Indian J. pure appl. Math., **10**(8) : 964-999, August 1979

ENUMERATION OF INCONGRUENT CYCLIC k-GONS

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(*Received* 26 *April* 1979)

If the circumference of a circle with an arbitrarily fixed radius is divided into n equal parts, we find in this paper, a formula for the number $R(n, k)$ of mutually incongruent convex k-gons that can be obtained by joining k of the n points of division. The problem was first raised by Richard H. Reis. The prologue gives an account of his contributions to the solution of the problem.

1. PROLOGUE

1.1. Men of letters are known for their apathy towards Mathematics in general and computational work in particular and they openly confess this with a sense of pride. I was, therefore, not a little surprised when I received a letter, dated April 13, 1978, from Richard H. Reis, Professor of English at the Southeastern Massachusetts University, N. Dartmouth (U.S.A.), posing a problem in Partition Theory. The problem was essentially this:

Given a regular k-gon with positive integers, summing to a given number n, written at the vertices, Reis was interested in finding $R(n, k)$—the number of such polygons when reflections and rotations were considered redundant.

Reis had been working on the problem for about one year and had already obtained formulae for $R(n, k)$ for values of $k \leqslant 5$, and the asymptotic result true for any fixed k and large n.

Remembering that Mr. Martin Gardner had discussed the necklace-stringing problem in an article in Scientific American some years earlier, Reis wrote to Gardner to find if he knew someone who could tell him if his problem was new. Mr. Gardner identified his problem as one in Partition Theory (of which Reis had never heard) and referred him to Professor George E. Andrews, who then referred him to me.

In the middle 30's (if I remember aright), I had come across a similar problem wherein k coins of which k_1 were of one kind, k_2 of a second kind, k_3 of a third kind and so on, were to be arranged round a circle at equal distances. Here it was left vague if rotations alone or both rotations and reflections were to be considered redundant.

Finding the problem too difficult to solve in all its generality, I had put it aside and forgotten all about it till I received this letter from Reis. The letter revived my interest in the problem and I decided to attack it seriously. But at the time I was busy giving finishing touches to my book "Selected Topics in Number Theory" which had been accepted for publication by the Abacus Press. I was, therefore, forced to postpone the study of the problem posed by Reis till I had submitted the typescript of my book to the publishers. In the mean time Reis continued his work and succeeded in obtaining some results with his crude empirical methods. I was highly impressed by his zeal and insight. But all that I was doing during this period was writing ecouraging letters to him without devoting any time to the problem myself. My letters to Reis did at least one thing; that is they goaded him on and more and more success came to him. It was sometime in August that he sent me a table of values of $R(n, k)$ for $k \leqslant 12$ and n going up to $k + 30$ for $k \leqslant 6$ but not beyond $k + 17$ in other cases. The most remarkable relation to which he was led by a study of his table, was that

$$R(n, k) = R(n, n - k), \ 1 \leqslant k < n.$$

Like the rest of the material, I put this aside also.

On October 20, 1978 I was able to dispatch the final typescript of my book to the publishers and started looking at the problem of Reis with all seriousness. I requested Reis to send me all the results he had obtained. While I told him about the method I was going to use, I thought it might be best for me to study the problem independently, for then we could compare our results. Finding a few days later that my results agreed with those in the table of Reis, I looked into his papers to see what method he had used. I was surprised to find that we had both used the same procedure.

It was for the first time in November 1978 that Reis was able to give some really good general results. These gave $R(n, k)$ for $(n, k) = 1$ or 2 and also when k was an odd prime and n a multiple of k. But his formulae were expressed in a very complicated form. It did not take me long to give them an elegant shape. I decided to let Reis continue in his own independent way, while I went ahead in mine. Every letter from him from this time on brought some new results. By the middle of January, 1979, I had found the exact formula for $R(n, k)$ for all n and k and Reis had covered the same ground almost if not exactly.

I am sure, if Reis had some knowledge of Partition Theory, his insight would have enabled him to solve the problem without the least help from anyone. Simply because Euler's phi function had appeared in the formula for the number of necklaces with a given number of beads chosen from beads of two different colours without restriction, it was not enough reason to predict that it must appear in the solution of his problem. It could only be due to his insight that he could insist that it will and it did.

More than half the credit for solving the problem must go to Reis. My own contribution is the geometrical way of representing a decomposition of n into k parts and providing the proofs of the results we obtained independently.

1.2. Reis used the method of finite differences. To show how it worked, I consider the case $k = 4$.

For $m \geqslant 1$, the table computed by Reis gives

$m =$	1	2	3	4	5	6	...
$R(4m, 4) =$	1	8	29	72	145	256	...

Taking the differences as usual, we get

$$\begin{array}{ccccccccccccc} 1 & & 8 & & 29 & & 72 & & 145 & & 256 & & \cdots \\ & 7 & & 21 & & 43 & & 73 & & 111 & & & \\ & & 14 & & 22 & & 30 & & 38 & & & & \\ & & & 8 & & 8 & & 8 & & & & & \end{array}$$

Hence

$$R(4m, 4) = 1 + 7(m - 1; 1) + 14(m - 1; 2) + 8(m - 1; 3).$$

Here and in what follows, we write $(j; r)$ for $\binom{j}{r}$.

Similarly, we have

$$R(4m + 1, 4) = 1 + 9(m - 1; 1) + 16(m - 1; 2) + 8(m - 1; 3);$$
$$R(4m + 2, 4) = 3 + 13(m - 1; 1) + 18(m - 1; 2) + 8(m - 1; 3);$$
$$R(4m + 3, 4) = 4 + 16(m - 1; 1) + 20(m - 1; 2) + 8(m - 1; 3).$$

It will be seen that in each of the above cases, R is a cubic in m and, therefore, in n also. In general $R(n, k)$ is a polynomial in n of degree $(k - 1)$, not necessarily with integral coefficients.

The only drawback in this method is that quite a large number of values of $R(n, k)$ are needed before the formulae can be obtained and then for each k as many as k distinct formulae are necessary.

1.3. A few extracts from the letters I received from Reis, will be of interest to the reader.

April 13, 1978 :

Prof. Andrews informs me that he has not encountered this problem before, but he thinks that you may have studied it, if anyone has.

June 19, 1978 :

I believe that the general problem can be entirely solved.

June 30, 1978 :

Besides, I am not a Mathematician and do not know how to program a computer; such results as I have obtained have been produced with pencil, paper, and a small pocket calculator.

If I tried to write an article about my results so far, without the help of a skilled mathematician, I'd no doubt do it clumsily and it would not get printed.

Please let me know if you would be interested in helping me put my first draft into publishable form ...

December 26, 1978 :

I suspect that Euler's phi function will be involved somewhere.

I have never been able to understand the difference between partitions and distributions anyhow.

Yes, I suppose it is rather surprising for somebody trained in literary criticism to have some degree of mathematical talent, or even to be interested in Mathematics at all. My colleagues in our English department think I'm a Martian or something! On the other hand, my own explanation of my unusual combination of interests is that poetry, music, mathematics and chess (I am also interested in music and chess, by the way) share two features : all have pattern, and all have beauty. So perhaps I'm not so odd after all!

January 9, 1979 :

The conjecture (that my method of finding values of $R(n, k)$ would somehow or other turn out to involve Euler's phi function) now seems a safe bet, don't you think?

January 15, 1979 :

I have at last completed the set of algorithms whereby $R(n, k)$ can be found for any combination of k and n, including those with which I had been having trouble. As I had conjectured, Euler's phi function is involved, ...

February 3, 1979 :

For me, this (to exchange ideas in correspondence) has been a delightful and rewarding experience, in which I have learned a good deal about combinatorial mathematics that I didn't know before. And of course I warmly appreciate the kindness of your remarks about my mathematical talents, such as they are. I'm actually thinking of dreaming up a new problem, in order to have a reason for corresponding with you further!"

Needless to say that if I have discovered Reis, I am proud of my discovery.

In the following pages is given an account of how the final answer to the problem of Reis was obtained. Now that the expression for $R(n, k)$ is known, shorter proofs of the result should be possible.

2. Introduction

2.1. *Notations*

In what follows x denotes an arbitrary real number; other small letters denote positive integers unless stated otherwise.

(g, h) denotes as usual the g.c.d. of g and h.

As already stated, we write $(g; h)$ for $\binom{g}{h}$.

$\phi(m)$ denotes Euler's totient function and is the number of positive integers $\leqslant m$ which are prime to m.

$[x]$ denotes the largest integer $\leqslant x$.

2.2. *Partitions and Decompositions*

When a natural number n is expressed as a sum of one or more natural numbers and the order in which the summands are written is irrelevant, we get what we call a partition of n. When the order in which the summands are written is relevant, we have a decomposition of n. The total number of partitions of n is denoted by $p(n)$, while $p(n, k)$ denotes the number of partitions of n into exactly k summands.

In writing the parts in a partition, we write them in ascending order of magnitude and, when there is no cause for confusion, we omit the plus signs also. Thus the seven partitions of 5 are

$$11111, \quad 1112, \quad 122, \quad 113, \quad 23, \quad 14, \quad 5$$

and we have $p(5) = 7$.

On the other hand, the partitions of 10 into six parts are

$$111115, \quad 111124, \quad 111133, \quad 111223, \quad 112222$$

so that $p(10, 6) = 5$.

The decompositions of n into k parts are provided by the solutions of the Diophantine equation

$$u_1 + u_2 + \dots + u_k = n \qquad \dots(1)$$

in positive integers u.

