

Discussion questions

A Compare Δp and Δx for the two lowest energy levels of the one-dimensional particle in a box, and discuss how this relates to the uncertainty principle.

B On a graph of Δp versus Δx , sketch the regions that are allowed and forbidden by the Heisenberg uncertainty principle. Interpret the graph: Where does an atom lie on it? An elephant? Can either p or x be measured with perfect accuracy if we don't care about the other?

35.5 Electrons in electric fields

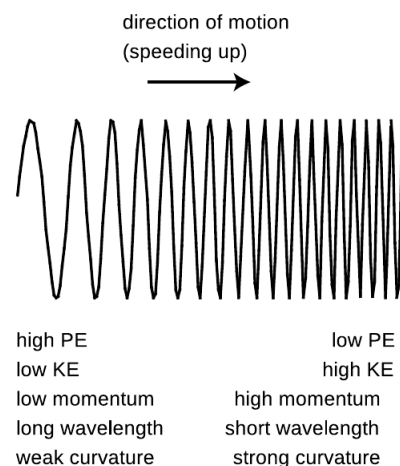
So far the only electron wave patterns we've considered have been simple sine waves, but whenever an electron finds itself in an electric field, it must have a more complicated wave pattern. Let's consider the example of an electron being accelerated by the electron gun at the back of a TV tube. The electron is moving from a region of low voltage into a region of higher voltage. Since its charge is negative, it loses PE by moving to a higher voltage, so its KE increases. As its potential energy goes down, its kinetic energy goes up by an equal amount, keeping the total energy constant. Increasing kinetic energy implies a growing momentum, and therefore a shortening wavelength, λ .

The wavefunction as a whole does not have a single well-defined wavelength, but the wave changes so gradually that if you only look at a small part of it you can still pick out a wavelength and relate it to the momentum and energy. (The picture actually exaggerates by many orders of magnitude the rate at which the wavelength changes.)

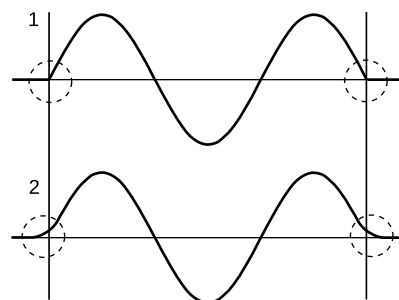
But what if the electric field was stronger? The electric field in a TV is only $\sim 10^5$ N/C, but the electric field within an atom is more like 10^{12} N/C. In figure k, the wavelength changes so rapidly that there is nothing that looks like a sine wave at all. We could get a general idea of the wavelength in a given region by measuring the distance between two peaks, but that would only be a rough approximation. Suppose we want to know the wavelength at point P. The trick is to construct a sine wave, like the one shown with the dashed line, which matches the curvature of the actual wavefunction as closely as possible near P. The sine wave that matches as well as possible is called the "osculating" curve, from a Latin word meaning "to kiss." The wavelength of the osculating curve is the wavelength that will relate correctly to conservation of energy.

Tunneling

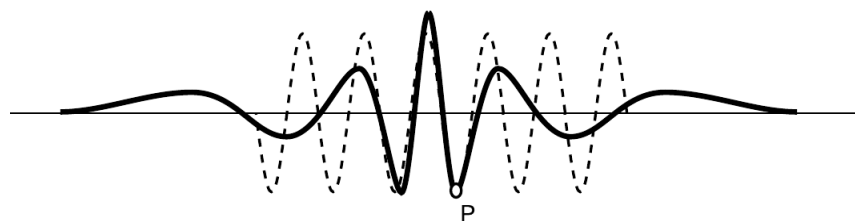
We implicitly assumed that the particle-in-a-box wavefunction would cut off abruptly at the sides of the box, 1/1, but that would be un-



j / An electron in a gentle electric field gradually shortens its wavelength as it gains energy.



l / 1. Kinks like this don't happen. 2. The wave actually penetrates into the classically forbidden region.



k / A typical wavefunction of an electron in an atom (heavy curve) and the osculating sine wave (dashed curve) that matches its curvature at point P.

physical. A kink has infinite curvature, and curvature is related to energy, so it can't be infinite. A physically realistic wavefunction must always “tail off” gradually, $1/2$. In classical physics, a particle can never enter a region in which its potential energy would be greater than the amount of energy it has available. But in quantum physics the wavefunction will always have a tail that reaches into the classically forbidden region. If it was not for this effect, called tunneling, the fusion reactions that power the sun would not occur due to the high potential energy that nuclei need in order to get close together! Tunneling is discussed in more detail in the next section.

35.6 \int ★ The Schrödinger equation

In section 35.5 we were able to apply conservation of energy to an electron's wavefunction, but only by using the clumsy graphical technique of osculating sine waves as a measure of the wave's curvature. You have learned a more convenient measure of curvature in calculus: the second derivative. To relate the two approaches, we take the second derivative of a sine wave:

$$\begin{aligned} \frac{d^2}{dx^2} \sin\left(\frac{2\pi x}{\lambda}\right) &= \frac{d}{dx} \left(\frac{2\pi}{\lambda} \cos \frac{2\pi x}{\lambda}\right) \\ &= -\left(\frac{2\pi}{\lambda}\right)^2 \sin \frac{2\pi x}{\lambda} \end{aligned}$$

Taking the second derivative gives us back the same function, but with a minus sign and a constant out in front that is related to the wavelength. We can thus relate the second derivative to the osculating wavelength:

$$[1] \quad \frac{d^2 \Psi}{dx^2} = -\left(\frac{2\pi}{\lambda}\right)^2 \Psi$$

This could be solved for λ in terms of Ψ , but it will turn out to be more convenient to leave it in this form.

Using conservation of energy, we have

$$\begin{aligned} E &= KE + PE \\ [2] \quad &= \frac{p^2}{2m} + PE \\ &= \frac{(h/\lambda)^2}{2m} + PE \end{aligned}$$

Note that both equation [1] and equation [2] have λ^2 in the denominator. We can simplify our algebra by multiplying both sides of equation [2] by Ψ to make it look more like equation [1]:

$$\begin{aligned} E \cdot \Psi &= \frac{(h/\lambda)^2}{2m} \Psi + PE \cdot \Psi \\ &= \frac{1}{2m} \left(\frac{h}{2\pi} \right)^2 \left(\frac{2\pi}{\lambda} \right)^2 \Psi + PE \cdot \Psi \\ &= -\frac{1}{2m} \left(\frac{h}{2\pi} \right)^2 \frac{d^2 \Psi}{dx^2} + PE \cdot \Psi \end{aligned}$$

Further simplification is achieved by using the symbol \hbar (h with a slash through it, read “h-bar”) as an abbreviation for $h/2\pi$. We then have the important result known as the **Schrödinger equation**:

$$E \cdot \Psi = -\frac{\hbar^2}{2m} \frac{d^2 \Psi}{dx^2} + PE \cdot \Psi$$

(Actually this is a simplified version of the Schrödinger equation, applying only to standing waves in one dimension.) Physically it is a statement of conservation of energy. The total energy E must be constant, so the equation tells us that a change in potential energy must be accompanied by a change in the curvature of the wavefunction. This change in curvature relates to a change in wavelength, which corresponds to a change in momentum and kinetic energy.

self-check D

Considering the assumptions that were made in deriving the Schrödinger equation, would it be correct to apply it to a photon? To an electron moving at relativistic speeds?

▷ Answer, p.

1047

Usually we know right off the bat how the potential energy depends on x , so the basic mathematical problem of quantum physics is to find a function $\Psi(x)$ that satisfies the Schrödinger equation for a given function $PE(x)$. An equation, such as the Schrödinger equation, that specifies a relationship between a function and its derivatives is known as a differential equation.

The study of differential equations in general is beyond the mathematical level of this book, but we can gain some important insights by considering the easiest version of the Schrödinger equation, in which the potential energy is constant. We can then rearrange the Schrödinger equation as follows:

$$\frac{d^2 \Psi}{dx^2} = \frac{2m(PE - E)}{\hbar^2} \Psi,$$

which boils down to

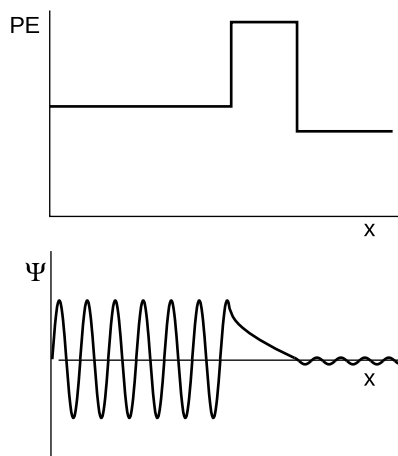
$$\frac{d^2 \Psi}{dx^2} = a\Psi,$$

where, according to our assumptions, a is independent of x . We need to find a function whose second derivative is the same as the original function except for a multiplicative constant. The only functions with this property are sine waves and exponentials:

$$\begin{aligned} \frac{d^2}{dx^2} [q \sin(rx + s)] &= -qr^2 \sin(rx + s) \\ \frac{d^2}{dx^2} [qe^{rx+s}] &= qr^2 e^{rx+s} \end{aligned}$$

The sine wave gives negative values of a , $a = -r^2$, and the exponential gives positive ones, $a = r^2$. The former applies to the classically allowed region with $PE < E$.

This leads us to a quantitative calculation of the tunneling effect discussed briefly in the preceding subsection. The wavefunction evidently tails off exponentially in the classically forbidden region. Suppose, as shown in figure m, a wave-particle traveling to the right encounters a barrier that it is classically forbidden to enter. Although the form of the Schrödinger equation we're using technically does not apply to traveling waves (because it makes no reference to time), it turns out that we can still use it to make a reasonable calculation of the probability that the particle will make it through the barrier. If we let the barrier's width be w , then the ratio of the



m / Tunneling through a barrier.

wavefunction on the left side of the barrier to the wavefunction on the right is

$$\frac{qe^{rx+s}}{qe^{r(x+w)+s}} = e^{-rw}.$$

Probabilities are proportional to the squares of wavefunctions, so the probability of making it through the barrier is

$$\begin{aligned} P &= e^{-2rw} \\ &= \exp\left(-\frac{2w}{\hbar}\sqrt{2m(PE - E)}\right) \end{aligned}$$

self-check E

If we were to apply this equation to find the probability that a person can walk through a wall, what would the small value of Planck's constant imply? ▷ Answer, p. 1047

Use of complex numbers

In a classically forbidden region, a particle's total energy, $PE + KE$, is less than its PE , so its KE must be negative. If we want to keep believing in the equation $KE = p^2/2m$, then apparently the momentum of the particle is the square root of a negative number. This is a symptom of the fact that the Schrödinger equation fails to describe all of nature unless the wavefunction and various other quantities are allowed to be complex numbers. In particular it is not possible to describe traveling waves correctly without using complex wavefunctions.

This may seem like nonsense, since real numbers are the only ones that are, well, real! Quantum mechanics can always be related to the real world, however, because its structure is such that the results of measurements always come out to be real numbers. For example, we may describe an electron as having non-real momentum in classically forbidden regions, but its average momentum will always come out to be real (the imaginary parts average out to zero), and it can never transfer a non-real quantity of momentum to another particle.

A complete investigation of these issues is beyond the scope of this book, and this is why we have normally limited ourselves to standing waves, which can be described with real-valued wavefunctions.

Summary

Selected vocabulary

wavefunction . . . the numerical measure of an electron wave, or in general of the wave corresponding to any quantum mechanical particle

Notation

\hbar Planck's constant divided by 2π (used only in optional section 35.6)

Ψ the wavefunction of an electron

Summary

Light is both a particle and a wave. Matter is both a particle and a wave. The equations that connect the particle and wave properties are the same in all cases:

$$E = hf$$
$$p = h/\lambda$$

Unlike the electric and magnetic fields that make up a photon-wave, the electron wavefunction is not directly measurable. Only the square of the wavefunction, which relates to probability, has direct physical significance.

A particle that is bound within a certain region of space is a standing wave in terms of quantum physics. The two equations above can then be applied to the standing wave to yield some important general observations about bound particles:

1. The particle's energy is quantized (can only have certain values).
2. The particle has a minimum energy.
3. The smaller the space in which the particle is confined, the higher its kinetic energy must be.

These immediately resolve the difficulties that classical physics had encountered in explaining observations such as the discrete spectra of atoms, the fact that atoms don't collapse by radiating away their energy, and the formation of chemical bonds.

A standing wave confined to a small space must have a short wavelength, which corresponds to a large momentum in quantum physics. Since a standing wave consists of a superposition of two traveling waves moving in opposite directions, this large momentum should actually be interpreted as an equal mixture of two possible momenta: a large momentum to the left, or a large momentum to the right. Thus it is not possible for a quantum wave-particle to be confined to a small space without making its momentum very uncertain. In general, the Heisenberg uncertainty principle states that

it is not possible to know the position and momentum of a particle simultaneously with perfect accuracy. The uncertainties in these two quantities must satisfy the approximate inequality

$$\Delta p \Delta x \gtrsim h.$$

When an electron is subjected to electric forces, its wavelength cannot be constant. The “wavelength” to be used in the equation $p = h/\lambda$ should be thought of as the wavelength of the sine wave that most closely approximates the curvature of the wavefunction at a specific point.

Infinite curvature is not physically possible, so realistic wavefunctions cannot have kinks in them, and cannot just cut off abruptly at the edge of a region where the particle’s energy would be insufficient to penetrate according to classical physics. Instead, the wavefunction “tails off” in the classically forbidden region, and as a consequence it is possible for particles to “tunnel” through regions where according to classical physics they should not be able to penetrate. If this quantum tunneling effect did not exist, there would be no fusion reactions to power our sun, because the energies of the nuclei would be insufficient to overcome the electrical repulsion between them.

Exploring further

The New World of Mr. Tompkins: George Gamow’s Classic Mr. Tompkins in Paperback, George Gamow. Mr. Tompkins finds himself in a world where the speed of light is only 30 miles per hour, making relativistic effects obvious. Later parts of the book play similar games with Planck’s constant.

The First Three Minutes: A Modern View of the Origin of the Universe, Steven Weinberg. Surprisingly simple ideas allow us to understand the infancy of the universe surprisingly well.

Three Roads to Quantum Gravity, Lee Smolin. The greatest embarrassment of physics today is that we are unable to fully reconcile general relativity (the theory of gravity) with quantum mechanics. This book does a good job of introducing the lay reader to a difficult, speculative subject, and showing that even though we don’t have a full theory of quantum gravity, we do have a clear outline of what such a theory must look like.

