1} = q + 4q^3 + 6q^5 + 8q^7 + 13q^9 + \ldots .$$

Incidentally, note that if $k \geq 1$ is a positive integer, then the k^{th} power of $q\Psi^4(q^2)$ defines $\delta_{4k}(n)$ by

$$q^k\Psi^{4k}(q^2) = \frac{\eta^{8k}(4\tau)}{\eta^{4k}(2\tau)} = \sum_{n=0}^{\infty} \delta_{4k}(n) q^{2n+k}.$$

Since spaces of modular forms are finite dimensional, one easily determines that two modular forms with the same level and weight are equal if their Fourier expansions agree for sufficiently many terms. We only need to check the first $\frac{k[SL_2(\mathbb{Z}):\Gamma']}{12}$ terms where k is the weight and Γ' is the relevant congruence subgroup [12] [28].

It turns out that $q\Psi^4(q^2)$ is the Eisenstein series on $\Gamma_0(4)$ defined by

$$q\Psi^4(q^2) = \sum_{n=0}^{\infty} \sigma_1(2n+1) q^{2n+1},$$

where $\sigma_k(m) = \sum_{d|m} d^k$ is the standard divisor function. We obtain

Theorem 3. *If $\delta_4(n)$ is the number of representations of n as a sum of 4 triangular numbers, then*

$$\delta_4(n) = \sigma_1(2n+1).$$

This result was known to Legendre [6],[14].

As a consequence of the multiplicativity of $\sigma_1(n)$, we establish the following interesting multiplicativity property for $\delta_4(n)$:

$$\delta_4(m)\delta_4(n) = \delta_4(2mn+m+n) \quad \text{when } (2m+1, 2n+1) = 1.$$

Incidentally, $q\Psi^4(q^2)$ is a basis element for the spaces of modular forms of level 4. Here we state a well known fact [12, p.184] that determines a polynomial basis for modular forms on $\Gamma_0(4)$ of either integral or half integral weight.

Proposition 3. *Define $\Theta(\tau)$ and $q\Psi^4(q^2))$ as above. $\Theta(\tau)$ has weight $\frac{1}{2}$ and $q\Psi^4(q^2)$ has weight 2. If $k \in \mathbb{Z}$, then $M_{\frac{k}{2}}(\Gamma_0(4))$ is the space of all isobaric polynomials in $\mathbb{C}[\Theta(\tau), q\Psi^4(q^2)]$ with weight $\frac{k}{2}$.*

This proposition says that all modular forms on $\Gamma_0(4)$ have Fourier expansions at $i\infty$ coming from triangular numbers and squares. Later in this paper we define generating functions for the generic figurate numbers as weight $k = \frac{1}{2}$ modular forms. Is there some general result which allows us to determine when the generating functions of a set of figurate numbers defines a polynomial basis for all modular forms (i.e. integral and half-integral weight) of given level and character?

The case $k = 6$:

We consider the modular form $q^3\Psi^6(q^4) \in M_3(\Gamma_0(8))$.

$$q^3\Psi^6(q^4) = \frac{\eta^{12}(8\tau)}{\eta^6(4\tau)} = q^3[\prod_{n=1}^{\infty} \frac{(1-q^{8n})^2}{(1-q^{4n})}]^6 = q^3 \sum_{n=0}^{\infty} \delta_6(n) q^{4n} = \sum_{n=0}^{\infty} \delta_6(n) q^{4n+3}.$$

The first few terms of the Fourier expansion of $q^3\Psi^6(q^4)$ are

$$q^3\Psi^6(q^4) = q^3 + 6q^7 + 15q^{11} + 26q^{15} + 45q^{19} + 66q^{23} + 82q^{27} + \ldots\ .$$

Define χ, the Dirichlet character $\mod 4$, by

$$\chi(1) = 1, \quad \chi(3) = -1,$$

define the generalized divisor function $\sigma_{2,\chi}(n)$ by

$$\sigma_{2,\chi}(n) = \sum_{d|n} \chi(d) d^2,$$

and define $G_{2,\chi}(\tau)$, a weight 3 Eisenstein series on $\Gamma_0(8)$, by

$$G_{2,\chi}(\tau) = \sum_{n=0}^{\infty} \sigma_{2,\chi}(4n+3) q^{4n+3}.$$

It turns out that our generating function $q^3\Psi^6(q^4)$ satisfies

$$q^3\Psi^6(q^4) = -\frac{1}{8} G_{2,\chi}(\tau).$$

We obtain

Theorem 4. *If $\delta_6(n)$ is the number of representations of n as a sum of 6 triangular numbers, then*

$$\delta_6(n) = -\frac{1}{8}\sigma_{2,\chi}(4n+3).$$

Note that this theorem naturally implies that $\delta_6(n)$ satisfies some interesting multiplicative properties.