It is well known that (1) has exactly $(n - 1; k - 1)$ such solutions. Hence n has exactly $(n - 1; k - 1)$ decompositions into k parts. Thus the ten decompositions of 6 into four parts are:

1113, 1131, 1311, 3111;

1122, 1212, 1221, 2112, 2121, 2211.

The frequency of a part in a partition is the number of times the part appears in the partition.

A partition in which

a_1 occurs as a part h_1 times;

a_2 occurs as a part h_2 times;

....

a_j occurs as a part h_j times;

a's all distinct, is said to be of the type

$$(h_1\ h_2\ \dots\ h_j). \qquad \dots(2)$$

Here, the order in which the h's are written is immaterial. Thus

1113344666 is a partition of 35 and it is of the type (2 2 3 3). The number of decompositions to which a partition of the type $(h_1\ h_2\ \dots\ h_j)$ leads is given by

$$\frac{(h_1 + h_2 + \dots + h_j)!}{h_1!\ h_2!\ \dots\ h_j!}. \qquad \dots(3)$$

In this paper, we will usually need to write out decompositions arising from a given partition and starting with a given element. Thus, the decompositions arising from the partition 111223 and starting with 2 are twenty in number, while there are thirty which start with 1 and only ten which start with 3. Since our interest will be in having to record the minimum number of decompositions, it will be best if we choose as our starting element one of those the frequency of which is the least.

It is noteworthy that the number of decompositions arising from a given partition, depends only on the frequencies of the parts and not on the size of those parts.

2.3 *Graphical Representation of a Decomposition*

Take a circle with an arbitrarily fixed radius. Divide the circumference into n equal parts. Call each part a step. Then any decomposition

$$c_1 c_2 \dots c_k \qquad \dots(4)$$

of n, can be represented graphically as follows:

Select one of the points of division as the starting point. From here move c_1 steps in the counter-clockwise direction and there put a mark. Then move ahead c_2 steps and again put a mark. Continue in this manner till you have finally moved c_k steps and put a mark. This will bring you to the starting point. Join the k marked points in order by straight lines to get a convex cyclic k-gon. The sides of this k-gon can be taken to represent the numbers $c_1, c_2, \dots c_k$; because they are proportional to the angles subtended by the sides at the centre of the circle. The k-gon, therefore, provides a graphical representation of the given decomposition of n. To make the representation unique, it will be necessary to indicate the starting point by an arrowhead or some such sign.

The following figure represents, for example, the decomposition

$$22143$$

of 12.

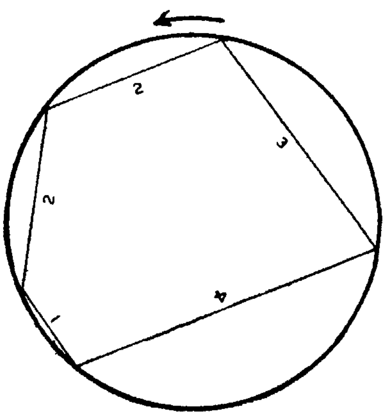

FIG. 1.

One may ask

What decompositions does the k-gon represent if the starting point is not indicated? And what if one is permitted to move in the clockwise direction also?

These questions are easy to answer and are left to the reader.

2.4. *Congruence of k-gons*

Let us represent the decompositions

1223, 2213, 2312

of 8, by quadrilaterals using equal circles.

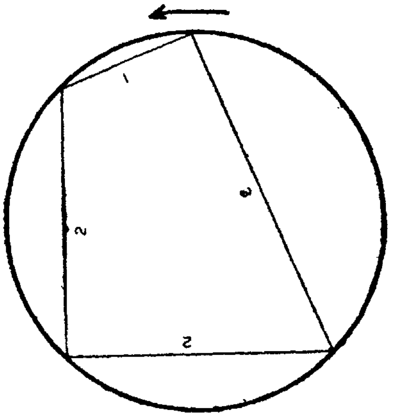

FIG. 2.1.

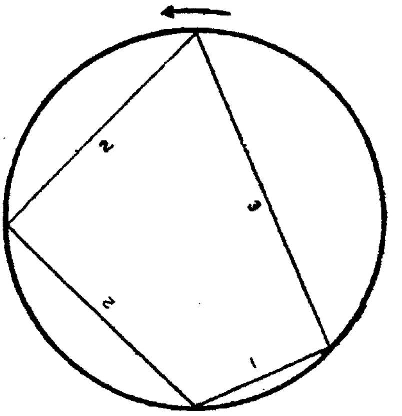

FIG. 2.2.

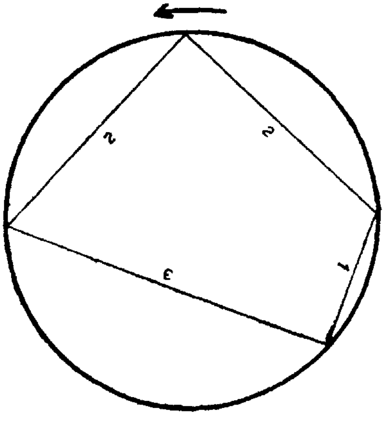

FIG. 2.3.

It will be seen that the quadrilateral in (2.1) representing the decomposition 1223, can be cut out of the paper and made to fit upon the quadrilateral in (2.3), representing the decomposition 2312, directly, that is by just rotating the paper; but it can be made to fit upon the quadrilateral in (2.2), representing the decomposition 2213, only if we first turn it upside-down and then rotate.

We say that the quadrilaterals in (2.1) and (2.3) are directly congruent, while those in (2.1) and (2.2) are invertedly so. But the three quadrilaterals are mutually congruent anyway.

The definitions can be extended to k-gons immediately.

The k-gons represented by the decompositions

$$c_1\, c_2 \,\ldots\, c_k;\; c_2\, c_3 \,\ldots\, c_k\, c_1;\; \ldots;\; c_k\, c_1 \,\ldots\, c_{k-1} \qquad \ldots(5)$$

of n, are all directly congruent among themselves. While any one of these is invertedly congruent to each of the k-gons represented by the decompositions

$$c_k\, c_{k-1} \ldots c_1;\ c_{k-1}\, c_{k-2} \ldots c_1\, c_k;\ \ldots\ c_1\, c_k \ldots c_3\, c_2 \qquad \ldots(6)$$

It is not implied here that the decompositions in (5) are all distinct. But if they are distinct in the case of (5), they are so in the case of (6) too. If any decomposition in (5) is identical with some decomposition in (6), then the decompositions in (6) are just a permutation of those in (5). When this happens, every two k-gons are both directly and invertedly congruent. In fact, each k-gon has now at least one axis of symmetry. For k odd, any axis of symmetry runs through one vertex and the middle point of the side opposite to it. When k is even, any axis of symmetry either runs through two opposite vertices or through the middle points of two opposite sides.

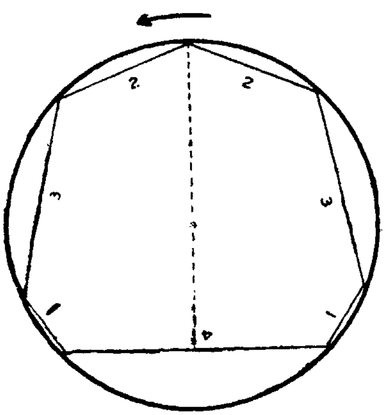

FIG. 3.1.

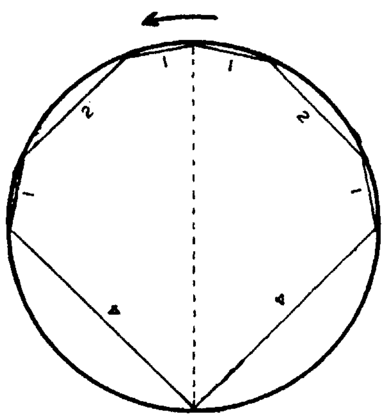

FIG. 3.2.

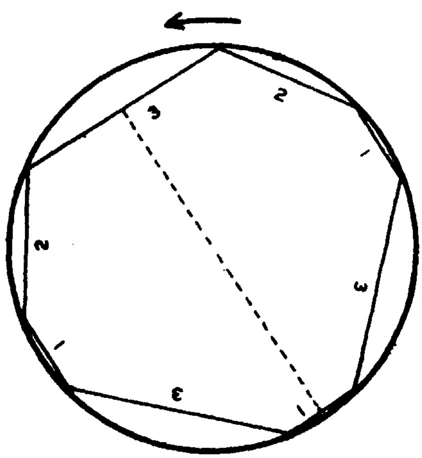

FIG. 3.3.

Definition — Two decompositions are said to be equivalent when the k-gons representing them are congruent (whether directly or invertedly).

2.5. *The Number of Symmetrical k-gons for a given n*

(i) When k is odd.

Starting from a vertex through which an axis of symmetry passes, we have in this case

$$c_1 = c_k;\ c_2 = c_{k-1};\ \dots;\ c_h = c_{h+2};\ c_{h+1} \text{ independent,}$$

where $h = (k - 1)/2$.