Problems

Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.

1 In a television, suppose the electrons are accelerated from rest through a voltage difference of 10^4 V. What is their final wavelength?
✓

2 Use the Heisenberg uncertainty principle to estimate the minimum velocity of a proton or neutron in a ^{208}Pb nucleus, which has a diameter of about 13 fm ($1 \text{ fm} = 10^{-15} \text{ m}$). Assume that the speed is nonrelativistic, and then check at the end whether this assumption was warranted.
✓

3 A free electron that contributes to the current in an ohmic material typically has a speed of 10^5 m/s (much greater than the drift velocity).

- (a) Estimate its de Broglie wavelength, in nm. ✓
- (b) If a computer memory chip contains 10^8 electric circuits in a 1 cm^2 area, estimate the linear size, in nm, of one such circuit. ✓
- (c) Based on your answers from parts a and b, does an electrical engineer designing such a chip need to worry about wave effects such as diffraction?
- (d) Estimate the maximum number of electric circuits that can fit on a 1 cm^2 computer chip before quantum-mechanical effects become important.

4 On page 998, I discussed the idea of hooking up a video camera to a visible-light microscope and recording the trajectory of an electron orbiting a nucleus. An electron in an atom typically has a speed of about 1% of the speed of light.

- (a) Calculate the momentum of the electron. ✓
- (b) When we make images with photons, we can't resolve details that are smaller than the photons' wavelength. Suppose we wanted to map out the trajectory of the electron with an accuracy of 0.01 nm. What part of the electromagnetic spectrum would we have to use?
- (c) As found in homework problem 12 on page 813, the momentum of a photon is given by $p = E/c$. Estimate the momentum of a photon having the necessary wavelength. ✓
- (d) Comparing your answers from parts a and c, what would be the effect on the electron if the photon bounced off of it? What does this tell you about the possibility of mapping out an electron's orbit around a nucleus?

5 Find the energy of a nonrelativistic particle in a one-dimensional box of length L , expressing your result in terms of L , the particle's mass m , the number of peaks and valleys n in the wavefunction, and fundamental constants. ✓

6 The Heisenberg uncertainty principle, $\Delta p \Delta x \gtrsim h$, can only be made into a strict inequality if we agree on a rigorous mathematical definition of Δx and Δp . Suppose we define the deltas in terms of the full width at half maximum (FWHM), which we first encountered on p. 481 and revisited on page 949 of this book. Now consider the lowest-energy state of the one-dimensional particle in a box. As argued on page 999, the momentum has equal probability of being h/L or $-h/L$, so the FWHM definition gives $\Delta p = 2h/L$.

- (a) Find Δx using the FWHM definition. Keep in mind that the probability distribution depends on the square of the wavefunction.
 (b) Find $\Delta x \Delta p$. ✓

7 If x has an average value of zero, then the standard deviation of the probability distribution $D(x)$ is defined by

$$\sigma^2 = \sqrt{\int D(x)x^2 dx},$$

where the integral ranges over all possible values of x .

Interpretation: if x only has a high probability of having values close to the average (i.e., small positive and negative values), the thing being integrated will always be small, because x^2 is always a small number; the standard deviation will therefore be small. Squaring x makes sure that either a number below the average ($x < 0$) or a number above the average ($x > 0$) will contribute a positive amount to the standard deviation. We take the square root of the whole thing so that it will have the same units as x , rather than having units of x^2 .

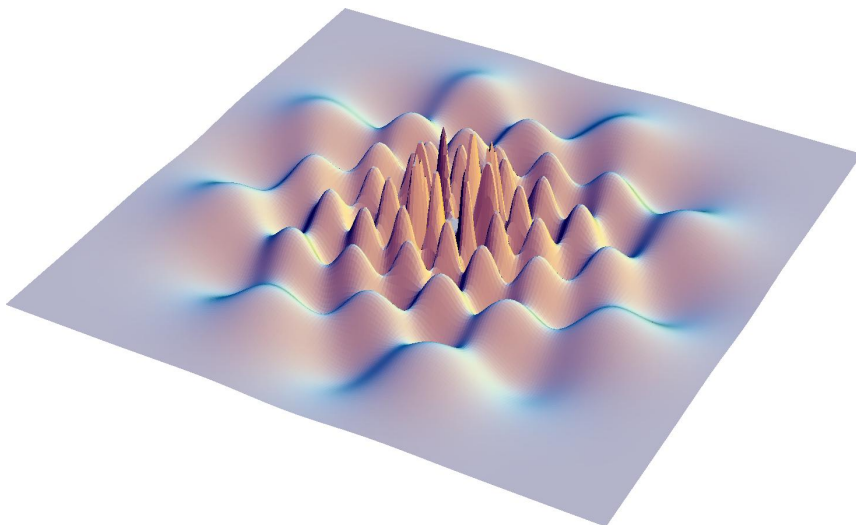
Redo problem 6 using the standard deviation rather than the FWHM.

Hints: (1) You need to determine the amplitude of the wave based on normalization. (2) You'll need the following definite integral:

$$\int_{-\pi/2}^{\pi/2} u^2 \cos^2 u du = (\pi^3 - 6\pi)/24. \quad \checkmark \quad \int$$

8 In section 35.6 we derived an expression for the probability that a particle would tunnel through a rectangular barrier, i.e., a region in which the interaction energy $U(x)$ has a graph that looks like a rectangle. Generalize this to a barrier of any shape. [Hints: First try generalizing to two rectangular barriers in a row, and then use a series of rectangular barriers to approximate the actual curve of an arbitrary function $U(x)$. Note that the width and height of the barrier in the original equation occur in such a way that all that matters is the area under the U -versus- x curve. Show that this is still true for a series of rectangular barriers, and generalize using an integral.] If you had done this calculation in the 1930's you could have become a famous physicist. ✓ ★

9 The electron, proton, and neutron were discovered, respectively, in 1897, 1919, and 1932. The neutron was late to the party, and some physicists felt that it was unnecessary to consider it as fundamental. Maybe it could be explained as simply a proton with an electron trapped inside it. The charges would cancel out, giving the composite particle the correct neutral charge, and the masses at least approximately made sense (a neutron is heavier than a proton). (a) Given that the diameter of a proton is on the order of 10^{-15} m, use the Heisenberg uncertainty principle to estimate the trapped electron's minimum momentum. ✓
(b) Find the electron's minimum kinetic energy. ✓
(c) Show via $E = mc^2$ that the proposed explanation may have a problem, because the contribution to the neutron's mass from the electron's kinetic energy would be comparable to the neutron's entire mass.



A wavefunction of an electron in a hydrogen atom.

Chapter 36

The Atom

You can learn a lot by taking a car engine apart, but you will have learned a lot more if you can put it all back together again and make it run. Half the job of reductionism is to break nature down into its smallest parts and understand the rules those parts obey. The second half is to show how those parts go together, and that is our goal in this chapter. We have seen how certain features of all atoms can be explained on a generic basis in terms of the properties of bound states, but this kind of argument clearly cannot tell us any details of the behavior of an atom or explain why one atom acts differently from another.

The biggest embarrassment for reductionists is that the job of putting things back together is usually much harder than the taking them apart. Seventy years after the fundamentals of atomic physics were solved, it is only beginning to be possible to calculate accurately the properties of atoms that have many electrons. Systems consisting of many atoms are even harder. Supercomputer manufacturers point to the folding of large protein molecules as a process whose calculation is just barely feasible with their fastest machines. The goal of this chapter is to give a gentle and visually oriented guide to some of the simpler results about atoms.

36.1 Classifying states

We'll focus our attention first on the simplest atom, hydrogen, with one proton and one electron. We know in advance a little of what we should expect for the structure of this atom. Since the electron is bound to the proton by electrical forces, it should display a set of discrete energy states, each corresponding to a certain standing wave pattern. We need to understand what states there are and what their properties are.

What properties should we use to classify the states? The most sensible approach is to use conserved quantities. Energy is one conserved quantity, and we already know to expect each state to have a specific energy. It turns out, however, that energy alone is not sufficient. Different standing wave patterns of the atom can have the same energy.

Momentum is also a conserved quantity, but it is not particularly appropriate for classifying the states of the electron in a hydrogen atom. The reason is that the force between the electron and the proton results in the continual exchange of momentum between them. (Why wasn't this a problem for energy as well? Kinetic energy and momentum are related by $KE = p^2/2m$, so the much more massive proton never has very much kinetic energy. We are making an approximation by assuming all the kinetic energy is in the electron, but it is quite a good approximation.)

Angular momentum does help with classification. There is no transfer of angular momentum between the proton and the electron, since the force between them is a center-to-center force, producing no torque.

Like energy, angular momentum is quantized in quantum physics. As an example, consider a quantum wave-particle confined to a circle, like a wave in a circular moat surrounding a castle. A sine wave in such a "quantum moat" cannot have any old wavelength, because an integer number of wavelengths must fit around the circumference, C , of the moat. The larger this integer is, the shorter the wavelength, and a shorter wavelength relates to greater momentum and angular momentum. Since this integer is related to angular momentum, we use the symbol ℓ for it:

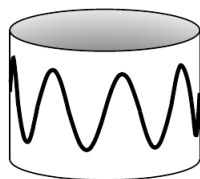
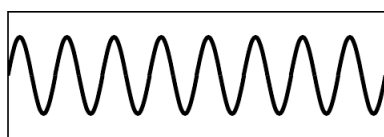
$$\lambda = \frac{C}{\ell}$$

The angular momentum is

$$L = rp.$$

Here, $r = C/2\pi$, and $p = h/\lambda = h\ell/C$, so

$$\begin{aligned} L &= \frac{C}{2\pi} \cdot \frac{h\ell}{C} \\ &= \frac{h}{2\pi} \ell \end{aligned}$$



a / Eight wavelengths fit around this circle: $\ell = 8$.

In the example of the quantum moat, angular momentum is quantized in units of $h/2\pi$, and this turns out to be a completely general fact about quantum physics. That makes $h/2\pi$ a pretty important number, so we define the abbreviation $\hbar = h/2\pi$. This symbol is read “h-bar.”

quantization of angular momentum

The angular momentum of a particle due to its motion through space is quantized in units of \hbar .

self-check A

What is the angular momentum of the wavefunction shown on page 1011? ▷ Answer, p. 1047

36.2 Angular momentum in three dimensions

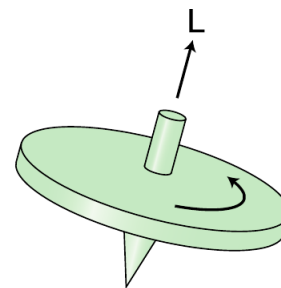
Up until now we’ve only worked with angular momentum in the context of rotation in a plane, for which we could simply use positive and negative signs to indicate clockwise and counterclockwise directions of rotation. A hydrogen atom, however, is unavoidably three-dimensional. Let’s first consider the generalization of angular momentum to three dimensions in the classical case, and then consider how it carries over into quantum physics.

Three-dimensional angular momentum in classical physics

If we are to completely specify the angular momentum of a classical object like a top, b , in three dimensions, it’s not enough to say whether the rotation is clockwise or counterclockwise. We must also give the orientation of the plane of rotation or, equivalently, the direction of the top’s axis. The convention is to specify the direction of the axis. There are two possible directions along the axis, and as a matter of convention we use the direction such that if we sight along it, the rotation appears clockwise.

Angular momentum can, in fact, be defined as a vector pointing along this direction. This might seem like a strange definition, since nothing actually moves in that direction, but it wouldn’t make sense to define the angular momentum vector as being in the direction of motion, because every part of the top has a different direction of motion. Ultimately it’s not just a matter of picking a definition that is convenient and unambiguous: the definition we’re using is the only one that makes the total angular momentum of a system a conserved quantity if we let “total” mean the vector sum.

As with rotation in one dimension, we cannot define what we mean by angular momentum in a particular situation unless we pick a point as an axis. This is really a different use of the word “axis” than the one in the previous paragraphs. Here we simply mean a



b / The angular momentum vector of a spinning top.

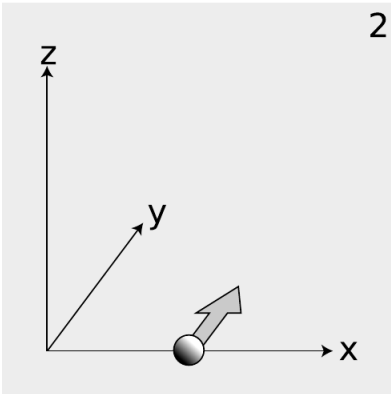
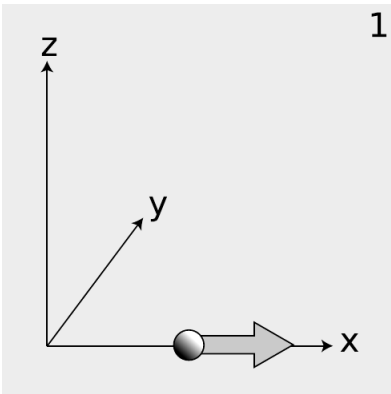
point from which we measure the distance r . In the hydrogen atom, the nearly immobile proton provides a natural choice of axis.

Three-dimensional angular momentum in quantum physics

Once we start to think more carefully about the role of angular momentum in quantum physics, it may seem that there is a basic problem: the angular momentum of the electron in a hydrogen atom depends on both its distance from the proton and its momentum, so in order to know its angular momentum precisely it would seem we would need to know both its position and its momentum simultaneously with good accuracy. This, however, might seem to be forbidden by the Heisenberg uncertainty principle.

Actually the uncertainty principle does place limits on what can be known about a particle's angular momentum vector, but it does not prevent us from knowing its magnitude as an exact integer multiple of \hbar . The reason is that in three dimensions, there are really three separate uncertainty principles:

$$\begin{aligned}\Delta p_x \Delta x &\gtrsim \hbar \\ \Delta p_y \Delta y &\gtrsim \hbar \\ \Delta p_z \Delta z &\gtrsim \hbar\end{aligned}$$



Now consider a particle, $c/1$, that is moving along the x axis at position x and with momentum p_x . We may not be able to know both x and p_x with unlimited accuracy, but we can still know the particle's angular momentum about the origin exactly. Classically, it is zero, because the particle is moving directly away from the origin: if it was to be nonzero, we would need both a nonzero x and a nonzero p_y . In quantum terms, the uncertainty principle does not place any constraint on $\Delta x \Delta p_y$.