The case $k = 8$:

Here we consider the weight 4 modular form $q^2\Psi^8(q^2)$ on $\Gamma_0(4)$ defined by

$$q^2\Psi^8(q^2) = \frac{\eta^{16}(4\tau)}{\eta^8(2\tau)} = \sum_{n=0}^{\infty} \delta_8(n) q^{2n+2}.$$

Here are the first few terms of the Fourier expansion of $q^2\Psi^8(q^2)$

$$q^2\Psi^8(q^2) = q^2 + 8q^4 + 28q^6 + 64q^8 + 126q^{10} + 224q^{12} + \dots .$$

Here we define $E(\tau)$,a well known weight 4 Eisenstein series on $\Gamma_0(2)$,

$$E(\tau) = \sum_{n=1}^{\infty} \sigma_3^{\sharp}(n) q^n$$

where

$$\sigma_3^{\sharp}(n) = \sum_{\substack{d \mid n \\ \frac{n}{d} \text{ odd}}} d^3.$$

Replacing q by q^2 gives us an Eisenstein series $E(2\tau)$ on $\Gamma_0(4)$ with Fourier expansion

$$E(2\tau) = \sum_{n=1}^{\infty} \sigma_3^{\sharp}(n) q^{2n}.$$

By equating Fourier coefficients we find that

$$q^2\Psi^8(q^2) = E(2\tau).$$

This proves

Theorem 5. *If $\delta_8(n)$ is the number of representations of n as a sum of 8 triangular numbers, then*

$$\delta_8(n) = \sigma_3^{\sharp}(n+1).$$

As a consequence of the multiplicativity of $\sigma_3^{\sharp}(n)$, we get following multiplicative property for $\delta_8(n)$.

Corollary 3. *If $\delta_8(n)$ is the number of representations of n as a sum of 8 triangular numbers and if $(a+1, b+1) = 1$, then*

$$\delta_8(a)\delta_8(b) = \delta_8((a+1)(b+1)-1).$$

The case $k = 10$:

In this case we are interested in the weight 5 modular form on $\Gamma_0(8)$ given by $q^5\Psi^{10}(q^4)$

$$q^5\Psi^{10}(q^4) = \frac{\eta^{20}(8\tau)}{\eta^{10}(4\tau)} = \sum_{n=0}^{\infty} \delta_{10}(n) q^{4n+5}.$$

Unfortunately, $q^5\Psi^{10}(q^4)$ is not an Eisenstein series; it is a linear combination of an Eisenstein series and a cusp form with complex multiplication [24].

Let $F(\tau)$ be the modular form with complex multiplication by $Q(i)$ defined as

$$F(\tau) = \eta^4(\tau)\eta^2(2\tau)\eta^4(4\tau) = \sum_{n=1}^{\infty} a(n)q^n = q - 4q^2 + 16q^4 - 14q^5 - 64q^8 + \dots.$$

It turns out that

$$q^5\Psi^{10}(q^4) = \frac{1}{640}\left[G_{4,\chi}(\tau) - 2F(\tau) - \frac{1}{4}(F(\tau) \mid T_2)\right]. \tag{1}$$

Here $G_{4,\chi}(\tau)$ is the Eisenstein series defined by

$$G_{4,\chi}(\tau) = \sum_{n\equiv 1 \mod 4} \sigma_{4,\chi}(n) q^n$$

and

$$\sigma_{4,\chi}(n) = \sum_{d|n} \chi(d) d^4.$$

The character χ is the same Dirichlet character $\mod 4$ which occured when $k = 6$. Now we can state the explicit formula for $\delta_{10}(n)$.