The related decomposition can be written in the form

$$c_1\ c_2\ \dots\ c_h\ c_{h+1}\ c_h\ \dots\ c_2\ c_1. \qquad \dots(7)$$

The number of symmetric k-gons for the given n will, therefore, be the same as the number of solutions in positive integers of the equation

$$2c_1 + 2c_2 + \dots + 2c_h + c_{h+1} = n. \qquad \dots(8)$$

If n is odd, so also must c_{h+1} be. The equation can, therefore, be written

$$c_1 + c_2 + \dots + c_h + (c_{h+1} + 1)/2 = (n + 1)/2. \qquad \dots(9)$$

Hence the number of symmetric k-gons is

$$\left(\frac{n-1}{2};\ \frac{k-1}{2}\right). \qquad \dots(10)$$

If n is even, so also is c_{h+1}, and (8) can be written as

$$c_1 + c_2 + \dots + (c_{h+1}/2) = n/2. \qquad \dots(11)$$

In this case, therefore, the number of symmetric k-gons is

$$\left(\frac{n-2}{2};\ \frac{k-1}{2}\right). \qquad \dots(12)$$

Results (10) and (12) can be combined into the single result

$$\left(\left[\frac{n-1}{2}\right];\ \left[\frac{k}{2}\right]\right). \qquad \dots(13)$$

(ii) When k is even.

(a) When n is even.

If the k-gon has an axis of symmetry passing through two opposite vertices, then we have starting from one of these vertices

$$c_1 = c_k;\ c_2 = c_{k-1};\ \dots;\ c_j = c_{j+1};\quad \text{with}\quad j = k/2.$$

The corresponding decomposition is

$$c_1\, c_2 \,\dots\, c_{j-1}\, c_j\, c_j\, c_{j-1} \,\dots\, c_1. \qquad \dots(14)$$

The number of such decompositions is the same as the number of solutions of the Diophantine equation

$$c_1 + c_2 + \dots + c_j = n/2 \qquad \dots(15)$$

which is given by

$$\left(\frac{n}{2} - 1;\ \frac{k}{2} - 1\right). \qquad \dots(16)$$

Note that the decompositions

$$c_1\, c_2 \dots c_j\, c_j \dots c_2\, c_1 \quad \text{and} \quad c_j\, c_{j-1} \dots c_1\, c_1 \dots c_{j-1}\, c_j$$

are equivalent. Also they are distinct unless

$$[c_1, c_2, \dots, c_{j-1}, c_j] = [c_j, c_{j-1}, \dots, c_2, c_1]. \qquad \dots(\text{A})$$

Hence all the solutions do not give incongruent symmetric k-gons.

If the k-gon has an axis of symmetry passing through the middle points of two opposite sides, then starting with one of these sides, we have (we use d's to distinguish them from the c's used in the foregoing case)

$$d_2 = d_k;\ d_3 = d_{k-1};\ \dots;\ d_j = d_{j+2}; \quad \text{with} \quad j = k/2$$

and d_1 and d_{j+1} free.

The corresponding decomposition is

$$d_1 d_2 \dots d_j\, d_{j+1}\, d_j \dots d_2.$$

The number of such decompositions is given by the number of positive integral solutions of the Diophantine equation

$$d_1 + 2d_2 + \dots + 2d_j + d_{j+1} = n$$

which can be written as

$$\frac{d_1 + 1}{2} + d_2 + \dots + d_j + \frac{d_{j+1} + 1}{2} = \frac{n}{2} + 1 \qquad \dots(17)$$

when d_1 and d_{j+1} are both odd; and as

$$\frac{d_1}{2} + d_2 + \dots + d_j + \frac{d_{j+1}}{2} = \frac{n}{2} \qquad \dots(18)$$

when d_1 and d_{j+1} are both even.

Evidently (17) leads to $\left(\frac{n}{2}; \frac{k}{2}\right)$ and (18) to $\left(\frac{n}{2} - 1; \frac{k}{2}\right)$ decompositions.

Note that in this case the decompositions

$$d_1\, d_2\, \dots\, d_j\, d_{j+1}\, d_j\, \dots\, d_2 \quad \text{and} \quad d_{j+1}\, d_j\, \dots\, d_2\, d_1\, d_2\, \dots\, d_j$$

are equivalent and distinct also except when

$$[d_1, d_2, \dots, d_{j+1}] = [d_{j+1}, d_j, \dots, d_1]. \qquad \dots(B)$$

The total number of decompositions obtained from (15), (17) and (18) is readily seen to be

$$2\left(\frac{n}{2}; \frac{k}{2}\right). \qquad \dots(19)$$

We assert that the number of symmetric k-gons obtained from these decompositions is only

$$\left(\frac{n}{2}; \frac{k}{2}\right). \qquad \dots(20)$$

This will follow if we can show that the decompositions satisfying relations A and B consist of pairs of equivalents.

The following examples cover all the four cases that can arise.

(1) $n \equiv 2 \pmod 4$, $k \equiv 0 \pmod 4$;

(2) $n \equiv 0 \pmod 4$, $k \equiv 0 \pmod 4$;

(3) $n \equiv 0 \pmod 4$, $k \equiv 2 \pmod 4$;

(4) $n \equiv 2 \pmod 4$, $k \equiv 2 \pmod 4$.

Case 1 — Take $n = 18$, $k = 8$.

The set A is empty.

The set B consists of the following decompositions:

1	1	6	1	1	1	6	1	6	1	1	1	6	1	1	1
1	2	4	2	1	2	4	2	4	2	1	2	4	2	1	2
1	3	2	3	1	3	2	3	2	3	1	3	2	3	1	3
3	1	4	1	3	1	4	1	4	1	3	1	4	1	3	1
3	2	2	2	3	2	2	2	2	2	3	2	2	2	3	2
5	1	2	1	5	1	2	1	2	1	5	1	2	1	5	1

We have written the pairs of equivalents in one line.

Case 2 — Take $n = 20$, $k = 8$.

Set A consists of the two pairs of equivalents

1	4	4	1	1	4	4	1	4	1	1	4	4	1	1	4
2	3	3	2	2	3	3	2	3	2	2	3	3	2	2	3

Set B consists of eight pairs of equivalents :

1	1	7	1	1	1	7	1	7	1	1	1	7	1	1	1
1	2	5	2	1	2	5	2	5	2	1	2	5	2	1	2
1	3	3	3	1	3	3	3	3	3	1	3	3	3	1	3
1	4	1	4	1	4	1	4	4	1	4	1	4	1	4	1
2	1	6	1	2	1	6	1	6	1	2	1	6	1	2	1
2	2	4	2	2	2	4	2	4	2	2	2	4	2	2	2
2	3	2	3	2	3	2	3	3	2	3	2	3	2	3	2
3	1	5	1	3	1	5	1	5	1	3	1	5	1	3	1

Case 3 — Take $n = 20$, $k = 10$.

Each element of set A pairs with an element of set B :

A										*B*									
1	1	6	1	1	1	1	6	1	1	6	1	1	1	1	6	1	1	1	1
1	2	4	2	1	1	2	4	2	1	4	2	1	1	2	4	2	1	1	2
1	3	2	3	1	1	3	2	3	1	2	3	1	1	3	2	3	1	1	3
2	1	4	1	2	2	1	4	1	2	4	1	2	2	1	4	1	2	2	1
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	1	2	1	3	3	1	2	1	3	2	1	3	3	1	2	1	3	3	1

Case 4 — Take $n = 18$, $k = 10$.

Again each element of A pairs with an element of B :

1	1	5	1	1	1	1	5	1	1	5	1	1	1	1	5	1	1	1	1
1	2	3	2	1	1	2	3	2	1	3	2	1	1	2	3	2	1	1	2
1	3	1	3	1	1	3	1	3	1	1	3	1	1	3	1	3	1	1	3
2	1	3	1	2	2	1	3	1	2	3	1	2	2	1	3	1	2	2	1
2	2	1	2	2	2	2	1	2	2	1	2	2	2	2	1	2	2	2	2
3	1	1	1	3	3	1	1	1	3	1	1	3	3	1	1	1	3	3	1

The reader will find that the general case needs no new technique.

(b) When n is odd.

The Diophantine equation we have now to consider is

$$d_1 + 2d_2 + \ldots + 2d_j + d_{j+1} = n.$$

Since n is odd, we can avoid duplication by assuming d_1 to be odd and d_{j+1} to be even.

The number of symmetric k-gons is readily found to be

$$\left(\frac{n-1}{2};\ \frac{k}{2}\right). \qquad \ldots(21)$$

Putting together (20) and (21), we can state that

For k even, the number of symmetric k-gons is given by

$$\left(\left[\frac{n}{2}\right]; \left[\frac{k}{2}\right]\right).$$

3. The Problem of Reis

3.1. By far the best way of stating the problem of Reis will be to ask:

If a circle is drawn with an arbitrarily fixed radius and its circumference is divided into n equal parts, find $R(n, k)$—the number of mutually incongruent convex k-gons that can be obtained by joining k of the n points of division.

Alternatively, one can ask

What is the number $R(n, k)$ of equivalence classes into which the $(n-1; k-1)$ decompositions of n into k parts can be decomposed.

We can easily prove two interesting theorems concerning $R(n, k)$.

Theorem 1 — For $n > k$,

$$R(n, k) \geqslant (n-1; k-1)/(2k).$$

PROOF: Since no equivalence class into which the decompositions of n into k parts can be decomposed can have more than $2k$ elements, the theorem follows immediately.

Theorem 2 — For each $k < n$,

$$R(n, k) = R(n, n-k).$$

PROOF: Every time we select k of the n points of division on our basic circle, we are left with $(n-k)$ points which when joined to form a convex $(n-k)$-gon produce a unique figure corresponding to the given k-gon. Moreover, if the k-gons are incongruent, so also are the corresponding $(n-k)$-gons. Hence the theorem follows.