Suppose, on the other hand, a particle finds itself, as in figure $c/2$, at a position x along the x axis, and it is moving parallel to the y axis with momentum p_y . It has angular momentum $x p_y$ about the z axis, and again we can know its angular momentum with unlimited accuracy, because the uncertainty principle only relates x to p_x and y to p_y . It does not relate x to p_y .

As shown by these examples, the uncertainty principle does not restrict the accuracy of our knowledge of angular momenta as severely as might be imagined. However, it does prevent us from knowing all three components of an angular momentum vector simultaneously. The most general statement about this is the following theorem, which we present without proof:

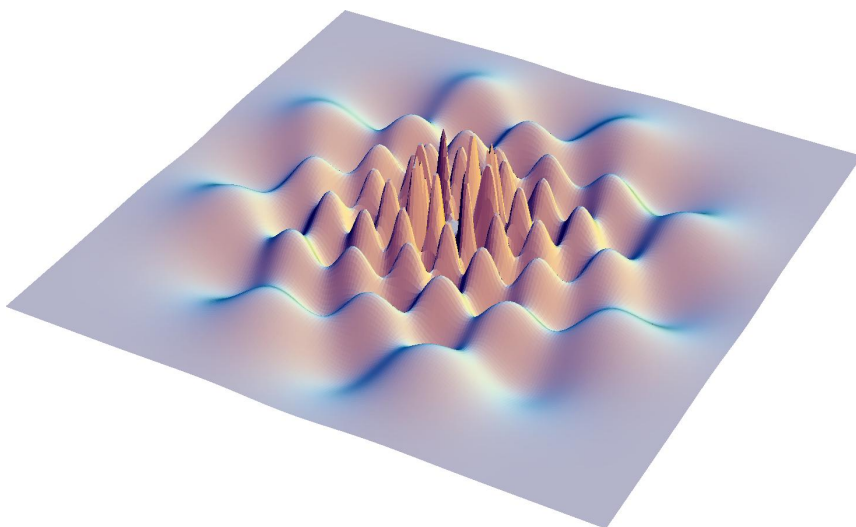
- $c/1$. This particle is moving directly away from the axis, and has no angular momentum.
 2. This particle has angular momentum.

the angular momentum vector in quantum physics

The most that can be known about an angular momentum vector is its magnitude and one of its three vector components. Both are quantized in units of \hbar .

36.3 The hydrogen atom

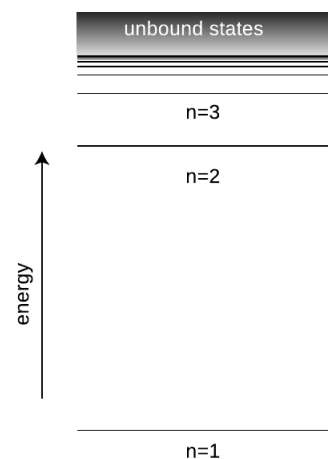
Deriving the wavefunctions of the states of the hydrogen atom from first principles would be mathematically too complex for this book, but it's not hard to understand the logic behind such a wavefunction in visual terms. Consider the wavefunction from the beginning of the chapter, which is reproduced below. Although the graph looks three-dimensional, it is really only a representation of the part of the wavefunction lying within a two-dimensional plane. The third (up-down) dimension of the plot represents the value of the wavefunction at a given point, not the third dimension of space. The plane chosen for the graph is the one perpendicular to the angular momentum vector.



Each ring of peaks and valleys has eight wavelengths going around in a circle, so this state has $L = 8\hbar$, i.e., we label it $\ell = 8$. The wavelength is shorter near the center, and this makes sense because when the electron is close to the nucleus it has a lower PE, a higher KE, and a higher momentum.

Between each ring of peaks in this wavefunction is a nodal circle, i.e., a circle on which the wavefunction is zero. The full three-dimensional wavefunction has nodal spheres: a series of nested spherical surfaces on which it is zero. The number of radii at which nodes occur, including $r = \infty$, is called n , and n turns out to be closely

d / A wavefunction of a hydrogen atom.



e / The energy of a state in the hydrogen atom depends only on its n quantum number.

related to energy. The ground state has $n = 1$ (a single node only at $r = \infty$), and higher-energy states have higher n values. There is a simple equation relating n to energy, which we will discuss in section 36.4.

The numbers n and ℓ , which identify the state, are called its quantum numbers. A state of a given n and ℓ can be oriented in a variety of directions in space. We might try to indicate the orientation using the three quantum numbers $\ell_x = L_x/\hbar$, $\ell_y = L_y/\hbar$, and $\ell_z = L_z/\hbar$. But we have already seen that it is impossible to know all three of these simultaneously. To give the most complete possible description of a state, we choose an arbitrary axis, say the z axis, and label the state according to n , ℓ , and ℓ_z .

Angular momentum requires motion, and motion implies kinetic energy. Thus it is not possible to have a given amount of angular momentum without having a certain amount of kinetic energy as well. Since energy relates to the n quantum number, this means that for a given n value there will be a maximum possible ℓ . It turns out that this maximum value of ℓ equals $n - 1$.

In general, we can list the possible combinations of quantum numbers as follows:

n can equal 1, 2, 3, ...
ℓ can range from 0 to $n - 1$, in steps of 1
ℓ_z can range from $-\ell$ to ℓ , in steps of 1

Applying these rules, we have the following list of states:

$n = 1, \ell = 0, \ell_z = 0$	one state
$n = 2, \ell = 0, \ell_z = 0$	one state
$n = 2, \ell = 1, \ell_z = -1, 0, \text{ or } 1$	three states
...	...

self-check B

Continue the list for $n = 3$.

▷ Answer, p. 1047

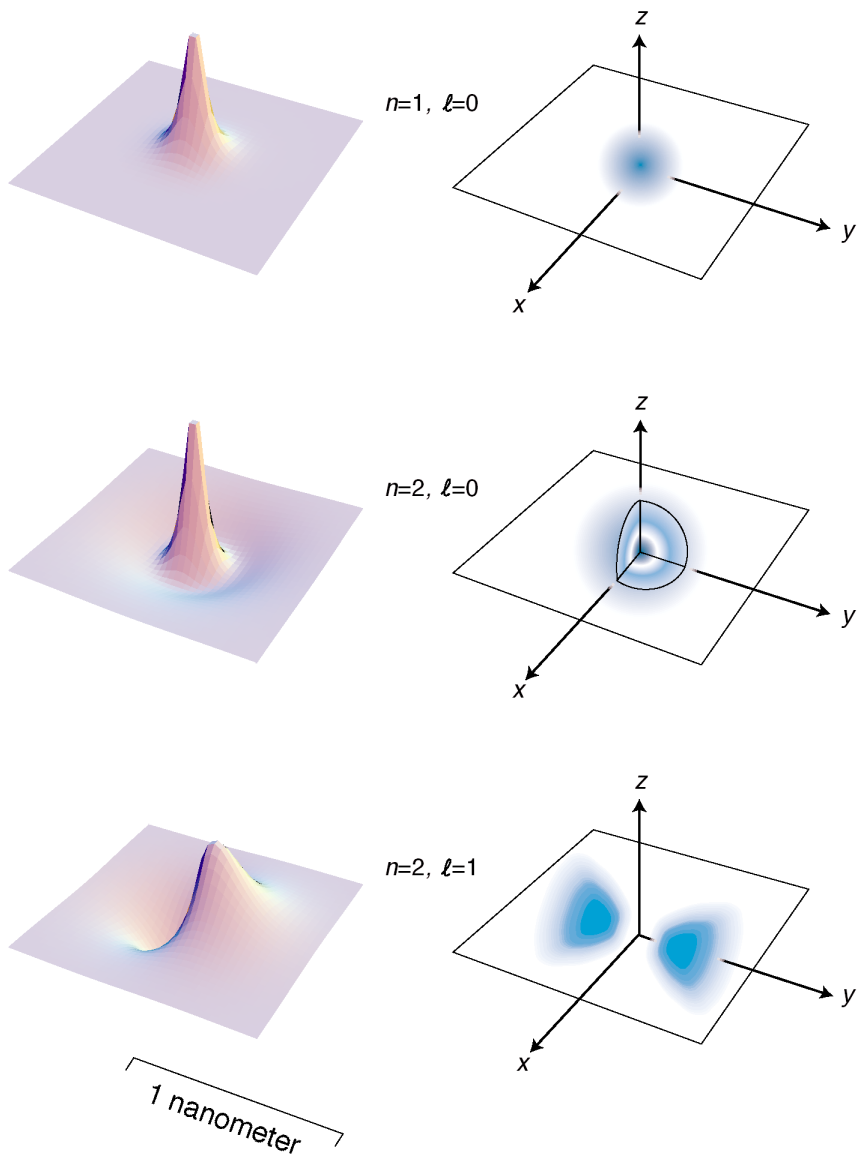
Figure f shows the lowest-energy states of the hydrogen atom. The left-hand column of graphs displays the wavefunctions in the $x - y$ plane, and the right-hand column shows the probability distribution in a three-dimensional representation.

Discussion questions

A The quantum number n is defined as the number of radii at which the wavefunction is zero, including $r = \infty$. Relate this to the features of the figures on the facing page.

B Based on the definition of n , why can't there be any such thing as an $n = 0$ state?

C Relate the features of the wavefunction plots in figure f to the corresponding features of the probability distribution pictures.



f / The three lowest-energy states of hydrogen.

D How can you tell from the wavefunction plots in figure f which ones have which angular momenta?

E Criticize the following incorrect statement: “The $\ell = 8$ wavefunction in figure d has a shorter wavelength in the center because in the center the electron is in a higher energy level.”

F Discuss the implications of the fact that the probability cloud in of the $n = 2, \ell = 1$ state is split into two parts.

36.4 ★ Energies of states in hydrogen

History

The experimental technique for measuring the energy levels of an atom accurately is spectroscopy: the study of the spectrum of light emitted (or absorbed) by the atom. Only photons with certain energies can be emitted or absorbed by a hydrogen atom, for example, since the amount of energy gained or lost by the atom must equal the difference in energy between the atom's initial and final states. Spectroscopy had become a highly developed art several decades before Einstein even proposed the photon, and the Swiss spectroscopist Johann Balmer determined in 1885 that there was a simple equation that gave all the wavelengths emitted by hydrogen. In modern terms, we think of the photon wavelengths merely as indirect evidence about the underlying energy levels of the atom, and we rework Balmer's result into an equation for these atomic energy levels:

$$E_n = -\frac{2.2 \times 10^{-18} \text{ J}}{n^2},$$

This energy includes both the kinetic energy of the electron and the electrical energy. The zero-level of the electrical energy scale is chosen to be the energy of an electron and a proton that are infinitely far apart. With this choice, negative energies correspond to bound states and positive energies to unbound ones.

Where does the mysterious numerical factor of $2.2 \times 10^{-18} \text{ J}$ come from? In 1913 the Danish theorist Niels Bohr realized that it was exactly numerically equal to a certain combination of fundamental physical constants:

$$E_n = -\frac{mk^2e^4}{2\hbar^2} \cdot \frac{1}{n^2},$$

where m is the mass of the electron, and k is the Coulomb force constant for electric forces.

Bohr was able to cook up a derivation of this equation based on the incomplete version of quantum physics that had been developed by that time, but his derivation is today mainly of historical interest. It assumes that the electron follows a circular path, whereas the whole concept of a path for a particle is considered meaningless in our more complete modern version of quantum physics. Although Bohr was able to produce the right equation for the energy levels, his model also gave various wrong results, such as predicting that the atom would be flat, and that the ground state would have $\ell = 1$ rather than the correct $\ell = 0$.

Approximate treatment

A full and correct treatment is impossible at the mathematical level of this book, but we can provide a straightforward explanation for the form of the equation using approximate arguments.

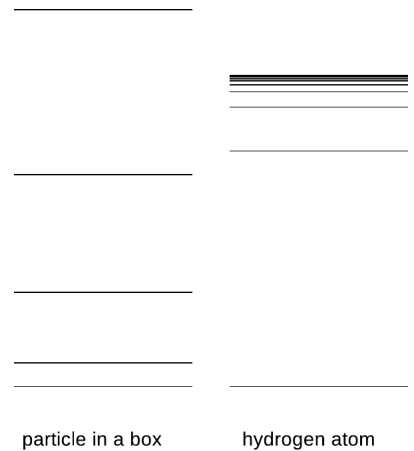
A typical standing-wave pattern for the electron consists of a central oscillating area surrounded by a region in which the wavefunction tails off. As discussed in section 35.6, the oscillating type of pattern is typically encountered in the classically allowed region, while the tailing off occurs in the classically forbidden region where the electron has insufficient kinetic energy to penetrate according to classical physics. We use the symbol r for the radius of the spherical boundary between the classically allowed and classically forbidden regions. Classically, r would be the distance from the proton at which the electron would have to stop, turn around, and head back in.

If r had the same value for every standing-wave pattern, then we'd essentially be solving the particle-in-a-box problem in three dimensions, with the box being a spherical cavity. Consider the energy levels of the particle in a box compared to those of the hydrogen atom, g. They're qualitatively different. The energy levels of the particle in a box get farther and farther apart as we go higher in energy, and this feature doesn't even depend on the details of whether the box is two-dimensional or three-dimensional, or its exact shape. The reason for the spreading is that the box is taken to be completely impenetrable, so its size, r , is fixed. A wave pattern with n humps has a wavelength proportional to r/n , and therefore a momentum proportional to n , and an energy proportional to n^2 . In the hydrogen atom, however, the force keeping the electron bound isn't an infinite force encountered when it bounces off of a wall, it's the attractive electrical force from the nucleus. If we put more energy into the electron, it's like throwing a ball upward with a higher energy — it will get farther out before coming back down. This means that in the hydrogen atom, we expect r to increase as we go to states of higher energy. This tends to keep the wavelengths of the high energy states from getting too short, reducing their kinetic energy. The closer and closer crowding of the energy levels in hydrogen also makes sense because we know that there is a certain energy that would be enough to make the electron escape completely, and therefore the sequence of bound states cannot extend above that energy.