Theorem 6. *If $\delta_{10}(n)$ is the number of representations of n as a sum of 10 triangular numbers, then*

$$\delta_{10}(n) = \frac{1}{640}[\sigma_{4,\chi}(4n+5) - a(4n+5)].$$

Proof. By equation (1), we obtain

$$\delta_{10}(n) = \frac{1}{640}[\sigma_{4,\chi}(4n+5) - 2a(4n+5) - \frac{1}{4}a(8n+10)].$$

Since 2 divides the level, $T_2 = U_2$ and one easily verifies that

$$F(\tau) \mid U_2 = -4F(\tau).$$

This implies that $-4a(n) = a(2n)$. Consequently, we obtain

$$\delta_{10}(n) = \frac{1}{640}[\sigma_{4,\chi}(4n+5) - a(4n+5)].$$

□

Incidentally, since all forms with complex multiplication are lacunary, (i.e. the arithmetic density of their non-zero Fourier coefficients is 0) this theorem implies that $\delta_{10}(n) = \frac{1}{640}\sigma_{4,\chi}(4n+5)$ almost always. For lacunarity, the reader should consult [9], [24], [25].

The case $k = 12$:

Here we consider the weight 6 modular form on $\Gamma_0(4)$ with Fourier expansion $q^3\Psi^{12}(q^2)$,

$$q^3\Psi^{12}(q^2) = \frac{\eta^{24}(4\tau)}{\eta^{12}(2\tau)} = \sum_{n=0}^{\infty} \delta_{12}(n)q^{2n+3}.$$

Here are the first few terms of the Fourier expansion of $q^3\Psi^{12}(q^2)$:

$$q^3\Psi^{12}(q^2) = q^3 + 12q^5 + 66q^7 + 232q^9 + 627q^{11} + 1452q^{13} + \dots.$$

The space of cusp forms $S_6(\Gamma_0(4))$ is 1 dimensional and is spanned by the $\eta-$product

$$\eta^{12}(2\tau) = q - 12q^3 + 54q^5 - 88q^7 - 99q^9 - + \dots.$$

Our form $q^3\Psi^{12}(q^2)$ satisfies the following identity:

$$q^3\Psi^{12}(q^2) = \frac{1}{256}[E(\tau) - \eta^{12}(2\tau)]$$

where $E(\tau) = \sum_{n=0}^{\infty} \sigma_5(2n+1)q^{2n+1}$. Consequently, we have proved the following theorem.

Theorem 7. *If $\eta^{12}(2\tau) = \sum_{n=1}^{\infty} a(n)q^n$ and $\delta_{12}(n)$ is the number of representations of n as a sum of 12 triangular numbers, then*

$$\delta_{12}(n) = \frac{\sigma_5(2n+3) - a(2n+3)}{256}.$$

As a simple consequence of this formula, we obtain the following mod 256 congruence for $a(n)$, the Fourier coefficients of $\eta^{12}(2\tau)$:

$$a(2n+1) \equiv \sigma_5(2n+1) \mod 256.$$

4. The Ramanujan function $\tau(n)$, the Leech Lattice, and $\delta_{24}(n)$.

In this section we derive a formula for $\delta_{24}(n)$, the number of representations of n as a sum of 24 triangular numbers. It turns out that for odd n we obtain an interesting relation between $\delta_{24}(n-3)$ and N_{2n}, the number of lattice points on the Leech lattice with norm $2n$ [3, p, 131]. As a corollary to the formula for $\delta_{24}(n)$, we get interesting congruences for the Ramanujan $\tau(n)$ function. Recall that $\tau(n)$ is defined to be the n^{th} Fourier coefficient of the unique normalized weight 12 cusp form on $SL_2(\mathbb{Z})$, $\Delta(\tau) = \eta^{24}(\tau)$.

The congruential behavior of $\tau(n)$ has been studied extensively by Rankin, Serre and Swinnerton-Dyer [16], [17], [19], [26], [29],[30]. The theory of $\ell-$adic Galois representations due to Deligne and Serre [4], [5] provide a theoretical interpretation of these congruences. Here we recall the congruences for $\tau(n) \mod 256$ and $\mod 691$:

$$\tau(2n+1) \equiv \sigma_{11}(2n+1) \mod 256$$

and

$$\tau(n) \equiv \sigma_{11}(n) \mod 691.$$

Kolberg [13], extended the $\mod 256$ congruence by proving

$$\tau(8n+1) \equiv \sigma_{11}(8n+1) \mod 2^{11},$$

$$\tau(8n+3) \equiv 1217\sigma_{11}(8n+3) \mod 2^{13},$$

$$\tau(8n+5) \equiv 1537\sigma_{11}(8n+5) \mod 2^{12}$$

and

$$\tau(8n+7) \equiv 705\sigma_{11}(8n+7) \mod 2^{14}.$$

We will see that these congruences are closely related to the formula for $\delta_{24}(n)$. Here we consider the weight 12 modular form on $\Gamma_0(4)$

$$q^6\Psi^{24}(q^2) = \frac{\eta^{48}(4\tau)}{\eta^{24}(2\tau)} = \sum_{n=0}^{\infty} \delta_{24}(n) q^{2n+6}.$$