Evidently

$$R(n, n) = 1 \quad \text{and} \quad R(n, n-1) = 1.$$

We can, therefore, take

$$R(n, 0) = 1 \quad \text{and} \quad R(n, 1) = 1.$$

3.2. *Evaluation of* $R(n, k)$

As one of our definitions of $R(n, k)$ itself suggests, one way of evaluating $R(n, k)$ will be to write out all the decompositions of n into k parts and decompose them into equivalence classes. But the labour this will involve will be prohibitive even for small values of n and k. A short-cut will be to consider only those decompositions which start with a suitably chosen element and break these up into equivalence classes. The number of classes so obtained will be $R(n, k)$. We have already stated how such an element can best be chosen.

The following example will show how one proceeds along these lines.

Take $n = 11$, $k = 5$.

The partitions of 11 into 5 parts are:

1. 11117	6. 11234
2. 11126	7. 11333
3. 11135	8. 12224
4. 11144	9. 12233
5. 11225	10. 22223

First note that the contribution which the decompositions arising from any of these partitions, make to $R(n, k)$ depends only on the type of the partition and not on the size of the parts. We will, therefore, do well to put together those partitions which are of one type.

Type	*Partitions*		
(1 4)	11117;	22223	
(1 1 3)	11126;	11135;	12224
(2 3)	11144;	11333	
(1 2 2)	11225;	12233	
(1 1 1 2)	11234		

We need consider only one member of each type, say the first from the left. Moreover, we take only those decompositions which start with one of the most suitable elements.

Partition	*Decompositions*	*Classes*	*Number of classes*
11117	71111	71111	1
11126	61112 61121 61211 62111	{61112, 62111} {61121, 61211}	2
11144	41114 41141 41411 44111	{41114, 44111} {41141, 41411}	2

Partition	*Decompositions*	*Classes*	*Number of classes*
11225	51122 51212 51221 52112 52121 52211	51122 } 52211 51212 } 52121 51221 52112	4
11234	41123 41132 41213 41231 41312 41321 42113 42131 42311 43112 43121 43211	41123 } 43211 41132 } 42311 41213 } 43121 41231 } 41321 41312 } 42131 42113 } 43112	6

Hence

$$R(11, 5) = 2.1 + 3.2 + 2.2 + 2.4 + 1.6 = 26.$$

From the above, it will be clear, that to find $R(n, k)$, we have to determine two things:

One : What contribution does a given type of partition make to $R(n, k)$?

Two : How many partitions of n into k parts belong to that type?

For one, we need not consider the given n at all. It will be enough to consider the least n for which a partition of that type exists. The importance of knowing such an n will be realized a little later. We will denote such an n by n_0.

Example — For the type (1 2 2 3), the n_0 is the least number which can be written in the form

$$3u_1 + 2u_2 + 2u_3 + u_4$$

with u's all distinct positive integers. Evidently, for n_0 we must take $u_1 = 1$, $u_2 = 2$, $u_3 = 3$, $u_4 = 4$.

This gives $n_0 = 17$.

3.3. It will not be out of place here to give a few easy-to-prove rules, for determining $C(T)$—the contribution which a partition of type T will make to R for any n.

(i) When T has at least three odd frequencies, no symmetric polygons can arise. Each class will, therefore, contain the same number of decompositions, if the g.c.d. of the frequencies is 1. To find this number, it will be enough to consider only one of the classes.

Example — Take $T = (5\ 3\ 3\ 2)$.

Consider the partition 1111122233344.

The number of decompositions starting with 4 is given by

$$12!/(5!\ 3!\ 3!\ 1!).$$

The class to which the decomposition 4433322211111 belongs has the four members

4433322211111, 4333222111114, 4111112223334, 4411111222333.

Hence $\qquad C(5\ 3\ 3\ 2) = 12!/(5!\ 3!\ 3!\ 1!\ 4).$

(ii) When the partition has just one unrepeated summand and the frequencies of all other summands are even, we consider the decompositions which start with the unrepeated element. Then each ordinary class has two decompositions belonging to it, while each decomposition representable by a symmetric k-gon forms a class by itself.

Let $\quad T = (1\ 2a_1\ 2a_2\ \dots\ 2a_j)$; where each $a > 0$.

Then, we readily have

$$2C(T) = (2a_1 + 2a_2 + \dots + 2a_j)!/(2a_1)!\,(2a_2)!\,\dots\,(2a_j)! + (a_1 + a_2 + \dots + a_j)!/a_1!\,a_2!\,\dots\,a_j!.$$

Example — Take $T = (1\ 2\ 2\ 4)$, then

$$2C(1\ 2\ 2\ 4) = 8!/2!\,2!\,4! + 4!/1!\,1!\,2!.$$

(iii) When $T = (1\ 2a_1 - 1\ 2a_2\ 2a_3\ \dots\ 2a_j)$; with each $a > 0$;

we have
$$2C(T) = (2a_1 - 1 + 2a_2 + 2a_3 + \dots + 2a_j)!/(2a_1 - 1)!\,(2a_2)!\,\dots\,(2a_j)! + (a_1 - 1 + a_2 + \dots + a_j)!/(a_1 - 1)!\,a_2!\,\dots\,a_j!$$

Recall that $0!$ is taken as 1.

This rule covers the case when the partition has two unrepeated summands.

Example — $2C(1\ 3\ 2\ 4) = 9!/3!\,2!\,4! + 4!/1!\,1!\,2!.$

(iv) When there are three or more unrepeated summands in the partition, $C(T)$ is half the number of decompositions which start with one of the unrepeated summands, this one remaining fixed.

Example — $T = (1\ 1\ 1\ 2\ 3)$, $C(T) = \frac{1}{2}(7!/1!\ 1!\ 2!\ 3!)$.

3.4. We give below, for reference, $C(T)$ for each T, when $k = 5$ or 6.

$k = 5$

T	$C(T)$
(5)	1
(1 4)	1
(2 3)	2
(1 1 3)	2
(1 2 2)	4
(1 1 1 2)	6
(1 1 1 1 1)	12

$k = 6$

T	$C(T)$
(6)	1
(1 5)	1
(2 4)	3
(3 3)	3
(1 1 4)	3
(1 2 3)	6
(2 2 2)	11
(1 1 1 3)	10
(1 1 2 2)	16
(1 1 1 1 2)	30
(1 1 1 1 1 1)	60

Note that the total number of T's for any k is $p(k)$.

We leave it to the reader to compute $C(T)$'s for $k = 7$.

3.5. *The Number of Partitions of n of a given Type T*

Suppose, we wish to find the number of those partitions of n which are of the type (2 3).

Let the summand which is repeated twice in the partition be denoted by u and that which is repeated thrice by v. Then the required number of partitions is the same as the number of solutions of the Diophantine equation

$$2u + 3v = n \qquad \ldots(22)$$

where u and v are distinct positive integers.

From the theory of partitions, it is well known that the number of solutions of (22) including those with $u = v$, is the same as the coefficient of

x^n in the ascending power expansion of $x^5/(1 - x^2)(1 - x^3)$.

Since the number of solutions of (22) with $u = v$, is in the same manner the same as the coefficient of

x^n in the ascending power expansion of $x^5/(1 - x^5)$,

it follows that the required number is the coefficient of

$$x^n \text{ in } x^5\{(1 - x^2)^{-1}(1 - x^3)^{-1} - (1 - x^5)^{-1}\}.$$

Writing $p(n, (2\ 3))$ for the number of partitions of n which are of the type (2 3), we thus have

$$\sum_{n\geqslant 5} p(n, (2\ 3))\, x^n = x^5 \frac{(1-x^5)-(1-x^2)(1-x^3)}{(1-x^2)(1-x^3)(1-x^5)}. \qquad ...(23)$$

The expression on the right of (23) is called the generating function of $p(n, (2\ 3))$.

For any given k, the best denominator to use for all types of partitions is

$$D_k = (1-x)(1-x^2)\ldots(1-x^k) \qquad ...(24)$$

and this we shall use henceforth. We shall thus be left to record only the numerators of generating functions. For this we shall adopt the notation:

$$(b_0 + b_1 + b_2 + \ldots + b_j)_x = b_0 + b_1 x + b_2 x^2 + \ldots + b_j x^j \qquad ...(25)$$

where the b's are integers not necessarily positive.

Thus, (23) will take the form

$$\sum_{n\geqslant 7} p(n, (2\ 3))\, x^n = x^7(1+0-1-2+1+0+1+2-2)_x/D_5. \qquad ...(26)$$

This implies that (22) has no solution for $n < 7$. (So the least n has asserted itself).

We shall denote the numerator in the generator of $p(n, T)$ for any type T of partitions by $P(T)$.

If $\quad T = (a_1\ a_2\ a_3\ \ldots\ a_j)$

where $\quad a_1 \leqslant a_2 \leqslant a_3 \leqslant \ldots \leqslant a_j$; and $a_1 + a_2 + a_3 + \ldots + a_j = k$;

then $P(T)$ is of the form

$$x^{n_0}(1 + b_1 + b_2 + \ldots + b_r)_x$$

where

$$n_0 = a_j + 2a_{j-1} + 3a_{j-2} + \ldots + ja_1 \qquad ...(27)$$

and

$$r = (k+1; 2) - n_0. \qquad ...(28)$$

We might also state that the total number of solutions of the Diophantine equation

$$a_1u_1 + a_2u_2 + a_3u_3 + \ldots + a_ju_j = n$$

in positive integers u is the coefficient of x^n in the expansion (in ascending powers of x) of

$$x^k/(1-x^{a_1})(1-x^{a_2})\ldots(1-x^{a_j}). \qquad ...(29)$$

Write (29) with D_k as the denominator and let $Q(T)$ denote the numerator. Then $Q(T)$ will be of the form

$$x^k(1 + c_1 + c_2 + \ldots + c_s)_x, \; s = (k; 2) \qquad \ldots(30)$$

where the c's are integers but not necessarily positive.