When the electron is at the maximum classically allowed distance r from the proton, it has zero kinetic energy. Thus when the electron is at distance r , its energy is purely electrical:

$$[1] \quad E = -\frac{ke^2}{r}$$

Now comes the approximation. In reality, the electron's wavelength cannot be constant in the classically allowed region, but we pretend that it is. Since n is the number of nodes in the wavefunction, we can interpret it approximately as the number of wavelengths that fit across the diameter $2r$. We are not even attempting a derivation



g / The energy levels of a particle in a box, contrasted with those of the hydrogen atom.

that would produce all the correct numerical factors like 2 and π and so on, so we simply make the approximation

$$[2] \quad \lambda \sim \frac{r}{n}.$$

Finally we assume that the typical kinetic energy of the electron is on the same order of magnitude as the absolute value of its total energy. (This is true to within a factor of two for a typical classical system like a planet in a circular orbit around the sun.) We then have

$$[3] \quad \begin{aligned} &\text{absolute value of total energy} \\ &= \frac{ke^2}{r} \\ &\sim K \\ &= p^2/2m \\ &= (h/\lambda)^2/2m \\ &\sim h^2n^2/2mr^2 \end{aligned}$$

We now solve the equation $ke^2/r \sim h^2n^2/2mr^2$ for r and throw away numerical factors we can't hope to have gotten right, yielding

$$[4] \quad r \sim \frac{h^2n^2}{mke^2}.$$

Plugging $n = 1$ into this equation gives $r = 2$ nm, which is indeed on the right order of magnitude. Finally we combine equations [4] and [1] to find

$$E \sim -\frac{mk^2e^4}{h^2n^2},$$

which is correct except for the numerical factors we never aimed to find.

Discussion questions

A States of hydrogen with n greater than about 10 are never observed in the sun. Why might this be?

B Sketch graphs of r and E versus n for the hydrogen atom, and compare with analogous graphs for the one-dimensional particle in a box.

36.5 Electron spin

It's disconcerting to the novice ping-pong player to encounter for the first time a more skilled player who can put spin on the ball. Even though you can't see that the ball is spinning, you can tell something is going on by the way it interacts with other objects in its environment. In the same way, we can tell from the way electrons interact with other things that they have an intrinsic spin of their own. Experiments show that even when an electron is not moving through space, it still has angular momentum amounting to $\hbar/2$.

This may seem paradoxical because the quantum moat, for instance, gave only angular momenta that were integer multiples of \hbar , not half-units, and I claimed that angular momentum was always quantized in units of \hbar , not just in the case of the quantum moat. That whole discussion, however, assumed that the angular momentum would come from the motion of a particle through space. The $\hbar/2$ angular momentum of the electron is simply a property of the particle, like its charge or its mass. It has nothing to do with whether the electron is moving or not, and it does not come from any internal motion within the electron. Nobody has ever succeeded in finding any internal structure inside the electron, and even if there was internal structure, it would be mathematically impossible for it to result in a half-unit of angular momentum.

We simply have to accept this $\hbar/2$ angular momentum, called the “spin” of the electron — Mother Nature rubs our noses in it as an observed fact. Protons and neutrons have the same $\hbar/2$ spin, while photons have an intrinsic spin of \hbar . In general, half-integer spins are typical of material particles. Integral values are found for the particles that carry forces: photons, which embody the electric and magnetic fields of force, as well as the more exotic messengers of the nuclear and gravitational forces. The photon is particularly important: it has spin 1.

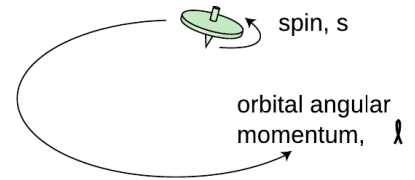
As was the case with ordinary angular momentum, we can describe spin angular momentum in terms of its magnitude, and its component along a given axis. We notate these quantities, in units of \hbar , as s and s_z , so an electron has $s = 1/2$ and $s_z = +1/2$ or $-1/2$.

Taking electron spin into account, we need a total of four quantum numbers to label a state of an electron in the hydrogen atom: n , ℓ , ℓ_z , and s_z . (We omit s because it always has the same value.) The symbols ℓ and ℓ_z include only the angular momentum the electron has because it is moving through space, not its spin angular momentum. The availability of two possible spin states of the electron leads to a doubling of the numbers of states:

$n = 1,$	$\ell = 0,$	$\ell_z = 0,$	$s_z = +1/2$ or $-1/2$	two states
$n = 2,$	$\ell = 0,$	$\ell_z = 0,$	$s_z = +1/2$ or $-1/2$	two states
$n = 2,$	$\ell = 1,$	$\ell_z = -1, 0,$ or $1,$	$s_z = +1/2$ or $-1/2$	six states
...				...

36.6 Atoms with more than one electron

What about other atoms besides hydrogen? It would seem that things would get much more complex with the addition of a second electron. A hydrogen atom only has one particle that moves around much, since the nucleus is so heavy and nearly immobile. Helium, with two, would be a mess. Instead of a wavefunction whose square tells us the probability of finding a single electron at any given lo-



\hbar / The top has angular momentum both because of the motion of its center of mass through space and due to its internal rotation. Electron spin is roughly analogous to the intrinsic spin of the top.

cation in space, a helium atom would need to have a wavefunction whose square would tell us the probability of finding two electrons at any given combination of points. Ouch! In addition, we would have the extra complication of the electrical interaction between the two electrons, rather than being able to imagine everything in terms of an electron moving in a static field of force created by the nucleus alone.

Despite all this, it turns out that we can get a surprisingly good description of many-electron atoms simply by assuming the electrons can occupy the same standing-wave patterns that exist in a hydrogen atom. The ground state of helium, for example, would have both electrons in states that are very similar to the $n = 1$ states of hydrogen. The second-lowest-energy state of helium would have one electron in an $n = 1$ state, and the other in an $n = 2$ state. The relatively complex spectra of elements heavier than hydrogen can be understood as arising from the great number of possible combinations of states for the electrons.

A surprising thing happens, however, with lithium, the three-electron atom. We would expect the ground state of this atom to be one in which all three electrons settle down into $n = 1$ states. What really happens is that two electrons go into $n = 1$ states, but the third stays up in an $n = 2$ state. This is a consequence of a new principle of physics:

the Pauli exclusion principle

Only one electron can ever occupy a given state.

There are two $n = 1$ states, one with $s_z = +1/2$ and one with $s_z = -1/2$, but there is no third $n = 1$ state for lithium's third electron to occupy, so it is forced to go into an $n = 2$ state.

It can be proved mathematically that the Pauli exclusion principle applies to any type of particle that has half-integer spin. Thus two neutrons can never occupy the same state, and likewise for two protons. Photons, however, are immune to the exclusion principle because their spin is an integer. Material objects can't pass through each other, but beams of light can. With a little oversimplification, we can say that the basic reason is that the exclusion principle applies to one but not to the other.¹

Photons are made out electric and magnetic fields, which are directly measurable, but the wavefunction of a spin-1/2 particle is not observable (p. 991). The exclusion principle offers a more fundamental explanation of this difference between light and matter. We saw in example 2 on p. 971 that in a typical light wave, a huge num-

¹There are also electrical forces between atoms, but as argued on page ??, the attractions and repulsions tend to cancel out.

ber of photons overlap one another within a volume of one cubic wavelength, and this is what allows us to measure the amplitude and phase of the wave with a device like an antenna. But for electrons, the exclusion principle prevents us from having more than *one* particle in such a volume, so we can't perform this type of classical measurement of the wave.

Deriving the periodic table

We can now account for the structure of the periodic table, which seemed so mysterious even to its inventor Mendeleev. The first row consists of atoms with electrons only in the $n = 1$ states:

- H 1 electron in an $n = 1$ state
- He 2 electrons in the two $n = 1$ states

The next row is built by filling the $n = 2$ energy levels:

- Li 2 electrons in $n = 1$ states, 1 electron in an $n = 2$ state
- Be 2 electrons in $n = 1$ states, 2 electrons in $n = 2$ states
- ...
- O 2 electrons in $n = 1$ states, 6 electrons in $n = 2$ states
- F 2 electrons in $n = 1$ states, 7 electrons in $n = 2$ states
- Ne 2 electrons in $n = 1$ states, 8 electrons in $n = 2$ states

In the third row we start in on the $n = 3$ levels:

- Na 2 electrons in $n = 1$ states, 8 electrons in $n = 2$ states, 1 electron in an $n = 3$ state
- ...

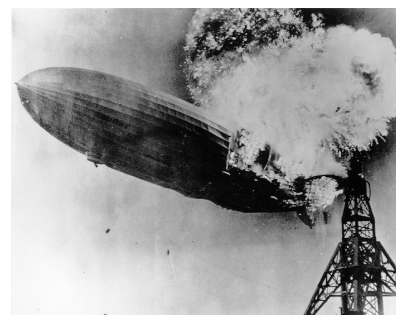
We can now see a logical link between the filling of the energy levels and the structure of the periodic table. Column 0, for example, consists of atoms with the right number of electrons to fill all the available states up to a certain value of n . Column I contains atoms like lithium that have just one electron more than that.

This shows that the columns relate to the filling of energy levels, but why does that have anything to do with chemistry? Why, for example, are the elements in columns I and VII dangerously reactive? Consider, for example, the element sodium (Na), which is so reactive that it may burst into flames when exposed to air. The electron in the $n = 3$ state has an unusually high energy. If we let a sodium atom come in contact with an oxygen atom, energy can be released by transferring the $n = 3$ electron from the sodium to one of the vacant lower-energy $n = 2$ states in the oxygen. This energy is transformed into heat. Any atom in column I is highly reactive for the same reason: it can release energy by giving away the electron that has an unusually high energy.

Column VII is spectacularly reactive for the opposite reason: these atoms have a single vacancy in a low-energy state, so energy is released when these atoms steal an electron from another atom.

I	II	III	IV	V	VI	VII	0
1 H							2 He
3 Li	4 Be	5 B	6 C	7 N	8 O	9 F	10 Ne
11 Na	...						

i / The beginning of the periodic table.



j / Hydrogen is highly reactive.

It might seem as though these arguments would only explain reactions of atoms that are in different rows of the periodic table, because only in these reactions can a transferred electron move from a higher- n state to a lower- n state. This is incorrect. An $n = 2$ electron in fluorine (F), for example, would have a different energy than an $n = 2$ electron in lithium (Li), due to the different number of protons and electrons with which it is interacting. Roughly speaking, the $n = 2$ electron in fluorine is more tightly bound (lower in energy) because of the larger number of protons attracting it. The effect of the increased number of attracting protons is only partly counteracted by the increase in the number of repelling electrons, because the forces exerted on an electron by the other electrons are in many different directions and cancel out partially.

Neutron stars

example 1

Here's an exotic example that doesn't even involve atoms. When a star runs out of fuel for its nuclear reactions, it begins to collapse under its own weight. Since Newton's law of gravity depends on the inverse square of the distance, the gravitational forces become stronger as the star collapses, which encourages it to collapse even further. The final result depends on the mass of the star, but let's consider a star that's only a little more massive than our own sun. Such a star will collapse to the point where the gravitational energy being released is sufficient to cause the reaction $p + e^- \rightarrow n + \nu$ to occur. (As you found in homework problem 10 on page 812, this reaction can only occur when there is some source of energy, because the mass-energy of the products is greater than the mass-energy of the things being consumed.) The neutrinos fly off and are never heard from again, so we're left with a star consisting only of neutrons!

Now the exclusion principle comes into play. The collapse can't continue indefinitely. The situation is in fact closely analogous to that of an atom. A lead atom's cloud of 82 electrons can't shrink down to the size of a hydrogen atom, because only two electrons can have the lowest-energy wave pattern. The same happens with the neutron star. No physical repulsion keeps the neutrons apart. They're electrically neutral, so they don't repel or attract one another electrically. The gravitational force is attractive, and as the collapse proceeds to the point where the neutrons come within range of the strong nuclear force ($\sim 10^{-15}$ m), we even start getting nuclear attraction. The only thing that stops the whole process is the exclusion principle. The star ends up being only a few kilometers across, and has the same billion-ton-per-teaspoon density as an atomic nucleus. Indeed, we can think of it as one big nucleus (with atomic number zero, because there are no protons).

As with a spinning figure skater pulling her arms in, conservation of angular momentum makes the star spin faster and faster. The

whole object may end up with a rotational period of a fraction of a second! Such a star sends out radio pulses with each revolution, like a sort of lighthouse. The first time such a signal was detected, radio astronomers thought that it was a signal from aliens.

Summary

Selected vocabulary

quantum number	a numerical label used to classify a quantum state
spin	the built-in angular momentum possessed by a particle even when at rest

Notation

n	the number of radial nodes in the wavefunction, including the one at $r = \infty$
\hbar	$h/2\pi$
\mathbf{L}	the angular momentum vector of a particle, not including its spin
ℓ	the magnitude of the L vector, divided by \hbar
ℓ_z	the z component of the L vector, divided by \hbar ; this is the standard notation in nuclear physics, but not in atomic physics
s	the magnitude of the spin angular momentum vector, divided by \hbar
s_z	the z component of the spin angular momentum vector, divided by \hbar ; this is the standard notation in nuclear physics, but not in atomic physics

Other terminology and notation

m_ℓ	a less obvious notation for ℓ_z , standard in atomic physics
m_s	a less obvious notation for s_z , standard in atomic physics

Summary

Hydrogen, with one proton and one electron, is the simplest atom, and more complex atoms can often be analyzed to a reasonably good approximation by assuming their electrons occupy states that have the same structure as the hydrogen atom's. The electron in a hydrogen atom exchanges very little energy or angular momentum with the proton, so its energy and angular momentum are nearly constant, and can be used to classify its states. The energy of a hydrogen state depends only on its n quantum number.

In quantum physics, the angular momentum of a particle moving in a plane is quantized in units of \hbar . Atoms are three-dimensional, however, so the question naturally arises of how to deal with angular momentum in three dimensions. In three dimensions, angular momentum is a vector in the direction perpendicular to the plane of motion, such that the motion appears clockwise if viewed along the direction of the vector. Since angular momentum depends on both position and momentum, the Heisenberg uncertainty principle limits the accuracy with which one can know it. The most that can

be known about an angular momentum vector is its magnitude and one of its three vector components, both of which are quantized in units of \hbar .

In addition to the angular momentum that an electron carries by virtue of its motion through space, it possesses an intrinsic angular momentum with a magnitude of $\hbar/2$. Protons and neutrons also have spins of $\hbar/2$, while the photon has a spin equal to \hbar .

Particles with half-integer spin obey the Pauli exclusion principle: only one such particle can exist in a given state, i.e., with a given combination of quantum numbers.