The first few terms of the Fourier expansion are

$$q^6\Psi^{24}(q^2) = q^6 + 24q^8 + 276q^{10} + 2048q^{12} + 11178q^{14} + \dots.$$

We obtain the following identity

$$q^6\Psi^{24}(q^2) = \frac{1}{176896}[\frac{1}{2048}E(\tau) - \Delta(2\tau) - 2072\Delta(4\tau)].$$

$E(\tau)$ is the weight 12 Eisenstein series defined by

$$E(\tau) = \sum_{n=1}^{\infty} \sigma_{11}^{\sharp}(n) q^n$$

where $\sigma_{11}^{\sharp}(n) = \sum_{\substack{d \mid n \\ \frac{n}{d} \text{odd}}} d^{11}$. Note that $\sigma_{11}^{\sharp}(2n+6) = 2^{11}\sigma_{11}^{\sharp}(n+3)$. This identity proves the following formula for $\delta_{24}(n)$.

Theorem 8. *If $\delta_{24}(n)$ is the number of representations of n as a sum of 24 triangular numbers, then*

$$\delta_{24}(n) = \frac{1}{176896}[\sigma_{11}^{\sharp}(n+3) - \tau(n+3) - 2072\tau(\frac{n+3}{2})].$$

Before proceeding to congruences, we note the connection between $\delta_{24}(n)$ and the Leech lattice. The 24 dimensional Leech lattice is well known for its nice symmetric solution to the sphere packing problem. Let N_m be the number of points on this lattice with norm m. The lattice theta function $\Theta(\tau)$ is defined by

$$\Theta(\tau) = \sum_{m=0}^{\infty} N_m q^m.$$

It turns out that

$$\Theta(\tau) = 1 + 196560q^4 + 16773120q^6 + 398034000q^8 + \dots$$

is a weight $k = 12$ modular form. Using techniques from modular form theory [3, p. 51], it is known that N_{2m} is given by

$$N_{2m} = \frac{65520}{691}(\sigma_{11}(m) - \tau(m)).$$

We obtain the following corollary to Theorem 8 connecting the representation of $n-3$ as a sum of 24 triangular numbers with the number of points on the Leech lattice with norm $2n$.

Corollary 4. *If $n \geq 3$ odd, then*

$$N_{2n} = N_6 \ \delta_{24}(n-3).$$

□

Now we list congruences for $\tau(n)$ that are consequences of this formula (Note: 176896=256.691). The proofs are left to the reader.

$$\tau(n) + 2072\tau(\frac{n}{2}) \equiv \sigma_{11}^{\sharp}(n) \mod 176896 \tag{2}$$

$$\tau(n) \equiv \sigma_{11}^{\sharp}(n) \mod 8 \tag{3}$$

$$\tau(2^k n) \equiv \sigma_{11}(n) \sum_{i=0}^{k} (-2072)^i \ 2048^{k-i} \mod 176896, \text{ where } (2,n) = 1 \tag{4}$$

$$\tau(2^k n) \equiv (-24)^k \sigma_{11}(n) \mod 256, \text{where } (2,n) = 1 \tag{5}$$

Along these lines we discuss the existence of many formal convolution identities. It will be clear that this phenomenon occurs often when one considers modular forms that are linear combinations of two eigenforms.

First we state and prove a general theorem.

Theorem 9. *Let $F(n), G(n)$, and $H(n)$ be functions on the non-negative integers such that $F(1) = G(1) = 1$. Suppose α is a non-negative integer where $F(\alpha) \neq G(\alpha)$ and such that*

$$F(\alpha^{n+2}) = F(\alpha)F(\alpha^{n+1}) + H(\alpha)F(\alpha^n)$$

and

$$G(\alpha^{n+2}) = G(\alpha)G(\alpha^{n+1}) + H(\alpha)G(\alpha^n)$$

for all positive integers n. Then

$$\frac{F(\alpha^{n+1}) - G(\alpha^{n+1})}{F(\alpha) - G(\alpha)} = \sum_{k=0}^{n} F(\alpha^n)G(\alpha^{n-k}).$$