To obtain $P(T)$ from $Q(T)$, we have to remove somehow from $Q(T)$ all terms of degree less than n_0 in x. We have also to bear in mind that the coefficient of x^{n_0} in the final answer has to be 1. How this is done, we illustrate by an example in the next sub-section.

3.6. *The Generating Functions for* $R(n, 5)$ *and* $R(n, 6)$

Write $Q(T) = x^k N(T)$, $P(T) = x^{n_0} M(T)$.

(i) Case $k = 5$.

We take each type in turn.

Type (5), we have

$$n_0(5) = 5.$$

$$N(5) = (1 - 1 - 1 + 0 + 0 + 2 + 0 + 0 - 1 - 1 + 1)_x = M(5).$$

Type (1 4), we have

$$n_0(1\;4) = 6.$$

$$N(1\;4) = (1 + 0 - 1 - 1 + 0 + 0 + 0 + 1 + 1 + 0 - 1)_x.$$

To get rid of the 5th degree term from $Q(1\;4)$, we subtract $M(5)$ from $N(1\;4)$. This we do most conveniently as follows:

$$\begin{array}{rl} 1 + 0 - 1 - 1 + 0 + 0 + 0 + 1 + 1 + 0 - 1 & N(1\;4) \\ -1 + 1 + 1 + 0 + 0 - 2 - 0 - 0 + 1 + 1 - 1 & -M(5) \\ \hline 1 + 0 - 1 + 0 - 2 + 0 + 1 + 2 + 1 - 2 & \end{array}$$

This means that

$$M(1\;4) = (1 + 0 - 1 + 0 - 2 + 0 + 1 + 2 + 1 - 2)_x$$

Type (2 3) : $n_0(2\;3) = 7$;

$$N(2\;3) = (1 - 1 + 0 + 0 - 1 + 0 + 1 + 0 + 0 + 1 - 1)_x.$$

Processing:

$$\begin{array}{rl} 1 - 1 + 0 + 0 - 1 + 0 + 1 + 0 + 0 + 1 - 1 & N(2\;3) \\ -1 + 1 + 1 - 0 - 0 - 2 - 0 - 0 + 1 + 1 - 1 & -M(5) \\ \hline 1 + 0 - 1 - 2 + 1 + 0 + 1 + 2 - 2 & \end{array}$$

Hence

$$M(2\;3) = (1 + 0 - 1 - 2 + 1 + 0 + 1 + 2 - 2)_x.$$

Type (1 1 3) : $n_0(1\ 1\ 3) = 8$;

$$N(1\ 1\ 3) = (1+1+0+0-1-2-1+0+0+1+1)_x.$$

Processing :

$$\begin{array}{rl}
1+1+0+0-1-2-1+0+0+1+1 & N(1\ 1\ 3)\\
-1+1+1-0-0-2-0-0+1+1-1 & -M(5)\\
-2-0+2+0+4+0-2-4-2+4 & -2M(1\ 4)\\
-1+0+1+2-1+0-1-2+2 & -M(2\ 3)\\
\hline
)\,2+0+2-2-2-4-2+6 & \text{Divide by } 2\\
\hline
1+0+1-1-1-2-1+3 &
\end{array}$$

Thus

$$M(1\ 1\ 3) = (1+0+1-1-1-2-1+3)_x.$$

Type (1 2 2) : $n_0(1\ 2\ 2) = 9$.

$$N(1\ 2\ 2) = (1+0+1-1+0-2+0-1+1+0+1)_x.$$

Processing :

$$\begin{array}{rl}
1+0+1-1+0-2+0-1+1+0+1 & N(1\ 2\ 2)\\
-1+1+1+0+0-2+0+0+1+1-1 & -M(5)\\
-1+0+1+0+2+0-1-2-1+2 & -M(1\ 4)\\
-2+0+2+4-2+0-2-4+4 & -2M(2\ 3)\\
\hline
)\,2+2-2-2-2-4+6 & \text{Divide by } 2\\
\hline
1+1-1-1-1-2+3 &
\end{array}$$

i.e. $M(1\ 2\ 2) = (1+1-1-1-1-2+3)_x.$

Type (1 1 1 2) : $n_0(1\ 1\ 1\ 2) = 11$.

$$N(1\ 1\ 1\ 2) = (1+2+3+3+2+0-2-3-3-2-1)_x.$$

Processing :

$$\begin{array}{rl}
1+2+3+3+2+0-2-3-3-2-1 & N(1\ 1\ 1\ 2)\\
-1+1+1+0+0-2+0+0+1+1-1 & -M(5)\\
-3+0+3+0+6+0-3-6-3+6 & -3M(1\ 4)\\
-4+0+4+8-4+0-4-8+8 & -4M(2\ 3)\\
-6+0-6+6+6+12+6-18 & -6M(1\ 1\ 3)\\
-6-6+6+6+6+12-18 & -6M(1\ 2\ 2)\\
\hline
)\,6+6+6+6-24 & \text{Divide by } 6\\
\hline
1+1+1+1-4 &
\end{array}$$

so that $M(1\ 1\ 1\ 2) = (1+1+1+1-4)_x.$

Type (1 1 1 1 1) : $n_0(1\ 1\ 1\ 1\ 1) = 15$.

$$N(1\ 1\ 1\ 1\ 1) = (1 + 4 + 9 + 15 + 20 + 22 + 20 + 15 + 9 + 4 + 1)_x.$$

Processing :

1	+ 4	+ 9	+ 15	+ 20	+ 22	+ 20	+ 15	+ 9	+ 4	+ 1	$N(1\,1\,1\,1\,1)$
− 1	+ 1	+ 1	+ 0	+ 0	− 2	+ 0	+ 0	+ 1	+ 1	− 1	$-M(5)$
	− 5	+ 0	+ 5	+ 0	+ 10	+ 0	− 5	− 10	− 5	+ 10	$-5M(1\,4)$
		− 10	+ 0	+ 10	+ 20	− 10	+ 0	− 10	− 20	+ 20	$-10M(2\,3)$
			− 20	+ 0	− 20	+ 20	+ 20	+ 40	+ 20	− 60	$-20M(1\,1\,3)$
				− 30	− 30	+ 30	+ 30	+ 30	+ 60	− 90	$-30M(1\,2\,2)$
						− 60	− 60	− 60	− 60	+ 240	$-60M(1\,1\,1\,2)$
										) 120	Divide by 120
										1	

Thus $\qquad M(1\ 1\ 1\ 1\ 1) = (1)_x$.

Using the table of $C(T)$'s in section 3.4, we have

$1\ M(5)$	=	1	− 1	− 1	+ 0	+ 0	+ 2	+ 0	+ 0	− 1	− 1	+ 1
$1\ M(1\ 4)$	=		1	+ 0	− 1	+ 0	− 2	+ 0	+ 1	+ 2	+ 1	− 2
$2\ M(2\ 3)$	=			2	+ 0	− 2	− 4	+ 2	+ 0	+ 2	+ 4	− 4
$2\ M(1\ 1\ 3)$	=				2	+ 0	+ 2	− 2	− 2	− 4	− 2	+ 6
$4\ M(1\ 2\ 2)$	=					4	+ 4	− 4	− 4	− 4	− 8	+ 12
$6\ M(1\ 1\ 1\ 2)$	=							6	+ 6	+ 6	+ 6	− 24
$12\ M(1\ 1\ 1\ 1\ 1)$	=											12
		1	+ 0	+ 1	+ 1	+ 2	+ 2	+ 2	+ 1	+ 1	+ 0	+ 1

This shows that $R(n, 5)$ is the coefficient of x^n in

$$x^5(1 + 0 + 1 + 1 + 2 + 2 + 2 + 1 + 1 + 0 + 1)_x/D_5.$$

Letting

$$(1 + 0 + 1 + 1 + 2 + 2 + 2 + 1 + 1 + 0 + 1)_x$$
$$= D_5 \sum_{j \geqslant 5} R(j, 5)\, x^{j-5}, \qquad \ldots(31)$$

and comparing the coefficients of like powers of x on the two sides, the values of $R(n, 5)$ can be computed in succession. In fact for values of $n > 15$, one gets a recurrence relation from which the values of $R(n, 5)$ can be readily computed. This is about half as laborious as the method of Reis.

Proceeding on the same lines, one can show that

$$(1 + 0 + 2 + 2 + 5 + 4 + 9 + 6 + 9 + 6 + 7 + 3 + 4 + 1 + 1)_x$$
$$= D_6 \sum_{j \geqslant 6} R(j, 6)\, x^{j-6}. \qquad \ldots(32)$$

3.7. *Closed Formulae for* $R(n, k)$

The break-through came unexpectedly, when I tried to express the generating functions for $R(n, 5)$ and $R(n, 6)$ in terms of N's in place of M's.