We can enumerate the lowest-energy states of hydrogen as follows:

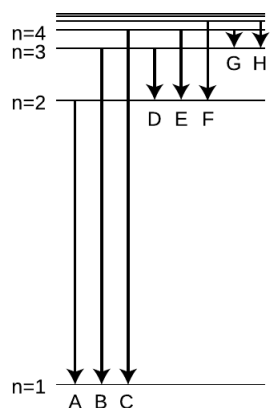
$n = 1,$	$\ell = 0,$	$l_z = 0,$	$s_z = +1/2$ or $-1/2$	two states
$n = 2,$	$\ell = 0,$	$l_z = 0,$	$s_z = +1/2$ or $-1/2$	two states
$n = 2,$	$\ell = 1,$	$l_z = -1, 0,$ or $1,$	$s_z = +1/2$ or $-1/2$	six states
...				...

The periodic table can be understood in terms of the filling of these states. The nonreactive noble gases are those atoms in which the electrons are exactly sufficient to fill all the states up to a given n value. The most reactive elements are those with one more electron than a noble gas element, which can release a great deal of energy by giving away their high-energy electron, and those with one electron fewer than a noble gas, which release energy by accepting an electron.

Problems

Key

- ✓ A computerized answer check is available online.
- ∫ A problem that requires calculus.
- ★ A difficult problem.



Problem 2.

1 (a) A distance scale is shown below the wavefunctions and probability densities illustrated in figure f on page 1017. Compare this with the order-of-magnitude estimate derived in section 36.4 for the radius r at which the wavefunction begins tailing off. Was the estimate in section 36.4 on the right order of magnitude?

(b) Although we normally say the moon orbits the earth, actually they both orbit around their common center of mass, which is below the earth's surface but not at its center. The same is true of the hydrogen atom. Does the center of mass lie inside the proton or outside it?

2 The figure shows eight of the possible ways in which an electron in a hydrogen atom could drop from a higher energy state to a state of lower energy, releasing the difference in energy as a photon. Of these eight transitions, only D, E, and F produce photons with wavelengths in the visible spectrum.

(a) Which of the visible transitions would be closest to the violet end of the spectrum, and which would be closest to the red end? Explain.

(b) In what part of the electromagnetic spectrum would the photons from transitions A, B, and C lie? What about G and H? Explain.

(c) Is there an upper limit to the wavelengths that could be emitted by a hydrogen atom going from one bound state to another bound state? Is there a lower limit? Explain.

3 Before the quantum theory, experimentalists noted that in many cases, they would find three lines in the spectrum of the same atom that satisfied the following mysterious rule: $1/\lambda_1 = 1/\lambda_2 + 1/\lambda_3$. Explain why this would occur. Do not use reasoning that only works for hydrogen — such combinations occur in the spectra of all elements. [Hint: Restate the equation in terms of the energies of photons.]

4 Find an equation for the wavelength of the photon emitted when the electron in a hydrogen atom makes a transition from energy level n_1 to level n_2 . [You will need to have read optional section 36.4.] ✓

5 Estimate the angular momentum of a spinning basketball, in units of \hbar . Explain how this result relates to the correspondence principle.

6 Assume that the kinetic energy of an electron in the $n = 1$ state of a hydrogen atom is on the same order of magnitude as the absolute value of its total energy, and estimate a typical speed at

which it would be moving. (It cannot really have a single, definite speed, because its kinetic and interaction energy trade off at different distances from the proton, but this is just a rough estimate of a typical speed.) Based on this speed, were we justified in assuming that the electron could be described nonrelativistically?

7 The wavefunction of the electron in the ground state of a hydrogen atom, shown in the top left of figure f on p. 1017, is

$$\Psi = \pi^{-1/2} a^{-3/2} e^{-r/a},$$

where r is the distance from the proton, and $a = \hbar^2/kme^2 = 5.3 \times 10^{-11}$ m is a constant that sets the size of the wave. The figure doesn't show the proton; let's take the proton to be a sphere with a radius of $b = 0.5$ fm.

(a) Reproduce figure f in a rough sketch, and indicate, relative to the size of your sketch, some idea of how big a and b are.

(b) Calculate symbolically, without plugging in numbers, the probability that at any moment, the electron is inside the proton. [Hint: Does it matter if you plug in $r = 0$ or $r = b$ in the equation for the wavefunction?] ✓

(c) Calculate the probability numerically. ✓

(d) Based on the equation for the wavefunction, is it valid to think of a hydrogen atom as having a finite size? Can a be interpreted as the size of the atom, beyond which there is nothing? Or is there any limit on how far the electron can be from the proton?

8 Use physical reasoning to explain how the equation for the energy levels of hydrogen,

$$E_n = -\frac{mk^2e^4}{2\hbar^2} \cdot \frac{1}{n^2},$$

should be generalized to the case of an atom with atomic number Z that has had all its electrons removed except for one. ★

9 This question requires that you read optional section 36.4. A muon is a subatomic particle that acts exactly like an electron except that its mass is 207 times greater. Muons can be created by cosmic rays, and it can happen that one of an atom's electrons is displaced by a muon, forming a muonic atom. If this happens to a hydrogen atom, the resulting system consists simply of a proton plus a muon.

(a) How would the size of a muonic hydrogen atom in its ground state compare with the size of the normal atom?

(b) If you were searching for muonic atoms in the sun or in the earth's atmosphere by spectroscopy, in what part of the electromagnetic spectrum would you expect to find the absorption lines?

10 Consider a classical model of the hydrogen atom in which the electron orbits the proton in a circle at constant speed. In this

model, the electron and proton can have no intrinsic spin. Using the result of problem 14 in ch. 24, show that in this model, the atom's magnetic dipole moment D_m is related to its angular momentum by $D_m = (-e/2m)L$, regardless of the details of the orbital motion. Assume that the magnetic field is the same as would be produced by a circular current loop, even though there is really only a single charged particle. [Although the model is quantum-mechanically incorrect, the result turns out to give the correct quantum mechanical value for the contribution to the atom's dipole moment coming from the electron's orbital motion. There are other contributions, however, arising from the intrinsic spins of the electron and proton.]

11 Hydrogen is the only element whose energy levels can be expressed exactly in an equation. Calculate the ratio λ_E/λ_F of the wavelengths of the transitions labeled E and F in problem 2 on p. 1028. Express your answer as an exact fraction, not a decimal approximation. In an experiment in which atomic wavelengths are being measured, this ratio provides a natural, stringent check on the precision of the results. ✓

Exercise 36: Quantum versus classical randomness

1. Imagine the classical version of the particle in a one-dimensional box. Suppose you insert the particle in the box and give it a known, predetermined energy, but a random initial position and a random direction of motion. You then pick a random later moment in time to see where it is. Sketch the resulting probability distribution by shading on top of a line segment. Does the probability distribution depend on energy?
2. Do similar sketches for the first few energy levels of the quantum mechanical particle in a box, and compare with 1.
3. Do the same thing as in 1, but for a classical hydrogen atom in two dimensions, which acts just like a miniature solar system. Assume you're always starting out with the same fixed values of energy and angular momentum, but a position and direction of motion that are otherwise random. Do this for $L = 0$, and compare with a real $L = 0$ probability distribution for the hydrogen atom.
4. Repeat 3 for a nonzero value of L , say $L = \hbar$.
5. Summarize: Are the classical probability distributions accurate? What qualitative features are possessed by the classical diagrams but not by the quantum mechanical ones, or vice-versa?

Hints for volume 2

Hints for chapter 21

Page 617, problem 8:

The force on the lithium ion is the vector sum of all the forces of all the quadrillions of sodium and chlorine atoms, which would obviously be too laborious to calculate. Nearly all of these forces, however, are canceled by a force from an ion on the opposite side of the lithium.

Page 619, problem 21:

The approach is similar to the one used for the other problem, but you want to work with voltage and electrical energy rather than force.

Hints for chapter 22

Page 663, problem 13:

Since we have $t \ll r$, the volume of the membrane is essentially the same as if it was unrolled and flattened out, and the field's magnitude is nearly constant.

Hints for chapter 24

Page 723, problem 3:

A stable system has low energy; energy would have to be added to change its configuration.

Hints for chapter 27

Page 833, problem 7:

Apply the equivalence principle.

Solutions to selected problems for volume 2

Solutions for chapter 21

Page 618, problem 10:

$$\Delta t = \Delta q / I = e / I = 0.160 \mu\text{s}$$

Page 619, problem 17:

It's much more practical to measure voltage differences. To measure a current, you have to break the circuit somewhere and insert the meter there, but it's not possible to disconnect the circuits sealed inside the board.

Page 622, problem 34:

In series, they give 11 k Ω . In parallel, they give $(1/1 \text{ k}\Omega + 1/10 \text{ k}\Omega)^{-1} = 0.9 \text{ k}\Omega$.

Page 622, problem 35:

The actual shape is irrelevant; all we care about is what's connected to what. Therefore, we can draw the circuit flattened into a plane. Every vertex of the tetrahedron is adjacent to every other vertex, so any two vertices to which we connect will give the same resistance. Picking two arbitrarily, we have this:



This is unfortunately a circuit that cannot be converted into parallel and series parts, and that's

what makes this a hard problem! However, we can recognize that by symmetry, there is zero current in the resistor marked with an asterisk. Eliminating this one, we recognize the whole arrangement as a triple parallel circuit consisting of resistances R , $2R$, and $2R$. The resulting resistance is $R/2$.

Solutions for chapter 22

Page 662, problem 4:

Let the square's sides be of length a . The field at the center is the vector sum of the fields that would have been produced individually by the three charges. Each of these individual fields is kq/r^2 , where $r_1 = a/\sqrt{2}$ for the two charges q_1 , and $r_2 = a/2$ for q_2 . Vector addition can be done by adding components. Let x be horizontal and y vertical. The y components cancel by symmetry. The sum of the x components is

$$E_x = \frac{kq_1}{r_1^2} \cos 45^\circ + \frac{kq_1}{r_1^2} \cos 45^\circ - \frac{kq_2}{r_2^2}.$$

Substituting $\cos 45^\circ = 1/\sqrt{2}$ and setting this whole expression equal to zero, we find $q_2/q_1 = 1/\sqrt{2}$.

Solutions for chapter 23

Page 695, problem 7:

To make the units make sense, we need to make sure that both sides of the \approx sign have the same units, and also that both terms on the right-hand side have the same units. Everything is unitless except for the second term on the right, so we add a factor of c^{-2} to fix it:

$$\gamma \approx 1 + \frac{v^2}{2c^2}.$$

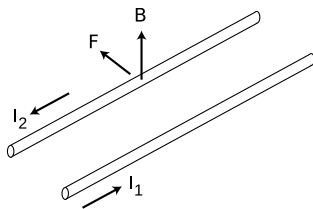
Solutions for chapter 24

Page 725, problem 11:

(a) Current means how much charge passes by a given point per unit time. During a time interval Δt , all the charge carriers in a certain region behind the point will pass by. This region has length $v\Delta t$ and cross-sectional area A , so its volume is $Av\Delta t$, and the amount of charge in it is $Avnq\Delta t$. To find the current, we divide this amount of charge by Δt , giving $I = Avnq$. (b) A segment of the wire of length L has a force QvB acting on it, where $Q = ALnq$ is the total charge of the moving charge carriers in that part of the wire. The force per unit length is $ALnqvB/L = AnqvB$. (c) Dividing the two results gives $F/L = IB$.

Page 725, problem 12:

(a) The figure shows the case where the currents are in opposite directions.



The field vector shown is one made by wire 1, which causes an effect on wire 2. It points up because wire 1's field pattern is clockwise as view from along the direction of current I_1 . For simplicity, let's assume that the current I_2 is made by positively charged particles moving in

the direction of the current. (You can check that the final result would be the same if they were negatively charged, as would actually be the case in a metal wire.) The force on one of these positively charged particles in wire 2 is supposed to have a direction such that when you sight along it, the B vector is clockwise from the v vector. This can only be accomplished if the force on the particle in wire 2 is in the direction shown. Wire 2 is repelled by wire 1.

To verify that wire 1 is also repelled by wire 2, we can either go through the same type of argument again, or we can simply apply Newton's third law.

Similar arguments show that the force is attractive if the currents are in the same direction.

(b) The force on wire 2 is $F/L = I_2 B$, where $B = 2kI_1/c^2 r$ is the field made by wire 1 and r is the distance between the wires. The result is

$$F/L = 2kI_1 I_2 / c^2 r.$$

Page 726, problem 16:

(a) Based on our knowledge of the field pattern of a current-carrying loop, we know that the magnetic field must be either into or out of the page. This makes sense, since that would mean the field is always perpendicular to the plane of the electrons' motion; if it was in their plane of motion, then the angle between the v and B vectors would be changing all the time, but we see no evidence of such behavior. With the field turned on, the force vector is apparently toward the center of the circle. Let's analyze the force at the moment when the electrons have started moving, which is at the right side of the circle. The force is to the left. Since the electrons are negatively charged particles, we know that if we sight along the force vector, the B vector must be counterclockwise from the v vector. The magnetic field must be out of the page. (b) Looking at figure f on page 701, we can tell that the current in the coils must be counterclockwise as viewed from the perspective of the camera. (c) Electrons are negatively charged, so to produce a counterclockwise current, the electrons in the coils must be going clockwise, i.e., they are counterrotating compared to the beam. (d) The current in the coils is keep the electrons in the beam from going straight, i.e. the force is a repulsion. This makes sense by comparison with figure w in section 23.2: like charges moving in opposite directions repel one another.

Page 727, problem 19:

The trick is to imagine putting together two identical solenoids to make one double-length solenoid. The field of the doubled solenoid is given by the vector sum of the two solenoids' individual fields. At points on the axis, symmetry guarantees that the individual fields lie along the axis, and similarly for the total field. At the center of one of the mouths, we thus have two parallel field vectors of equal strength, whose sum equals the interior field. But the interior field of the doubled solenoid is the same as that of the individual ones, since the equation for the field only depends on the number of turns per unit length. Therefore the field at the center of a solenoid's mouth equals exactly half the interior field.

Page 727, problem 21:

(a) Plugging in, we find

$$\sqrt{\frac{1-w}{1+w}} = \sqrt{\frac{1-u}{1+u}} \sqrt{\frac{1-v}{1+v}}.$$

(b) First let's simplify by squaring both sides.

$$\frac{1-w}{1+w} = \frac{1-u}{1+u} \cdot \frac{1-v}{1+v}.$$

For convenience, let's write A for the right-hand side of this equation. We then have

$$\frac{1-w}{1+w} = A$$

$$1-w = A + Aw.$$

Solving for w ,

$$w = \frac{1-A}{1+A}$$

$$= \frac{(1+u)(1+v) - (1-u)(1-v)}{(1+u)(1+v) + (1-u)(1-v)}$$

$$= \frac{2(u+v)}{2(1+uv)}$$

$$= \frac{u+v}{1+uv}$$

(c) This is all in units where $c = 1$. The correspondence principle says that we should get $w \approx u + v$ when both u and v are small compared to 1. Under those circumstances, uv is the product of two very small numbers, which makes it very, very small. Neglecting this term in the denominator, we recover the nonrelativistic result.