Proof. Define two formal power series $f(q)$ and $g(q)$ by

$$f(q) = \sum_{n=0}^{\infty} F(\alpha^n)q^n$$

and

$$g(q) = \sum_{n=0}^{\infty} G(\alpha^n)q^n.$$

By hypothesis, the coefficients of $f(q)$ and $g(q)$ are second-order linear recurrances in n. As a consequence, we obtain the following identities:

$$f(q) = \frac{1}{1 - F(\alpha)q - H(\alpha)q^2} \tag{6}$$

and

$$g(q) = \frac{1}{1 - G(\alpha)q - H(\alpha)q^2}. \tag{7}$$

To prove the theorem it suffices to show that

$$f(q) - g(q) = [F(\alpha) - G(\alpha)]qf(q)g(q).$$

By (6) and (7) we find that

$$\frac{1}{g(q)} - \frac{1}{f(q)} = [F(\alpha) - G(\alpha)]q$$

which proves the theorem.

∎

At this time we recall the Dirichlet convolution of two arithmetical functions.

Definition. If $F(n)$ and $G(n)$ are two such functions, then the Dirichlet convolution $(F*G)(n)$ is an arithmetical function defined by

$$(F*G)(n)=\sum_{d|n}F(d)G(\frac{n}{d}).$$

For our purposes let $F(n)=\sigma_{11}(n)$, $G(n)=\tau(n)$ and $H(n)=-n^{11}$. Since $E_{12}(\tau)$ and $\Delta(\tau)$ are eigenforms of the Hecke operators T_p, the conditions of the theorem hold for $n=p$ a prime and when the arithmetical functions are chosen to be their Fourier coefficients. We obtain:

$$\frac{\sigma_{11}(p^{n+1})-\tau(p^{n+1})}{\sigma_{11}(p)-\tau(p)}=\sum_{k=0}^{n}\sigma_{11}(p^k)\tau(p^{n-k})=(\sigma_{11}*\tau)(p^k). \tag{8}$$

We now apply these ideas to the formula for $\delta_{24}(n)$ given in Theorem 8. Let $n+3=p^{k+1}$ where p is an odd prime. By Theorem 8 we find that

$$\delta_{24}(p^{k+1}-3)=\frac{1}{176896}[\sigma_{11}(p^{k+1})-\tau(p^{k+1})]. \tag{9}$$

¿From the convolution identity (8) we obtain the following corollary to Theorem 8.

Corollary 5. *If p is an odd prime and k is a positive integer, then*

$$\frac{\delta_{24}(p^{k+1}-3)}{\delta_{24}(p-3)}=\sigma_{11}*\tau(p^k).$$

Proof. By (8) we obtain

$$\delta_{24}(p^{k+1}-3)=\frac{\sigma_{11}(p)-\tau(p)}{176896}(\sigma_{11}*\tau)(p^k)=\delta_{24}(p-3)(\sigma_{11}*\tau)(p^k).$$

□

5. Representations of integers as sums of figurate numbers

Here we suggest how modular form theory can be used to solve the general problem of calculating the number of representations of integers as sums of figurate numbers. The method requires defining a generating function that is a modular form of weight $\frac{1}{2}$.

The higher figurate numbers are given by the function

$$f_a(n)=\frac{an^2+(a-2)n}{2}.$$

Notice that $a=1$ gives triangular numbers and $a=2$ gives squares. Here we derive the generating functions for all of these figurate numbers using generalized Dedekind $\eta-$functions [21],[22].

Definition. The generalized Dedekind $\eta-$function is defined by

$$\eta_{\delta,g}(\tau)=e^{\pi i P_2(\frac{g}{\delta})\delta\tau}\prod_{\substack{n>0\\ n\equiv g \mod \delta}}(1-q^n)\prod_{\substack{n>0\\ n\equiv -g \mod \delta}}(1-q^n),$$

where $P_2(t)=\{t\}^2-\{t\}+\frac{1}{6}$ is the second Bernoulli polynomial and $\{t\}=t-[t]$ is the fractional part of t.

Theorem 3 of [22] gives a nice criterion for when functions $f(\tau)$ of the form

$$f(\tau)=\prod_{\substack{\delta\mid N\\ 0\le g<\delta}}\eta_{\delta,g}^{r_{\delta,g}}(\tau)$$

where

$$r_{\delta,g}\in\begin{cases}\frac{1}{2}\mathbb{Z} & \text{if } g=0 \text{ or } g=\frac{\delta}{2}\\ \mathbb{Z} & \text{otherwise}\end{cases}$$

are modular functions on $\Gamma_1(N)$.

We now define generating functions for the figurate numbers in terms of generalized Dedekind $\eta-$functions.