From the processing in 3.6, it will be seen that

$$\begin{aligned}
N(5) &= M(5) \\
N(1\ 4) &= M(5) + M(1\ 4) \\
N(2\ 3) &= M(5) + M(2\ 3) \\
N(1\ 1\ 3) &= M(5) + 2M(1\ 4) + M(2\ 3) + 2M(1\ 1\ 3) \\
N(1\ 2\ 2) &= M(5) + M(1\ 4) + 2M(2\ 3) + 2M(1\ 2\ 2) \\
N(1\ 1\ 1\ 2) &= M(5) + 3M(1\ 4) + 4M(2\ 3) + 6M(1\ 1\ 3) + 6M(1\ 2\ 2) \\
&\quad + 6M(1\ 1\ 1\ 2)
\end{aligned}$$

and finally

$$N(1\ 1\ 1\ 1\ 1) = M(5) + 5M(1\ 4) + 10M(2\ 3) + 20M(1\ 1\ 3) + 30M(1\ 2\ 2) + 60M(1\ 1\ 1\ 2) + 120M(1\ 1\ 1\ 1\ 1).$$

In the form of a matrix equation, these relations can be written as

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 2 & 0 & 0 & 0 \\
1 & 1 & 2 & 0 & 2 & 0 & 0 \\
1 & 3 & 4 & 6 & 6 & 6 & 0 \\
1 & 5 & 10 & 20 & 30 & 60 & 120
\end{bmatrix}
\begin{bmatrix}
M(5) \\ M(1\ 4) \\ M(2\ 3) \\ M(1\ 1\ 3) \\ M(1\ 2\ 2) \\ M(1\ 1\ 1\ 2) \\ M(1\ 1\ 1\ 1\ 1)
\end{bmatrix}
=
\begin{bmatrix}
N(5) \\ N(1\ 4) \\ N(2\ 3) \\ N(1\ 1\ 3) \\ N(1\ 2\ 2) \\ N(1\ 1\ 1\ 2) \\ N(1\ 1\ 1\ 1\ 1)
\end{bmatrix}$$

Hence, we get

$$10\begin{bmatrix} M(5) \\ M(1\ 4) \\ 2M(2\ 3) \\ 2M(1\ 1\ 3) \\ 4M(1\ 2\ 2) \\ 6M(1\ 1\ 1\ 2) \\ 12M(1\ 1\ 1\ 1\ 1) \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 10 & 0 & 0 & 0 & 0 & 0 \\ -20 & 0 & 20 & 0 & 0 & 0 & 0 \\ 20 & -20 & -10 & 10 & 0 & 0 & 0 \\ 40 & -20 & -40 & 0 & 20 & 0 & 0 \\ -60 & 60 & 50 & -30 & -30 & 10 & 0 \\ 24 & -30 & -20 & 20 & 15 & -10 & 1 \end{bmatrix} \begin{bmatrix} N(5) \\ N(1\ 4) \\ N(2\ 3) \\ N(1\ 1\ 3) \\ N(1\ 2\ 2) \\ N(1\ 1\ 1\ 2) \\ N(1\ 1\ 1\ 1\ 1) \end{bmatrix}$$

Whence, adding the entries in each column of the square matrix, it is readily seen that $10\ R(n, 5)$ is the coefficient of x^{n-5} in

$$\{4\ N(5) + 5\ N(1\ 2\ 2) + N(1\ 1\ 1\ 1\ 1)\}/D^5$$

i.e. in $$4(1 - x^5)^{-1} + 5(1 - x)^{-1}(1 - x^2)^{-2} + (1 - x)^{-5}. \qquad ...(33)$$

It simplifies matters, if we write (33) in the form

$$5(1 + x)(1 - x^2)^{-3} + (1 - x)^{-5} + 4(1 - x^5)^{-1}. \qquad ...(34)$$

Hence $$10\ R(n, 5) = 5([(n - 1)/2]; 2) + (n - 1; 4) + 4 \text{ if } (n, 5) = 5;$$
$$= 5([(n - 1)/2]; 2) + (n - 1; 4) \quad \text{otherwise.}$$

For $k = 6$, we get the matrix equation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 2 & 6 & 3 & 0 & 6 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 2 & 4 & 6 & 0 & 4 & 0 & 0 \\ 1 & 4 & 7 & 8 & 12 & 16 & 18 & 24 & 24 & 24 & 0 \\ 1 & 6 & 15 & 20 & 30 & 60 & 90 & 120 & 180 & 360 & 720 \end{bmatrix} M^* = N^*$$

where $M^* = \begin{bmatrix} M(6) \\ M(1\ 5) \\ M(2\ 4) \\ M(3\ 3) \\ M(1\ 1\ 4) \\ M(1\ 2\ 3) \\ M(2\ 2\ 2) \\ M(1\ 1\ 1\ 3) \\ M(1\ 1\ 2\ 2) \\ M(1\ 1\ 1\ 1\ 2) \\ M(1\ 1\ 1\ 1\ 1\ 1) \end{bmatrix}$ $N^* = \begin{bmatrix} N(6) \\ N(1\ 5) \\ N(2\ 4) \\ N(3\ 3) \\ N(1\ 1\ 4) \\ N(1\ 2\ 3) \\ N(2\ 2\ 2) \\ N(1\ 1\ 1\ 3) \\ N(1\ 1\ 2\ 2) \\ N(1\ 1\ 1\ 1\ 2) \\ N(1\ 1\ 1\ 1\ 1\ 1) \end{bmatrix}$

Proceeding as in the case of $k = 5$, we finally find that $12\,R(n, 6)$ is the coefficient of x^{n-6} in

$$2(1-x^6)^{-1} + 2(1-x^3)^{-2} + 4(1-x^2)^{-3} + 3(1-x)^{-2}(1-x^2)^{-2} + (1-x)^{-6}$$

i.e. in

$$6(1+x)(1-x^2)^{-4} + (1-x)^{-6} + (1-x^2)^{-3} + 2(1-x^3)^{-2} + 2(1-x^6)^{-1} \quad \ldots(35)$$

Results (34) and (35) are very suggestive and in view of the prediction made by Reis, led me to the conjecture:

"$2k\,R(n, k)$ is the coefficient of x^{n-k} in

$$k(1+x)(1-x^2)^{-[(k+2)/2]} + \sum_{d \mid g} \phi(d)(1-x^d)^{-n/d} \quad \ldots(36)$$

where $g = (n, k)$."

To prove the conjecture, all that is necessary is to show that the conjecture is not at variance with the fundamental relation

$$R(n, k) = R(n, n-k). \quad \ldots(37)$$

Let $n - k = h$, then we have to show that for each divisor d of g, relation (37) holds good.

Now, we have

$$2kh\,R(n,k) = hk\left(\left[\frac{k+h-t}{2}\right]; \left[\frac{k}{2}\right]\right) + h\sum_{d \mid g} \phi(d)\left(\frac{n}{d} - 1; \frac{k}{d} - 1\right)$$

where $t = 0$ or 1 according as k is even or odd; and

$$2hk\,R(n, h) = kh\left(\left[\frac{k+h-s}{2}\right]; \left[\frac{h}{2}\right]\right) + k\sum_{d\mid g}\phi(d)\left(\frac{n}{d}-1; \frac{h}{d}-1\right)$$

where $s = 0$ or 1 according as h is even or odd.

It is easy to see that

$$\left(\left[\frac{k+h-t}{2}\right]; \left[\frac{k}{2}\right]\right) = \left(\left[\frac{k+h-s}{2}\right]; \left[\frac{h}{2}\right]\right) \text{ for all } h \text{ and } k.$$

Also

$$\begin{aligned} h\left(\frac{n}{d}-1; \frac{k}{d}-1\right) &= h\left(\frac{k+h}{d}-1; \frac{k}{d}-1\right) \\ &= h\left(\frac{k+h}{d}-1; \frac{h}{d}\right) \\ &= h\left(\frac{k+h}{d}-1\right)!\Big/\left(\frac{k}{d}-1\right)!\left(\frac{h}{d}\right)! \\ &= d\left(\frac{k+h}{d}-1\right)!\Big/\left(\frac{k}{d}-1\right)!\left(\frac{h}{d}-1\right)! \\ &= k\left(\frac{k+h}{d}-1\right)!\Big/\left(\frac{h}{d}-1\right)!\left(\frac{k}{d}\right)! \end{aligned}$$

and we are through.

Of course, induction takes care of the rest.

4. THE FUNCTION $R'(n, k)$

4.1. Our account will not be complete, if we do not consider, at least very briefly, the function $R'(n, k)$ which is closely related to $R(n, k)$ which has been dealt with at length in the foregoing pages.

Two decompositions of n into k parts are defined to be weakly equivalent if the k-gons representing them are directly congruent.

$R'(n, k)$ denotes the number of equivalence classes into which the decompositions of n into k parts, can now be divided.

Besides replacing the table in section 3.4 by the following, no new technique is required.