Solutions for chapter 26

Page 810, problem 2:

(a) In the reaction $p + e^- \rightarrow n + \nu$, the charges on the left are $e + (-e) = 0$, and both charges on the right are zero. (b) The neutrino has negligible mass. The masses on the left add up to 1.6736×10^{-27} kg, which is less than the 1.6750×10^{-27} kg mass of the neutron on the right, so energy would be required from an external source in order to make this reaction happen.

Page 811, problem 7:

(a) The change in PE is $e\Delta V$, so the KE gained is $(1/2)mv^2 = eV$. Solving for v , we get $v = \sqrt{2eV/m}$. (b) Plugging in numbers, the velocity is 5.9×10^7 m/s. This is about 20% of the speed of light. (Since it's not that close to the speed of light, we'll get a reasonably accurate answer without taking into account Einstein's theory of relativity.)

Solutions for chapter 27

Page 833, problem 1:

At the center of each of the large triangle's sides, the angles add up to 180° because they form a straight line. Therefore $4s = S + 3 \times 180^\circ$, so $S - 180^\circ = 4(s - 180^\circ)$.

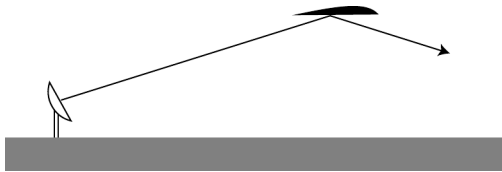
Page 833, problem 7:

By the equivalence principle, we can adopt a frame tied to the tossed clock, B, and in this frame there is no gravitational field. We see a desk and clock A go by. The desk applies a force to clock A, decelerating it and then reaccelerating it so that it comes back. We've already established that the effect of motion is to slow down time, so clock A reads a smaller time interval.

Solutions for chapter 28

Page 853, problem 4:

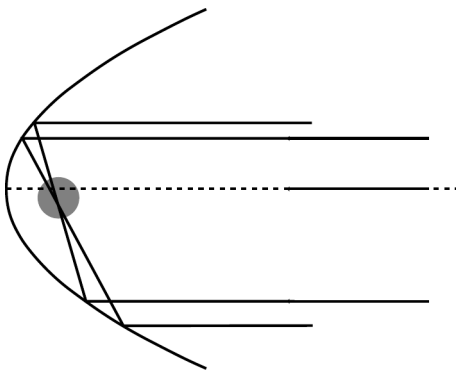
Because the surfaces are flat, you get specular reflection. In specular reflection, all the reflected rays go in one direction. Unless the plane is directly overhead, that direction won't be the right direction to make the rays come back to the radar station.



This is different from a normal plane, which has complicated, bumpy surfaces. These surfaces give diffuse reflection, which spreads the reflected rays randomly in more or less every possible direction.

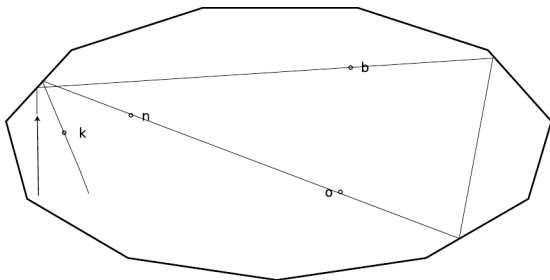
Page 853, problem 5:

- (a) The rays all cross at pretty much the same place, given the accuracy with which we can draw them.
- (b) It could be used to cook food, for instance. All the sunlight is concentrated in a small area.
- (c) Put the lightbulb at the point where the rays cross. The outgoing rays will then form a parallel beam going out to the right.



Page 853, problem 6:

It spells “bonk.”



Solutions for chapter 29

Page 868, problem 5:

The magnification is the ratio of the image’s size to the object’s size. It has nothing to do with the person’s location. The angular magnification, however, does depend on the person’s location, because things farther away subtend smaller angles. The distance to the actual object is not changed significantly, since it’s zillions of miles away in outer space, but the distance to the image does change if the observer’s point of view changes. If you can get closer to the image, the angular magnification is greater.

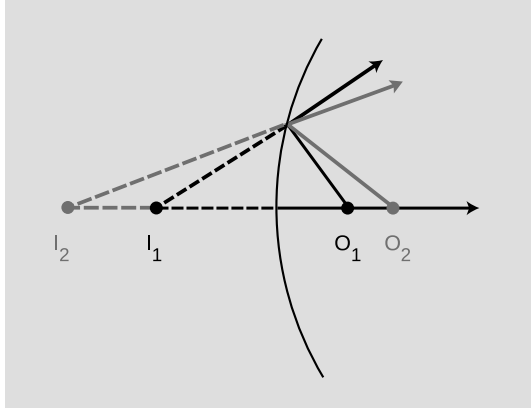
Solutions for chapter 30

Page 887, problem 1:

For a flat mirror, d_i and d_o are equal, so the magnification is 1, i.e., the image is the same size as the object.

Page 887, problem 2:

See the ray diagram below. Decreasing θ_o decreases θ_i , so the equation $\theta_f = \pm\theta_i + \pm\theta_o$ must have opposite signs on the right. Since θ_o is bigger than θ_i , the only way to get a positive θ_f is if the signs are $\theta_f = -\theta_i + \theta_o$. This gives $1/f = -1/d_i + 1/d_o$.

**Page 887, problem 3:**

(a) In problem #2 we found that the equation relating the object and image distances was of the form $1/f = -1/d_i + 1/d_o$. Let's make $f = 1.00$ m. To get a virtual image we need $d_o < f$, so let $d_o = 0.50$ m. Solving for d_i , we find $d_i = 1/(1/d_o - 1/f) = 1.00$ m. The magnification is $M = d_i/d_o = 2.00$. If we change d_o to 0.55 m, the magnification becomes 2.22. The magnification changes somewhat with distance, so the store's ad must be assuming you'll use the mirror at a certain distance. It can't have a magnification of 5 at all distances.

(b) Theoretically yes, but in practical terms no. If you go through a calculation similar to the one in part a, you'll find that the images of both planets are formed at almost exactly the same d_i , $d_i = f$, since $1/d_o$ is pretty close to zero for any astronomical object. The more distant planet has an image half as big ($M = d_i/d_o$, and d_o is doubled), but we're talking about *angular* magnification here, so what we care about is the angular size of the image compared to the angular size of the object. The more distant planet has half the angular size, but its image has half the angular size as well, so the angular magnification is the same. If you think about it, it wouldn't make much sense for the angular magnification to depend on the planet's distance — if it did, then determining astronomical distances would be much easier than it actually is!

Page 887, problem 4:

(a) This occurs when the d_i is infinite. Let's say it's a converging mirror creating a virtual image, as in problems 2 and 3. Then we'd get an infinite d_i if we put $d_o = f$, i.e., the object is at the focal point of the mirror. The image is infinitely large, but it's also infinitely far away, so its angular size isn't infinite; an angular size can never be more than about 180° since you can't see in back of your head!

(b) It's not possible to make the magnification infinite by having $d_o = 0$. The image location and object location are related by $1/f = 1/d_o - 1/d_i$, so $1/d_i = 1/d_o - 1/f$. If d_o is zero, then $1/d_o$ is infinite, $1/d_i$ is infinite, and d_i is zero as well. In other words, as d_o approaches zero, so does d_i , and d_i/d_o doesn't blow up. Physically, the mirror's curvature becomes irrelevant from the point of view of a tiny flea sitting on its surface: the mirror seems flat to the flea. So

physically the magnification would be 1, not infinity, for very small values of d_o .

Page 888, problem 10:

(a) The object distance is less than the focal length, so the image is virtual: because the object is so close, the cone of rays is diverging too strongly for the mirror to bring it back to a focus.

(b) Now the object distance is greater than the focal length, so the image is real. (c),(d) A diverging mirror can only make virtual images.

Solutions for chapter 31

Page 906, problem 1:

The refracted ray that was bent closer to the normal in the plastic when the plastic was in air will be bent farther from the normal in the plastic when the plastic is in water. It will become a diverging lens.

Page 906, problem 3:

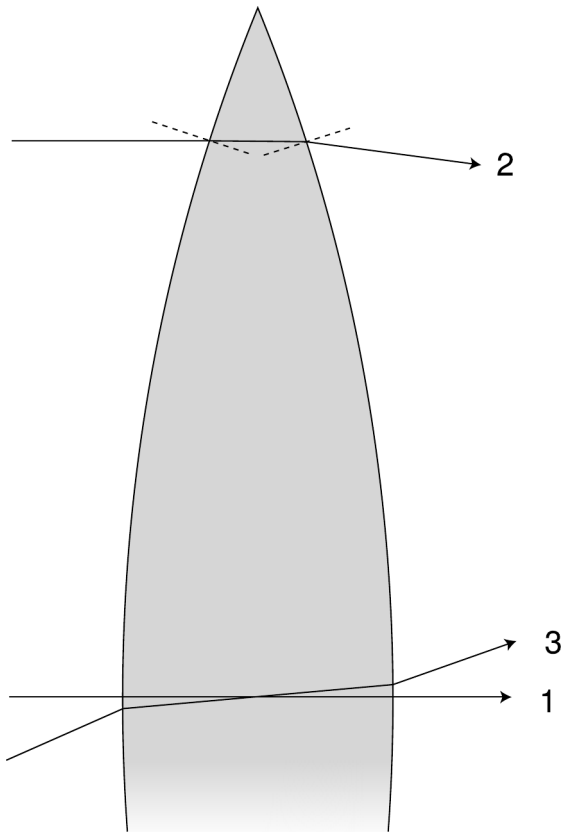
Refraction occurs only at the boundary between two substances, which in this case means the surface of the lens. Light doesn't get bent at all inside the lens, so the thickness of the lens isn't really what's important. What matters is the angles of the lens' surfaces at various points.

Ray 1 makes an angle of zero with respect to the normal as it enters the lens, so it doesn't get bent at all, and likewise at the back.

At the edge of the lens, 2, the front and back are not parallel, so a ray that traverses the lens at the edge ends up being bent quite a bit.

Although I drew both ray 1 and ray 2 coming in along the axis of the lens, it really doesn't matter. For instance, ray 3 bends on the way in, but bends an equal amount on the way out, so it still emerges from the lens moving in the same direction as the direction it originally had.

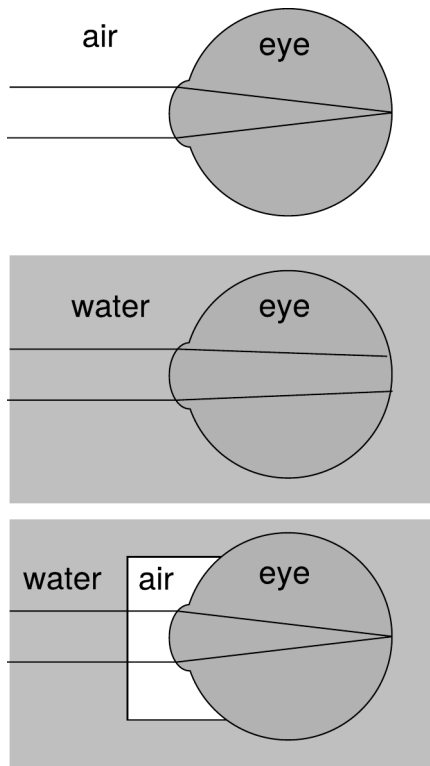
Summarizing and systematizing these observations, we can say that for a ray that enters the lens at the center, where the surfaces are parallel, the sum of the two deflection angles is zero. Since the total deflection is zero at the center, it must be larger away from the center.



Page 907, problem 7:

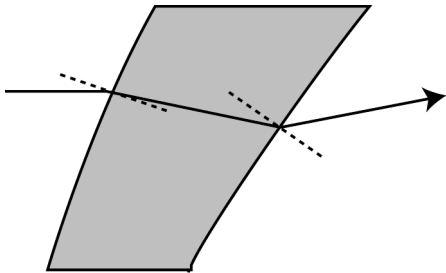
Normally, in air, your eyes do most of their focusing at the air-eye boundary. When you swim without goggles, there is almost no difference in speed at the water-eye interface, so light is not strongly refracted there (see figure), and the image is far behind the retina.

Goggles fix this problem for the following reason. The light rays cross a water-air boundary as they enter the goggles, but they're coming in along the normal, so they don't get bent. At the air-eye boundary, they get bent the same amount they normally would when you weren't swimming.



Page 907, problem 8:

(a) See the figure below. The first refraction clearly bends it inward. However, the back surface of the lens is more slanted, so the ray makes a bigger angle with respect to the normal at the back surface. The bending at the back surface is therefore greater than the bending at the front surface, and the ray ends up being bent *outward* more than inward.



(b) Lens 2 must act the same as lens 1. It's diverging. One way of knowing this is time-reversal symmetry: if we flip the original figure over and then reverse the direction of the ray, it's still a valid diagram.

Lens 3 is diverging like lens 1 on top, and diverging like lens 2 on the bottom. It's a diverging lens.

As for lens 4, any close-up diagram we draw of a particular ray passing through it will look exactly like the corresponding close-up diagram for some part of lens 1. Lens 4 behaves the same as lens 1.

Page 909, problem 13:

Since d_o is much greater than d_i , the lens-film distance d_i is essentially the same as f . (a) Splitting the triangle inside the camera into two right triangles, straightforward trigonometry

gives

$$\theta = 2 \tan^{-1} \frac{w}{2f}$$

for the field of view. This comes out to be 39° and 64° for the two lenses. (b) For small angles, the tangent is approximately the same as the angle itself, provided we measure everything in radians. The equation above then simplifies to

$$\theta = \frac{w}{f}$$

The results for the two lenses are $.70 \text{ rad} = 40^\circ$, and $1.25 \text{ rad} = 72^\circ$. This is a decent approximation.

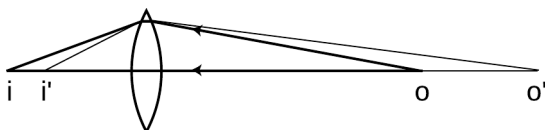
(c) With the 28-mm lens, which is closer to the film, the entire field of view we had with the 50-mm lens is now confined to a small part of the film. Using our small-angle approximation $\theta = w/f$, the amount of light contained within the same angular width θ is now striking a piece of the film whose linear dimensions are smaller by the ratio $28/50$. Area depends on the square of the linear dimensions, so all other things being equal, the film would now be overexposed by a factor of $(50/28)^2 = 3.2$. To compensate, we need to shorten the exposure by a factor of 3.2.