Theorem 10. *If* $a\ge 1$, *then*

$$q^{\frac{(a-2)^2}{8a}}\sum_{n=-\infty}^{\infty}q^{\frac{an^2+(a-2)n}{2}}=\frac{\eta(a\tau)\eta_{a,1}(2\tau)}{\eta_{a,1}(\tau)}.$$

Proof. By the Jacobi triple product identity [2]

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}z)(1+q^{2n-1}z^{-1})=\sum_{n=-\infty}^{\infty}q^{n^2}z^n,$$

where $z\neq 0$ and $\mid q\mid<1$. Replacing q by $q^{\frac{a}{2}}$ and z by $q^{\frac{a-2}{2}}$, we get

$$\sum_{n=-\infty}^{\infty}q^{\frac{an^2+(a-2)n}{2}}=\prod_{n=1}^{\infty}(1-q^{an})(1+q^{\frac{a(2n-1)+a-2}{2}})(1+q^{\frac{a(2n-1)-(a-2)}{2}})$$

$$=\prod_{n=1}^{\infty}(1-q^{an})(1+q^{an-1})(1+q^{an-(a-1)})$$

$$=q^{-\frac{a}{24}}\eta(a\tau)\prod_{n=1}^{\infty}\frac{(1-q^{2an-2})(1-q^{2an-2(a-1)})}{(1-q^{an-1})(1-q^{an-(a-1)})}$$

$$=q^{-\frac{a}{24}}q^{-aP_2(\frac{1}{a})+\frac{a}{2}P_2(\frac{1}{a})}\frac{\eta(a\tau)\eta_{a,1}(2\tau)}{\eta_{a,1}(\tau)}$$

$$=q^{-\frac{(a-2)^2}{8a}}\frac{\eta(a\tau)\eta_{a,1}(2\tau)}{\eta_{a,1}(\tau)}.$$

□

Notice that the generating functions constructed in the previous theorem are defined as a sum over all integers, in contrast to

$$\Psi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}.$$

This difference is minor because

$$\sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} = 2\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}.$$

Example: For pentagonal numbers, we let $a = 3$ in the last theorem. We obtain

$$q^{\frac{1}{24}} \sum_{n=-\infty}^{\infty} q^{\frac{3n^2+n}{2}} = \frac{\eta(3\tau)\eta_{3,1}(2\tau)}{\eta_{3,1}(\tau)}.$$

So if $\rho_k(n)$ is the number of representations of n as a sum of k pentagonal numbers, then we can calculate $\rho_k(n)$ using modular forms. Note that $\rho_k(n)$ will reflect the multiplicity of representations resulting from the possible choices of sign allowed for n in the generating function.

6. The number of lattice points in $k-$dimensional spheres

Gauss asked how many lattice points are contained in the circle of radius R centered at the origin. It is very difficult to calculate this number as a function of R asymptotically. Although this problem is difficult, it is easy to see that one can calculate the exact number of points in such a circle using $r_2(n)$. One simply must sum $r_2(n)$ for all n up to R.

We generalize this question by asking: How many lattice points are contained in a $k-$dimensional sphere with radius R centered at $(\frac{1}{2}, \frac{1}{2}, \dots \frac{1}{2})$? We are interested in this question because it gives a geometric meaning to $\delta_k(n)$.

Proposition 4. *The $k-$dimensional sphere of radius R centered at $(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ contains*

$$2^k \sum_{1 \le n \le \frac{R^2}{2} - \frac{k}{8}} \delta_k(n)$$

lattice points in $\mathbb{Z}^k$.

Proof. Consider concentric spheres centered at $(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$. If a sphere of radius r contains a lattice point $(x_1, x_2, \dots, x_k)$, then we have

$$(x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 + \cdots + (x_k - \frac{1}{2})^2 = r^2.$$

This implies that

$$\sum_{i=1}^{k} (x_i^2 - x_i) = r^2 - \frac{k}{4}.$$

Or equivalently,

$$\sum_{i=1}^{k} \frac{(x_i^2 - x_i)}{2} = \frac{r^2}{2} - \frac{k}{8}.$$

We get a representation of $\frac{r^2}{2} - \frac{k}{8}$ as a sum of k triangular numbers. The number of lattice points on this sphere is $2^k \delta_k(\frac{r^2}{2} - \frac{k}{8})$. Here the scalar 2^k accounts for the fact that x_i and $-x_i - 1$ define the same triangular number. The result now follows.

□

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