$k = 5$		$k = 6$	
T	$C'(T)$	T	$C'(T)$
(5)	1	(6)	1
(1 4)	1	(1 5)	1
(2 3)	2	(2 4)	3
(1 1 3)	4	(3 3)	4
(1 2 2)	6	(1 1 4)	5
(1 1 1 2)	12	(1 2 3)	10
(1 1 1 1 1)	24	(2 2 2)	16
		(1 1 1 3)	20
		(1 1 2 2)	30
		(1 1 1 1 2)	60
		(1 1 1 1 1 1)	120

We give only the final matrix in the case of $k = 6$:

$$\begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -18 & 0 & 18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 30 & -30 & -15 & 0 & 15 & 0 & 0 & 0 & 0 & 0 & 0 \\ 120 & -60 & -60 & -60 & 0 & 60 & 0 & 0 & 0 & 0 & 0 \\ 32 & 0 & -48 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 \\ -120 & 120 & 60 & 40 & -60 & -60 & 0 & 20 & 0 & 0 & 0 \\ -270 & 180 & 225 & 90 & -45 & -180 & -45 & 0 & 45 & 0 & 0 \\ 360 & -360 & -270 & -120 & 180 & 300 & 45 & -60 & -90 & 15 & 0 \\ -120 & 144 & 90 & 40 & -90 & -120 & -15 & 40 & 45 & -15 & 1 \end{bmatrix}$$

The column sums are

$$2 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1$$

From this we conclude that $6R'(n, 6)$ is the coefficient of x^{n-6} in

$$2(1 - x^6)^{-1} + 2(1 - x^3)^{-2} + (1 - x^2)^{-3} + (1 - x)^{-6}.$$

As in the case of $R(n, k)$, we finally get

$$k\, R'(n, k) = \sum_{d \mid (n, k)} \phi(d) \left(\frac{n}{d} - 1; \frac{k}{d} - 1 \right). \qquad \ldots(38)$$

This is in conformity with the fundamental relation

$$R'(n, k) = R'(n, n-k).$$

We further have

$$2\, R(n, k) = S(n, k) + R'(n, k) \qquad \ldots(39)$$

where $S(n, k)$ denotes the number of symmetric k-gons for the given n. A combinatorial proof of (39) is easy to give.

We might here note that the fundamental relation can be used to find the expression for $R'(n, k)$ for a given k after the values for $R'(n, m)$ have been obtained for each $m < k$, as the following example will illustrate.

Take $k = 10$.

Let

$$10\, R'(n, 10) = \sum_{d \mid 10} a_d \left(\frac{n}{d} - 1; \frac{k}{d} - 1 \right).$$

To determine the a's, take $n = 10, 11, 12, 15$ in turn.

We thus get the relations:

$$a_1 + a_2 + a_5 + a_{10} = 10\, R'(10, 0) = 10;$$

$$10\, a_1 = 10\, R'(11, 1) = 10;$$

$$55\, a_1 + 5\, a_2 = 10\, R'(12, 2) = 60;$$

$$2002\, a_1 + 2a_5 = 10\, R'(15, 5) = 2010.$$

These give

$$a_1 = 1 = \phi(1);\ a_2 = 1 = \phi(2);\ a_5 = 4 = \phi(5);$$

and $a_{10} = 4 = \phi(10)$.

4.2. There is an interesting formula for

$$B'(n) = \sum_{k=0}^{n} R'(n, k).$$

This we proceed to find.

We have

$$k\, R'(n, k) = \sum_{d \mid (n, k)} \phi(d) \left(\frac{n}{d} - 1; \frac{k}{d} - 1 \right). \qquad \ldots(40)$$

Therefore

$$(n-k)\,R'(n, n-k) = \sum_{d\,|\,(n,\,n-k)} \phi(d)\left(\frac{n}{d}-1;\frac{n-k}{d}-1\right)$$

or what is the same thing

$$(n-k)\,R'(n, k) = \sum_{d\,|\,(n,\,k)} \phi(d)\left(\frac{n}{d}-1;\frac{k}{d}\right). \qquad \ldots(41)$$

Adding (40) and (41), we get

$$n\,R'(n, k) = \sum_{d\,|\,(n,\,k)} \phi(d)\left(\frac{n}{d};\frac{k}{d}\right). \qquad \ldots(42)$$

Whence

$$n \sum_{k=0}^{n} R'(n, k) = \sum_{d\,|\,n} \phi(d)\,\{(n/d;\,0) + (n/d;\,1) + \ldots + (n/d;\,n/d)\}$$

$$n\,B'(n) = \sum_{d\,|\,n} \phi(d)\,2^{n/d}. \qquad \ldots(43)$$

We leave it to the reader to find a similar formula for

$$B(n) = \sum_{k=0}^{n} R(n, k).$$

It will thus be seen that the problem of Reis is directly related to the bead-stringing problem* when the beads are available in two different colours.

5. Tables

The tables that follow give for $n \leqslant 100$, $3 \leqslant k \leqslant 12$, the values of $S(n, k)$—the number of symmetric k-gons for any given n and k in the range and also the values of $R'(n, k)$ in the said range.

Note that

$$R'(n, 0) = 1 = R'(n, 1)$$

and $$R'(n, 2) = [n/2].$$

In preparing these tables, the Royal Society "Tables of Binomial Coefficients" [University Press, Cambridge (1954)] have been freely used.

*John Riordan (1958). An Introduction to Combinatorial Analysis. John Wiley & Sons, New York, p. 162.

TABLE I

Table for $S(n, k)$

(This table will enable the reader to find $S(n, k)$—the number of symmetric k-gons for $n \leqslant 100$, $k \leqslant 12$. The table actually gives the values of $(m; r)$ for $m \leqslant 50$, $r \leqslant 6$.)

m \ r	2	3	4	5	6
1					
2	1				
3	3	1			
4	6	4	1		
5	10	10	5	1	
6	15	20	15	6	1
7	21	35	35	21	7
8	28	56	70	56	28
9	36	84	126	126	84
10	45	120	210	252	210
11	55	165	330	462	462
12	66	220	495	792	924
13	78	286	715	1287	1716
14	91	364	1001	2002	3003
15	105	455	1365	3003	5005
16	120	560	1820	4368	8008
17	136	680	2380	6188	12376
18	153	816	3060	8568	18564
19	171	969	3876	11628	27132
20	190	1140	4845	15504	38760
21	210	1330	5985	20349	54264
22	231	1540	7315	26334	74613
23	253	1771	8855	33649	1 00947
24	276	2024	10626	42504	1 34596
25	300	2300	12650	53130	1 77100
26	325	2600	14950	65780	2 30230
27	351	2925	17550	80730	2 96010
28	378	3276	20475	98280	3 76740
29	406	3654	23751	1 18755	4 75020
30	435	4060	27405	1 42506	5 93775
31	465	4495	31465	1 69911	7 36281
32	496	4960	35960	2 01376	9 06192
33	528	5456	40920	2 37336	11 07568
34	561	5984	46376	2 78256	13 44904
35	595	6545	52360	3 24632	16 23160
36	630	7140	58905	3 76992	19 47792
37	666	7770	66045	4 35897	23 24784
38	703	8436	73815	5 01942	27 60681
39	741	9139	82251	5 75757	32 62623
40	780	9880	91390	6 58008	38 38380
41	820	10660	1 01270	7 49398	44 96388
42	861	11480	1 11930	8 50668	52 45786
43	903	12341	1 23410	9 62598	60 96454
44	946	13244	1 35751	10 86008	70 59052
45	990	14190	1 48995	12 21759	81 45060
46	1035	15180	1 63185	13 70754	93 66819
47	1081	16215	1 78365	15 33939	107 37573
48	1128	17296	1 94580	17 12304	122 71512
49	1176	18424	2 11876	19 06884	139 83816
50	1225	19600	2 30300	21 18760	158 90700

TABLE

Table of values

n \ k	3	4	5	6	7	8
3	1					
4	1	1				
5	2	1	1			
6	4	3	1	1		
7	5	5	3	1	1	
8	7	10	7	4	1	1
9	10	14	14	10	4	1
10	12	22	26	22	12	5
11	15	30	42	42	30	15
12	19	43	66	80	66	43
13	22	55	99	132	132	99
14	26	73	143	217	246	217
15	31	91	201	335	429	429
16	35	116	273	504	715	810
17	40	140	364	728	1144	1430
18	46	172	476	1038	1768	2438
19	51	204	612	1428	2652	3978
20	57	245	776	1944	3876	6310
21	64	285	969	2586	5538	9690
22	70	335	1197	3399	7752	14550
23	77	385	1463	4389	10659	21318
24	85	446	1771	5620	14421	30667
25	92	506	2126	7084	19228	43263
26	100	578	2530	8866	25300	60115
27	109	650	2990	10966	32890	82225
28	117	735	3510	13468	42288	1 11041
29	126	819	4095	16380	53820	1 48005
30	136	917	4751	19811	67860	1 95143
31	145	1015	5481	23751	84825	2 54475
32	155	1128	6293	28336	1 05183	3 28756
33	166	1240	7192	33566	1 29456	4 20732
34	176	1368	8184	39576	1 58224	5 34076
35	187	1496	9276	46376	1 92130	6 72452
36	199	1641	10472	54132	2 31880	8 40652
37	210	1785	11781	62832	2 78256	10 43460
38	222	1947	13209	72675	3 32112	12 87036
39	235	2109	14763	83661	3 94383	15 77532
40	247	2290	16451	95988	4 66089	19 22741