Page 910, problem 20:

One surface is curved outward and one inward. Therefore the minus sign applies in the lens-maker's equation. Since the radii of curvature are equal, the quantity $1/r_1 - 1/r_2$ equals zero, and the resulting focal length is infinite. A big focal length indicates a weak lens. An infinite focal length tells us that the lens is infinitely weak — it doesn't focus or defocus rays at all.

Page 911, problem 22:

(a) The situation being described requires a real image, since the rays need to converge at a point on Becky's neck. See the ray diagram drawn with thick lines, showing object location o and image location i .



If we move the object farther away, to o' the cone of rays intercepted by the lens (thin lines) is less strongly diverging, and the lens is able to bring it to a closer focus, at i' . In the diagrams, we see that a smaller θ_o leads to a larger θ_i , so the signs in the equation $\pm\theta_o \pm \theta_i = \theta_f$ must be the same, and therefore both positive, since θ_f is positive by definition. The equation relating the image and object locations must be $1/f = 1/d_o + 1/d_i$.

(b) The case with $d_i = f$ is not possible, because then we need $1/d_o = 0$, i.e., $d_o = \infty$. Although it is possible in principle to have an object so far away that it is practically at infinity, that is not possible in this situation, since Zahra can't take her lens very far away from the fire. By the way, this means that the *focal length* f is not where the *focus* happens — the focus happens at d_i .

For similar reasons, we can't have $d_o = f$.

Since all the variables are positive, we must have $1/d_o$ and $1/d_i$ both less than $1/f$. This implies that $d_o > f$ and $d_i > f$. Of the nine logical possibilities in the table, only this one is actually possible for this real image.

Solutions for chapter 32

Page 929, problem 1:

You don't want the wave properties of light to cause all kinds of funny-looking diffraction effects. You want to see the thing you're looking at in the same way you'd see a big object. Diffraction effects are most pronounced when the wavelength of the light is relatively large compared to the size of the object the light is interacting with, so red would be the worst. Blue light is near the short-wavelength end of the visible spectrum, which would be the best.

Page 930, problem 6:

For the size of the diffraction blob, we have:

$$\begin{aligned}\frac{\lambda}{d} &\sim \sin \theta \\ &\approx \theta \\ \theta &\sim \frac{700 \text{ nm}}{10 \text{ m}} \\ &\approx 10^{-7} \text{ radians}\end{aligned}$$

For the actual angular size of the star, the small-angle approximation gives

$$\begin{aligned}\theta &\sim \frac{10^9 \text{ m}}{10^{17} \text{ m}} \\ &= 10^{-8} \text{ radians}\end{aligned}$$

The diffraction blob is ten times bigger than the actual disk of the star, so we can never make an image of the star itself in this way.

Page 930, problem 7:

The equation, solved for θ , is $\theta = \sin^{-1}(m\lambda/d)$. The sine function only ranges from -1 to $+1$, so the inverse sine is undefined for $|m\lambda/d| > 1$, i.e., $|m| > d/\lambda$. Physically, we only get fringes out to angles of 90 degrees (the inverse sine of 1) on both sides, corresponding to values of m less than d/λ .

Page 931, problem 9:

- (a) You can tell it's a single slit because of the double-width central fringe.
- (b) Four fringes on the top pattern are about 23.5 mm, while five fringes on the bottom one are about 14.5 mm. The spacings are 5.88 and 2.90 mm, with a ratio of 2.03. A smaller d leads to larger diffraction angles, so the width of the slit used to make the bottom pattern was almost exactly twice as wide as the one used to make the top one.

Page 932, problem 11:

- (a) The patterns have two structures, a coarse one and a fine one. You can look up in the book which corresponds to w and which to d , or just use the fact that smaller features make bigger diffraction angles. The top and middle patterns have the same coarse spacing, so they have the same w . The fine structure in the top pattern has 7 fringes in 12.5 mm, for a spacing of 1.79 mm, while the middle pattern has 11 fringes in 41.5 mm, giving a spacing of 3.77 mm. The value of d for the middle pattern is therefore $(0.50 \text{ mm})(1.79/3.77) = 0.23 \text{ mm}$.
- (b) This one has about the same d as the top one (it's difficult to measure accurately because each group has only a small number of fringes), but the coarse spacing is different, indicating a different value of w . It has two coarse groupings in 23 mm, i.e., a spacing of 12.5 mm. The coarse groupings in the original pattern were about 23 mm apart, so there is a factor of two between the $w = 0.04 \text{ mm}$ of the top pattern and the $w = 0.08 \text{ mm}$ of the bottom one.

Answers to self-checks for volume 2

Answers to self-checks for chapter 21

Page 579, self-check A:

Either type can be involved in either an attraction or a repulsion. A positive charge could be involved in either an attraction (with a negative charge) or a repulsion (with another positive), and a negative could participate in either an attraction (with a positive) or a repulsion (with a negative).

Page 580, self-check B:

It wouldn't make any difference. The roles of the positive and negative charges in the paper would be reversed, but there would still be a net attraction.

Page 595, self-check C:

The large amount of power means a high rate of conversion of the battery's chemical energy into heat. The battery will quickly use up all its energy, i.e., "burn out."

Answers to self-checks for chapter 22

Page 641, self-check A:

The reasoning is exactly analogous to that used in example 1 on page 639 to derive an equation for the gravitational field of the earth. The field is $F/q_t = (kQq_t/r^2)/q_t = kQ/r^2$.

Page 653, self-check B:

$$\begin{aligned} E_x &= -\frac{dV}{dx} \\ &= -\frac{d}{dx} \left(\frac{kQ}{r} \right) \\ &= \frac{kQ}{r^2} \end{aligned}$$


Page 654, self-check C:

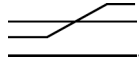
(a) The voltage (height) increases as you move to the east or north. If we let the positive x direction be east, and choose positive y to be north, then dV/dx and dV/dy are both positive. This means that E_x and E_y are both negative, which makes sense, since the water is flowing in the negative x and y directions (south and west).

(b) The electric fields are all pointing away from the higher ground. If this was an electrical map, there would have to be a large concentration of charge all along the top of the ridge, and especially at the mountain peak near the south end.

Answers to self-checks for chapter 23

Page 671, self-check A:

The diagram for the house looks like  because in the one dimension of space being represented, it has walls on both sides, and its existence also extends over a certain amount of time (left to right). If the dog is in the house at rest, then goes outside, and stays at rest in the

back yard for a while, the spacetime diagram looks like this: . An observer using another frame of reference has to agree that the dog went outside, because observers agree on intersections of world-lines, and the dog's world-line intersects the world-line of the house's back wall.

Page 681, self-check B:

At $v = 0$, we get $\gamma = 1$, so $t = T$. There is no time distortion unless the two frames of reference are in relative motion.

Answers to self-checks for chapter 24**Page 710, self-check A:**

An induced electric field can only be created by a *changing* magnetic field. Nothing is changing if your car is just sitting there. A point on the coil won't experience a changing magnetic field unless the coil is already spinning, i.e., the engine has already turned over.

Page 718, self-check B:

Both the time axis and the position axis have been turned around. Flipping the time axis means that the roles of transmitter and receiver have been swapped, and it also means that Alice and Betty are approaching one another rather than receding. The time experienced by the receiving observer is now the longer one, so the Doppler-shift factor has been inverted: the receiver now measures a Doppler shift of $1/2$ rather than 2 in frequency.

Answers to self-checks for chapter 25**Page 739, self-check A:**

Yes. The mass has the same kinetic energy regardless of which direction it's moving. Friction converts mechanical energy into heat at the same rate whether the mass is sliding to the right or to the left. The spring has an equilibrium length, and energy can be stored in it either by compressing it ($x < 0$) or stretching it ($x > 0$).

Page 739, self-check B:

Velocity, v , is the rate of change of position, x , with respect to time. This is exactly analogous to $I = \Delta q / \Delta t$.

Page 748, self-check C:

The impedance depends on the frequency at which the capacitor is being driven. It isn't just a single value for a particular capacitor.

Answers to self-checks for chapter 26**Page 758, self-check A:**

Yes. In U.S. currency, the quantum of money is the penny.

Page 779, self-check B:

Thomson was accelerating electrons, which are negatively charged. This apparatus is supposed to accelerated atoms with one electron stripped off, which have positive net charge. In both cases, a particle that is between the plates should be attracted by the forward plate and repelled by the plate behind it.

Page 787, self-check C:

The hydrogen-1 nucleus is simple a proton. The binding energy is the energy required to tear a nucleus apart, but for a nucleus this simple there is nothing to tear apart.

Page 795, self-check D:

The total momentum is zero before the collision. After the collision, the two momenta have reversed their directions, but they still cancel. Neither object has changed its kinetic energy, so the total energy before and after the collision is also the same.

Page 802, self-check E:

At $v = 0$, we have $\gamma = 1$, so the mass-energy is mc^2 as claimed. As v approaches c , γ approaches infinity, so the mass energy becomes infinite as well.

Answers to self-checks for chapter 28

Page 848, self-check A:

Only 1 is correct. If you draw the normal that bisects the solid ray, it also bisects the dashed ray.

Answers to self-checks for chapter 29

Page 856, self-check A:

You should have found from your ray diagram that an image is still formed, and it has simply moved down the same distance as the real face. However, this new image would only be visible from high up, and the person can no longer see his own image.

Page 861, self-check B:

Increasing the distance from the face to the mirror has decreased the distance from the image to the mirror. This is the opposite of what happened with the virtual image.

Answers to self-checks for chapter 30

Page 878, self-check A:

At the top of the graph, d_i approaches infinity when d_o approaches f . Interpretation: the rays just barely converge to the right of the mirror.

On the far right, d_i approaches f as d_o approaches infinity; this is the definition of the focal length.

At the bottom, d_i approaches negative infinity when d_o approaches f from the other side. Interpretation: the rays don't quite converge on the right side of the mirror, so they appear to have come from a virtual image point very far to the left of the mirror.

Answers to self-checks for chapter 31

Page 895, self-check A:

(1) If n_1 and n_2 are equal, Snell's law becomes $\sin \theta_1 = \sin \theta_2$, which implies $\theta_1 = \theta_2$, since both angles are between 0 and 90° . The graph would be a straight line along the diagonal of the graph. (2) The graph is farthest from the diagonal when the angles are large, i.e., when the ray strikes the interface at a grazing angle.

Page 899, self-check B:

(1) In 1, the rays cross the image, so it's real. In 2, the rays only appear to have come from the image point, so the image is virtual. (2) A ray is always closer to the normal in the medium with the higher index of refraction. The first left turn makes the ray closer to the normal, which is what should happen in glass. The second left turn makes the ray farther from the normal, and that's what should happen in air. (3) Take the topmost ray as an example. It will still take two right turns, but since it's entering the lens at a steeper angle, it will also leave at a steeper angle. Tracing backward to the image, the steeper lines will meet closer to the lens.

Answers to self-checks for chapter 32

Page 917, self-check A:

It would have to have a wavelength on the order of centimeters or meters, the same distance scale as that of your body. These would be microwaves or radio waves. (This effect can easily be noticed when a person affects a TV's reception by standing near the antenna.) None of this

contradicts the correspondence principle, which only states that the wave model must agree with the ray model when the ray model is applicable. The ray model is not applicable here because λ/d is on the order of 1.

Page 919, self-check B:

At this point, both waves would have traveled nine and a half wavelengths. They would both be at a negative extreme, so there would be constructive interference.

Page 923, self-check C:

Judging by the distance from one bright wave crest to the next, the wavelength appears to be about $2/3$ or $3/4$ as great as the width of the slit.

Page 924, self-check D:

Since the wavelengths of radio waves are thousands of times longer, diffraction causes the resolution of a radio telescope to be thousands of times worse, all other things being equal. (To compensate for the wavelength, it's desirable to make the telescope very large, as in figure z on page 924.)

(1 rectangle = $5 \text{ cm} \times 0.005 \text{ cm}^{-1} = 0.025$), but that would have been pointless, because we were just going to compare the two areas.

Answers to self-checks for chapter 33

Page 947, self-check A:

(1) Most people would think they were positively correlated, but they could be independent. (2) These must be independent, since there is no possible physical mechanism that could make one have any effect on the other. (3) These cannot be independent, since dying today guarantees that you won't die tomorrow.

Page 949, self-check B:

The area under the curve from 130 to 135 cm is about $3/4$ of a rectangle. The area from 135 to 140 cm is about 1.5 rectangles. The number of people in the second range is about twice as much. We could have converted these to actual probabilities

Answers to self-checks for chapter 34

Page 970, self-check A:

The axes of the graph are frequency and photon energy, so its slope is Planck's constant. It doesn't matter if you graph $e\Delta V$ rather than $E_s + e\Delta V$, because that only changes the y-intercept, not the slope.

Answers to self-checks for chapter 35

Page 991, self-check A:

Wavelength is inversely proportional to momentum, so to produce a large wavelength we would need to use electrons with very small momenta and energies. (In practical terms, this isn't very easy to do, since ripping an electron out of an object is a violent process, and it's not so easy to calm the electron down afterward.)

Page 1000, self-check B:

Under the ordinary circumstances of life, the accuracy with which we can measure the position and momentum of an object doesn't result in a value of $\Delta p \Delta x$ that is anywhere near the tiny order of magnitude of Planck's constant. We run up against the ordinary limitations on the accuracy of our measuring techniques long before the uncertainty principle becomes an issue.

Page 1000, self-check C:

The electron wave will suffer single-slit diffraction, and spread out to the sides after passing through the slit. Neither Δp nor Δx is zero for the diffracted wave.

Page 1004, self-check D:

No. The equation $KE = p^2/2m$ is nonrelativistic, so it can't be applied to an electron moving at relativistic speeds. Photons always move at relativistic speeds, so it can't be applied to them, either.

Page 1005, self-check E:

Dividing by Planck's constant, a small number, gives a large negative result inside the exponential, so the probability will be very small.

Answers to self-checks for chapter 36

Page 1013, self-check A:

If you trace a circle going around the center, you run into a series of eight complete wavelengths. Its angular momentum is $8\hbar$.

Page 1016, self-check B:

$n = 3, \ell = 0, \ell_z = 0$: one state

$n = 3, \ell = 1, \ell_z = -1, 0, \text{ or } 1$: three states

$n = 3, \ell = 2, \ell_z = -2, -1, 0, 1, \text{ or } 2$: five states

Photo Credits for Volume 2

Except as specifically noted below or in a parenthetical credit in the caption of a figure, all the illustrations in this book are under my own copyright, and are copyleft licensed under the same license as the rest of the book.

In some cases it's clear from the date that the figure is public domain, but I don't know the name of the artist or photographer; I would be grateful to anyone who could help me to give proper credit. I have assumed that images that come from U.S. government web pages are copyright-free, since products of federal agencies fall into the public domain. I've included some public-domain paintings; photographic reproductions of them are not copyrightable in the U.S. (*Bridgeman Art Library, Ltd. v. Corel Corp.*, 36 F. Supp. 2d 191, S.D.N.Y. 1999).

When "PSSC Physics" is given as a credit, it indicates that the figure is from the first edition of the textbook entitled *Physics*, by the Physical Science Study Committee. The early editions of these books never had their copyrights renewed, and are now therefore in the public domain. There is also a blanket permission given in the later PSSC College Physics edition, which states on the copyright page that "The materials taken from the original and second editions and the Advanced Topics of PSSC PHYSICS included in this text will be available to all publishers for use in English after December 31, 1970, and in translations after December 31, 1975."

Credits to Millikan and Gale refer to the textbooks *Practical Physics* (1920) and *Elements of Physics* (1927). Both are public domain. (The 1927 version did not have its copyright renewed.) Since it is possible that some of the illustrations in the 1927 version had their copyrights renewed and are still under copyright, I have only used them when it was clear that they were originally taken from public domain sources.

In a few cases, I have made use of images under the fair use doctrine. However, I am not a lawyer, and the laws on fair use are vague, so you should not assume that it's legal for you to use these images. In particular, fair use law may give you less leeway than it gives me, because I'm using the images for educational purposes, and giving the book away for free. Likewise, if the photo credit says "courtesy of ...," that means the copyright owner gave me permission to use it, but that doesn't mean you have permission to use it.

Photo credits to NEI refer to photos from the National Eye Institute, part of the National Institutes of Health, <http://www.nei.nih.gov/photo/>. "Items here are not copyrighted. However, we do ask that you credit as follows: National Eye Institute, National Institutes of Health (except where indicated otherwise)."

Cover Spider: Wikimedia Commons user Opoterser, CC-BY-SA. **Contents** See photo credits below, referring to the places where the images appear in the main text.

?? *Faraday*: Painting by Thomas Phillips, 1842. ?? *Binoculars*: Wikimedia commons, CC-BY-SA. ?? *Porro prisms*: Redrawn from a figure by Wikipedia user DrBob, CC-BY-SA. **Contents** *Insect's eye*: Wikimedia Commons, CC-BY-SA, user Reytan. **Contents** *Man's eye*: NEI. **537 Salamander**: Redrawn from an 18th century illustration by J.D. Meyer. **537 Hand**: Redrawn from a photo by Wikimedia Commons user Evan-Amos, CC-BY-SA. **575 Lightning**: C. Clark/NOAA photo library, public domain. **581 Knifefish**: Courtesy of Greg DeGreef. **582 Ampère**: Millikan and Gale, 1920. **585 Volta**: Millikan and Gale, 1920. **590 Ohm**: Millikan and Gale, 1920. **592 MRI**: Wikimedia Commons user Tomas Vendis, GFDL license. **595 Resistors**: Wikipedia user Afrank99, CC-BY-SA. **608 Crab**: From a photo by Hans Hillewaert, CC-BY-SA. **619 LP record**: Felipe Micaroni Lalli, CC-BY-SA. **619 Printed circuit board**: Bill Bertram, Wikipedia user Pixel8, CC-BY licensed. **634 GPS**: Wikipedia user HawaiianMama, CC-BY license. **634 Atomic clock on plane**: Copyright 1971, Associated press, used under U.S. fair use exception to copyright law. **635 Football pass**: Wikipedia user RMelon, CC-BY-SA licensed. **640 LIGO**: Wikipedia user Umptanum: "This image is copyrighted. The copyright holder has irrevocably released all rights to it.". **647 Hammerhead shark**: Wikimedia Commons user Littlegreenman, public domain. **653 Topographical maps**: Flat maps by USGS, public domain; perspective map by Wikipedia user Kbh3rd, CC-BY-SA licensed. **656 Avalanche transceiver**: Human figure: based on Pioneer 10 plaque, NASA, public domain. Dipole field: Wikimedia Commons user Geek3, CC-BY. **669 Cow**: Redrawn from a photo by Wikimedia Commons user Cgoodwin, CC-BY-SA. **669 Car**: Redrawn from a photo by Wikimedia Commons user Auregann, CC-BY-SA. **670 Hell**: Hieronymus Bosch, public domain. **671 Surveyors**: Don Swanson, public domain; Owais Khurshed, CC-BY-SA; Paul Peterson, public domain; British Royal Navy, public domain. **671 Pigeon**: Wikimedia commons user nitramtrebla, CC-BY-SA. **671 Mushroom cloud**: USGS, public domain. **673 S. Iijima and K. Fujiwara**, "An experiment for the potential blue shift at the Norikura Corona Station," *Annals of the Tokyo Astronomical Observatory, Second Series, Vol. XVII, 2 (1978) 68*: Used here under the U.S. fair use doctrine. **673 Horse**: From a public-domain photo by Eadweard Muybridge, 1872. **673 Satellite**: From a public-domain artist's conception of a GPS satellite, product of NASA. **674 Joan of Arc holding banner**: Ingres, 1854. **674 Joan of Arc interro-**

gated: Delaroché, 1856. **684 Muon storage ring at CERN:** (c) 1974 by CERN; used here under the U.S. fair use doctrine. **686 Colliding nuclei:** courtesy of RHIC. **708 Galaxies:** Ville Koistinen, CC-BY-SA. **708 Machine gunner's body:** Redrawn from a public-domain photo by Cpl. Sheila Brooks. **708 Machine gunner's head:** Redrawn from a sketch by Wenceslas Hollar, 17th century. **712 Hertz:** Public domain contemporary photograph. **715 Swan Lake:** Peter Gerstbach, GFDL 1.2. **716 C.S. Wu:** Smithsonian Institution, believed to be public domain. **736 Capacitors:** Wikipedia user de:Benutzer:Honina, CC-BY-SA licensed. **736 Inductors:** Wikipedia user de:Benutzer:Honina, CC-BY-SA licensed. **753 Curies:** Harper's Monthly, 1904. **757 Robert Millikan:** Clark Millikan, 1891, public domain. **760 J.J. Thomson:** Millikan and Gale, 1920.. **765 Becquerel:** Millikan and Gale, 1920. **765 Becquerel's photographic plate:** public domain. **767 Nuclear fuel pellets:** US DOE, public domain. **768 Rutherford:** public domain. **780 Nuclear power plant:** Wikipedia user Stefan Kuhn, CC-BY-SA licensed. **785 GAMMASPHERE:** courtesy of C.J. Lister/R.V.F. Janssens. **785 H bomb test:** public domain product of US DOE, Ivy Mike test. **785 Fatu Hiva Rainforest:** Wikipedia user Makemake, CC-BY-SA licensed. **785 fusion reactor:** "These images may be used free of charge for educational purposes but please use the acknowledgement 'photograph courtesy of EFDA-JET'". **785 Sun:** SOHO (ESA & NASA). **788 Chernobyl map:** CIA Handbook of International Economic Statistics, 1996, public domain. **789 Fifty-foot woman:** Public domain due to nonrenewal of copyright. **789 Zombies:** Public domain due to an error by the distributor in failing to place a copyright notice on the film. **792 Horses:** (c) 2004 Elena Filatova. **792 Polar bear:** U.S. Fish and Wildlife Service, public domain. **793 Crab Nebula:** "ESO Press Photos may be reproduced, if credit is given to the European Southern Observatory.". **799 Eclipse:** 1919, public domain. **799 Newspaper headline:** 1919, public domain. **801 Photo of PET scanner:** Wikipedia user Hg6996, public domain. **801 Ring of detectors in PET scanner:** Wikipedia user Damato, public domain. **801 PET body scan:** Jens Langner, public domain. **820 Einstein's ring:** I have lost the information about the source of the bitmapped image. I would be grateful to anyone who could put me in touch with the copyright owners.. **823 Artificial horizon:** NASA, public domain. **824 Pound and Rebka photo:** I presume this photo to be in the public domain, since it is unlikely to have had its copyright renewed. **825 Earth:** NASA, Apollo 17. Public domain. **825 Orion:** Wikipedia user Mouser, GFDL. **825 M100:** European Southern Observatory, CC-BY-SA. **825 Supercluster:** Wikipedia user Azcolvin429, CC-BY-SA. **828 Globular cluster:** Hubble Space Telescope. Hubble material is copyright-free and may be freely used as in the public domain without fee, on the condition that NASA and ESA is credited as the source of the material. The material was created for NASA by STScI under Contract NAS5-26555 and for ESA by the Hubble European Space Agency Information Centre. **831 Galaxies:** Hubble Space Telescope. Hubble material is copyright-free and may be freely used as in the public domain without fee, on the condition that NASA and ESA is credited as the source of the material. The material was created for NASA by STScI under Contract NAS5-26555 and for ESA by the Hubble European Space Agency Information Centre. **831 Cosmic microwave background image:** NASA/WMAP Science Team, public domain. **837 Rays of sunlight:** Wikipedia user PiccoloNamek, CC-BY-SA. **840 Jupiter and Io:** NASA/JPL/University of Arizona. **849 Ray-traced image:** Gilles Tran, Wikimedia Commons, public domain. **855 Narcissus:** Caravaggio, ca. 1598. **857 Prazinoscope:** Thomas B. Greenslade, Jr.. **859 The Sleeping Gypsy:** H. Rousseau, 1897. **862 Flower:** Based on a photo by Wikimedia Commons user Fir0002, CC-BY-SA. **862 Moon:** Wikimedia commons image. **863 Ladybug:** Redrawn from a photo by Wikimedia Commons user Gilles San Martin, CC-BY-SA. **883 Fish-eye lens:** Martin Dürschnabel, CC-BY-SA. **884 Hubble space telescope:** NASA, public domain. **888 Anamorphic image:** Wikipedia user Istvan Orosz, CC-BY. **891 Flatworm:** CC-BY-SA, Alejandro Sánchez Alvarado, Planaria.neuro.utah.edu. **891 Nautilus:** CC-BY-SA, Wikimedia Commons user Opencage, opencage.info. **891 Human eye:** Joao Estevao A. de Freitas, "There are no usage restrictions for this photo". **892 Cross-section of eye:** NEI. **892 Eye's anatomy:** After a public-domain drawing from NEI. **896 Water wave refracting:** Original photo from PSSC. **898 Ulcer:** Wikipedia user Aspersions, CC-BY-SA. **902 Jumping spider:** Photo: Wikipedia user Opoterser, CC-BY-SA. Line art: redrawn from M.F. Land, J. Exp. Biol. 51 (1969) 443. **913 Pleiades:** NASA/ESA/AURA/Caltech, public domain. **914 Diffraction of water waves:** Assembled from photos in PSSC. **914 Counterfactual lack of diffraction of water waves:** Assembled from photos in PSSC. **914 Diffraction of water waves:** Assembled from photos in PSSC. **915 Scaling of diffraction:** Assembled from photos in PSSC. **916 Huygens:** Contemporary painting?. **917 Diffraction of water waves:** Assembled from photos in PSSC. **918 Young:** Wikimedia Commons, "After a portrait by Sir Thomas Lawrence, From: Arthur Shuster & Arthur E. Shipley: Britain's Heritage of Science. London, 1917". **923 Single-slit diffraction of water waves:** PSSC. **923 Simulation of a single slit using three sources:** PSSC. **924 Pleiades:** NASA/ESA/AURA/Caltech, public domain. **924 Radio telescope:** Wikipedia user Hajor, CC-BY-SA. **941 Mount St. Helens:** public-domain image by Austin Post, USGS. **965 Ozone maps:** NASA/GSFC TOMS Team. **966 Digital camera image:** courtesy of Lyman Page. **973 Diffracted photons:** courtesy of Lyman Page. **987 Wicked Witch:** art by W.W. Denslow, 1900. Quote from *The Wizard of Oz*, L. Frank Baum, 1900. **998 Werner Heisenberg:** ca. 1927, believed to be public domain. **1023 Hindenburg:** Public domain product of the U.S. Navy.

Index

- aberration, 881
 - chromatic, 901
- absorption, 841
- absorption of waves, 525
- absorption spectrum, 996
- acceleration, 99
 - as a vector, 221
 - constant, 111
 - definition, 106
 - negative, 102
- alchemists, 293
- alchemy, 17, 576
- alpha decay, 781
 - nature of emitted particle, 768
- alpha particle, *see* alpha decay
- ammeter, 584
- ampere (unit), 582
- amplitude
 - defined, 460
 - peak-to-peak, 460
 - related to energy, 472
- anamorph, 889
- angular magnification, 862
- angular momentum
 - and the uncertainty principle, 1013
 - choice of axis theorem, 402
 - defined, 395
 - definition, 396
 - in three dimensions, 1013
 - introduction to, 393
 - quantization of, 1012
 - related to area swept out, 400
 - spin theorem, 403
- antielectron, 783
- antimatter, 783
- area, 109
 - operational definition, 41
 - scaling of, 43
- area under a curve
 - area under a-t graph, 111
 - under v-t graph, 109
- astrology, 17
- atom
 - raisin-cookie model of, 762
- atomic number
 - defined, 772
- atoms
 - helium, 1021
 - lithium, 1022
 - sodium, 1023
 - with many electrons, 1021
- averages, 946
 - rule for calculating, 946
- Avogadro's number, 441
- Bacon, Francis, 21
- Balmer, Johann, 1018
- beta decay, 783
 - nature of emitted particle, 768
- beta particle, *see* beta decay
- Big Bang, 276
 - described in general relativity, 829
- binding energy
 - nuclear, 787
- black hole, 448, 800, 826
 - event horizon, 826
 - formation, 828
 - information paradox, 827
 - singularity, 828
- Bohr
 - Niels, 916
- Bohr, Niels, 1018
- Boltzmann's constant, 441
- bond, *see* chemical bonds
- bound states, 995
- box
 - particle in a, 995
- Brahe, Tycho, 262
- brightness of light, 843
- Brownian motion, 756
- calculus
 - differential, 86
 - fundamental theorem of, 115
 - integral, 115
 - invention by Newton, 85
 - Leibnitz notation, 86
 - with vectors, 228