II
of $R'(n,k)$

9	10	11	12	k / n
1				9
1	1			10
5	1	1		11
19	6	1	1	12
55	22	6	1	13
143	73	26	7	14
335	201	91	31	15
715	504	273	116	16
1430	1144	728	364	17
2704	2438	1768	1038	18
4862	4862	3978	2652	19
8398	9252	8398	6310	20
14000	16796	16796	14000	21
22610	29414	32066	29414	22
35530	49742	58786	58786	23
54484	81752	1 04006	1 12720	24
81719	1 30752	1 78296	2 08012	25
1 20175	2 04347	2 97160	3 71516	26
1 73593	3 12455	4 82885	6 43856	27
2 46675	4 68754	7 66935	10 86601	28
3 45345	6 90690	11 93010	17 89515	29
4 76913	10 01603	18 20910	28 83289	30
6 50325	14 30715	27 31365	45 52275	31
8 76525	20 16144	40 32015	70 56280	32
11 68710	28 04880	58 64750	107 52060	33
15 42684	38 56892	84 14640	161 28424	34
20 17356	52 45128	119 20740	238 41480	35
26 15104	70 60984	166 89036	347 69374	36
33 62260	94 14328	231 07896	500 67108	37
42 89780	124 40668	316 66376	712 50060	38
54 33736	163 01164	429 75796	1002 76894	39
68 35972	211 91904	577 95036	1396 72312	40

(continued)

TABLE II

n \ k	3	4	5	6	7	8
41	260	2470	18278	1 09668	5 48340	23 30445
42	274	2670	20254	1 24936	6 42342	28 10385
43	287	2870	22386	1 41778	7 49398	33 72291
44	301	3091	24682	1 60468	8 70922	40 28183
45	316	3311	27151	1 81006	10 08436	47 90071
46	330	3553	29799	2 03665	11 63580	56 72645
47	345	3795	32637	2 28459	13 38117	66 90585
48	361	3960	35673	2 55704	15 33939	78 61662
49	376	4324	38916	2 85384	17 53074	92 03634
50	392	4612	42376	3 17860	19 97688	107 37826
51	409	4900	46060	3 53132	22 70100	124 85550
52	425	5213	49980	3 91560	25 72780	144 72178
53	442	5525	54145	4 33160	29 08360	167 23070
54	460	5863	58565	4 78341	32 79640	192 68210
55	477	6201	63251	5 27085	36 89595	221 37570
56	495	6566	68211	5 79852	41 41383	253 66335
57	514	6930	73458	6 36642	46 38348	289 89675
58	532	7322	79002	6 97914	51 84036	330 48639
59	551	7714	84854	7 63686	57 82194	375 84261
60	571	8135	91026	8 34472	64 36782	426 44141
61	590	8555	97527	9 10252	71 51980	482 75865
62	610	9005	1 04371	9 91597	79 32196	545 34355
63	631	9455	1 11569	10 78507	87 82075	614 74519
64	651	9936	1 19133	11 71552	97 06503	691 59400
65	672	10416	1 27076	12 70752	107 10624	776 52024
66	694	10928	1 35408	13 76738	117 99840	870 24440
67	715	11440	1 44144	14 89488	129 79824	973 48680
68	737	11985	1 53296	16 09696	142 56528	1087 06712
69	760	12529	1 62877	17 37362	156 36192	1211 80488
70	782	13107	1 72901	18 73179	171 25354	1348 62904
71	805	13685	1 83379	20 17169	187 30855	1498 46840
72	829	14298	1 94327	21 70092	204 59857	1662 37161
73	852	14910	2 05758	23 31924	223 19844	1841 38713
74	876	15558	2 17686	25 03494	243 18636	2036 69469
75	901	16206	2 30126	26 84802	264 64398	2249 47383
76	925	16891	2 43090	28 76676	287 65650	2481 04707
77	950	17575	2 56595	30 79140	312 31278	2732 73675
78	976	18297	2 70655	32 93095	338 70540	3006 02097
79	1001	19019	2 85285	35 18515	366 93085	3302 37765
80	1027	19780	3 00501	37 56376	397 08955	3623 45362

(continued)

9	10	11	12	k / n
85 44965	273 43888	770 60048	1926 50120	41
106 16489	350 34841	1019 18128	2632 89838	42
131 14465	445 89181	1337 67543	3567 13448	43
161 12057	563 92798	1743 03163	4793 35399	44
196 92535	708 93054	2255 68798	6391 11655	45
239 50355	886 17045	2900 17026	8458 85187	46
289 92535	1101 71633	3705 77311	11117 31933	47
349 39745	1362 65800	4707 33341	14514 30692	48
419 27666	1677 10664	5946 10536	18829 33364	49
501 08674	2054 46630	7470 74776	24279 96564	50
596 53210	2505 43370	9338 43470	31128 11660	51
707 51450	3042 32500	11616 10170	39688 39186	52
836 15350	3679 07540	14381 84020	50336 44070	53
984 80332	4431 62850	17726 45420	63519 85018	54
1156 07310	5317 93630	21755 19380	79769 04390	55
1352 85150	6358 41960	26589 68130	99711 37228	56
1578 32709	7575 96840	32370 04680	1 24085 18076	57
1836 01275	8996 48295	39257 29080	1 53757 80420	58
2129 77479	10648 87395	47435 89305	1 89743 57220	59
2463 85749	12565 69506	57116 68755	2 33226 57491	60
2842 91205	14783 14266	68540 02506	2 85583 43775	61
3272 03085	17341 79091	81979 24566	3 48411 91281	62
3756 77659	20286 59127	97744 48521	4 23559 43781	63
4303 21633	23667 72128	1 16186 84091	5 13158 68912	64
4917 96152	27540 58456	1 37702 92256	6 19663 15152	65
5608 20220	31966 78584	1 62739 81758	7 45891 00058	66
6381 74680	37014 13144	1 91800 49928	8 95068 99664	67
7247 06840	42757 74448	2 25449 70968	10 70886 31896	68
8213 34470	49280 06512	2 64320 34928	12 77548 35742	69
9290 50408	56672 12132	3 09120 40848	15 19842 24024	70
10489 27880	65033 52856	3 60640 47656	18 03202 38280	71
11821 25128	74473 93184	4 19761 86616	21 33789 76004	72
13298 90705	85113 00512	4 87465 39296	25 18571 19696	73
14935 69561	97082 08037	5 64840 85216	29 65414 78800	74
16746 08357	1 10524 14757	6 53097 23531	34 83185 25836	75
18745 61525	1 25595 68822	7 53573 73305	40 81858 08419	76
20950 98175	1 42466 67590	8 67751 57140	47 72633 64265	77
23380 08175	1 61322 63329	9 97266 73130	55 68073 00523	78
26052 09035	1 82364 63245	11 43923 60355	64 82233 75345	79
28987 53715	2 05811 59608	13 09709 63305	75 30830 87012	80

(continued)

TABLE II

n \ k	3	4	5	6	7	8
81	1054	20540	3 16316	40 06678	429 28600	3970 89550
82	1080	21340	3 32748	42 70396	463 62888	4346 53310
83	1107	22140	3 49812	45 47556	500 23116	4752 19602
84	1135	22981	3 67524	48 39212	539 21022	5189 91166
85	1162	23821	3 85901	51 45336	580 68792	5661 70722
86	1190	24703	4 04957	54 67063	624 79080	6169 82359
87	1219	25585	4 24711	58 04393	671 65011	6716 50110
88	1247	26510	4 45179	61 58460	721 40197	7304 21043
89	1276	27434	4 66378	65 29292	774 18748	7935 42167
90	1306	28402	4 88326	69 18108	830 15284	8612 85227
91	1335	29370	5 11038	73 24878	889 44948	9339 21945
92	1365	30383	5 34534	77 50908	952 23414	10117 50553
93	1396	31395	5 58831	81 96198	1018 66908	10950 69261
94	1426	32453	5 83947	86 62053	1088 92212	11842 04703
95	1457	33511	6 09901	91 48503	1163 16681	12794 83491
96	1489	34616	6 36709	96 56944	1241 58255	13812 62620
97	1520	35720	6 64392	101 87344	1324 35472	14898 99060
98	1552	36872	6 92968	107 41192	1411 67482	16057 82260
99	1585	38024	7 22456	113 18488	1503 74056	17293 01644
100	1617	39225	7 52876	119 20720	1600 75608	18608 81252

(continued)

9	10	11	12	k / n
32208 37534	2 31900 29720	14 96811 00920	87 31397 55800	81
35738 05950	2 60887 92574	17 07629 46120	101 03474 86044	82
39601 63350	2 93052 08790	19 44800 21970	116 68801 31820	83
43825 80852	3 28693 65932	22 11211 20870	134 51535 48264	84
48439 05066	3 68136 78508	25 10023 53420	154 78478 46090	85
53471 67930	4 11732 04254	28 44693 33876	177 79334 07614	86
58955 95494	4 59856 44198	32 18995 09386	203 86968 93324	87
64926 17730	5 12916 92408	36 37046 40476	233 37715 23300	88
71418 79503	5 71350 36024	41 03334 40536	266 71673 63484	89
78472 50409	6 35627 41159	46 22743 82376	304 33064 41754	90
86128 35715	7 06252 52863	52 00586 80173	346 70578 67820	91
94429 88555	7 83768 19906	58 42634 55503	394 37784 26497	92
1 03423 20895	8 68754 94706	65 55150 96418	447 93531 59533	93
1 13157 15697	9 61835 99743	73 44928 18878	508 02421 11469	94
1 23683 40413	10 63677 27559	82 19324 40173	575 35270 81211	95
1 35056 59175	11 74992 51760	91 86303 74311	650 69652 79992	96
1 47334 46260	12 96543 27088	102 54478 59696	734 90429 94488	97
1 60578 00980	14 29144 48180	114 33154 29776	828 90370 08568	98
1 74851 61178	15 73664 49604	127 32376 37706	933 70760 10664	99
1 90223 18084	17 31031 15760	141 62980 46436	1050 42106 70020	100