The Probability of a Run-Off Election when Three Equally-Favored Candidates Vie for Two Slots

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Suppose n voters each cast two votes for two candidates when three candidates are vying for two slots on a county commission. If the candidates are equally-favored, we may assume the two votes per voter are random. Hence there are $\binom{3}{2}^n$ possible ballot results. After presenting an example below, we will provide the number of ballot results that end in a tie for the second slot, necessitating a run-off election.

Example. For $n = 6$ voters, a ballot result that ends in a tie for the second slot can come in two types:

Type 1: {6,3,3}

	Candidate A	Candidate B	Candidate C	Totals
Voter 1	x	x		2
Voter 2	x	x		2
Voter 3	x	x		2
Voter 4	x		x	2
Voter 5	x		x	2
Voter 6	x		x	2
Totals	6	3	3	12

We have $\binom{3}{1}$ ways to choose the candidate that gets the 6 votes. We have $\binom{6}{3}$ ways to choose the voters who vote for the remaining two candidates. Hence there are 3(2)=60 ballot results of type 1.

Type 2: {4,4,4}

	Candidate A	Candidate B	Candidate C	Totals
Voter 1	x	x		2
Voter 2	x	x		2
Voter 3	x		x	2
Voter 4	x		x	2
Voter 5		x	x	2
Voter 6		x	x	2
Totals	6	3	3	12

We have $\binom{6}{4}$ ways to choose the voters who vote for candidate A. The voters who did not vote for candidate A must vote for candidates B and C. Then we have $\binom{4}{2}$ ways to choose the remaining voters that vote for candidates B and C. Hence there are 6(15)=90 ballot results of type 2.

Therefore, for $n = 6$ voters, there are 60+90=150 ballot results that necessitate a run-off election, and the probability $p(n)$ of a run-off in this case is given by

$$p(6) = \tfrac{150}{3^6} \approx .20576 \qquad \diamond$$

Theorem. The number of ballot results, say $r(n)$, that necessitate a run-off election is given by

$$r(n) = 3\sum_{k=0}^{u(n)} \binom{n}{2\lceil\frac{n-1}{2}\rceil - 2k}\binom{2\lceil\frac{n-1}{2}\rceil - 2k}{\lceil\frac{n-1}{2}\rceil - k} - 2\binom{n}{2n/3}\binom{2n/3}{n/3} I[3|n]$$

where $u(n) = \lfloor\frac{n+2}{2}\rfloor - \lfloor\frac{n+2}{3}\rfloor - 1$ and indicator function $I[statement]$ takes on value 1 if the statement is true and 0 otherwise.

Proof. A type 1 tie occurs when the last two candidates tie (as seen in the example above) and is of the form $(x,\ y,\ y)$. There $\binom{n}{x}$ ways to choose the voters who vote for the top winner, and then $\binom{x}{x/2}$ to choose the voters who vote for the remaining tied candidates. . Since there are 3 choices for the candidate who gets the x votes, we have $3\binom{n}{x}\binom{x}{x/2}$ number of ballots of the form $(x,\ y,\ y)$.

When a type 2 tie occurs, we have vote totals $\{2n/3,\ 2n/3, 2n/3\}$ which happens only when n is a multiple of 3. In this case, there are not 3 choices for the most votes, hence we must subtract $2\binom{n}{x}\binom{x}{x/2}$ from our total.

Since, in general, $n \geq x \geq y > 1$ and $x + 2y = 2n$, x can run from n (when n is even) or $n-1$ (when n is odd) down to $2(n - \lfloor\frac{2n}{3}\rfloor)$ in increments of 2. Note that

$$2\lceil\tfrac{n-1}{2}\rceil = \begin{cases} n & \text{for even } n \\ n-1 & \text{for odd } n \end{cases} \qquad (1)$$

and

$$\begin{aligned} 2\lceil\tfrac{n-1}{2}\rceil - 2u(n) &= 2\lceil\tfrac{n-1}{2}\rceil - 2\Big(\lfloor\tfrac{n+2}{2}\rfloor - \lfloor\tfrac{n+2}{3}\rfloor - 1\Big) \\ &= 2(n - \lfloor\tfrac{2n}{3}\rfloor). \qquad (2) \end{aligned}$$

Therefore, we have

$$r(n) = \sum_x 3\binom{n}{x}\binom{x}{x/2} - 2\binom{n}{2n/3}\binom{2n/3}{n/3} I[3|n]$$

where x runs from $2(n - \lfloor \frac{2n}{3} \rfloor)$ up to n or $n-1$ in increments of 2.

However, reversing the order of summation, and using the identities in (1) and (2), we obtain

$$r(n) = 3\sum_{k=0}^{u(n)} \binom{n}{2\lceil \frac{n-1}{2} \rceil - 2k}\binom{2\lceil \frac{n-1}{2} \rceil - 2k}{\lceil \frac{n-1}{2} \rceil - k} - 2\binom{n}{2n/3}\binom{2n/3}{n/3} I[3|n]. \qquad \square$$

Let $r(0) = 1$ because if no voters show up on election day all three candidates are tied with 0 votes each and a run-off is needed. Also, $r(1) = 0$, since two of the three candidates will clearly win with one vote each. Thus we have the following sequence of values for $r(n)$, $n =$ 0,..., 40 :

1, 0, 6, 6, 18, 90, 150, 420, 1890, 3570, 10206, 42966, 87318, 252252, 1019304, 2172456, 6319170, 24810786, 54712086, 159906318, 614406078, 1390381278, 4077926034, 15403838346, 35579546262, 104633453340, 389788932240, 915500037120, 2698033909680, 9934966920960, 23662402088400, 69853956443460, 254691586091970, 613858758011010, 1814737822473510, 6560221819413870, 15974712728822070, 47281700283443910, 169643060817818490, 416825614497182130, 1234958443451187390.

Corollary. When there are n voters, the probability $p(n)$ of a run-off election is given by

$$p(n) = \frac{r(n)}{3^n}.$$

Appendix

Generation of sequence values for $r(n)$ using *Maple*

```
>ind:=n->piecewise(n mod 3=0,1,0);
               ind := n -> piecewise(n mod 3 = 0, 1, 0)


>u:=n->floor(n/2+1)-floor(n/3+2/3)-1;
         u := n -> floor(1/2 n + 1) - floor(1/3 n + 2/3) - 1

> r:=n->3*sum(binomial(n,2*ceil((n-1)/2)-2*k)*binomial(2*ceil((n-1)/2)-2*k,ceil((n-
1)/2)-k),k=0..u(n))-ind(n)*2*binomial(n,2*n/3)*binomial(2*n/3,n/3);

              /u(n)
              |-----
              | \
  r := n -> 3 |  )   binomial(n, 2 ceil(1/2 n - 1/2) - 2 k)
              | /
              |-----
              \k = 0

                                                                   \
                                                                   |
                                                                   |
        binomial(2 ceil(1/2 n - 1/2) - 2 k, ceil(1/2 n - 1/2) - k)|
                                                                   |
                                                                   |
                                                                   /

         - 2 ind(n) binomial(n, 2/3 n) binomial(2/3 n, 1/3 n)

> seq(r(n),n=2..40);

  6, 6, 18, 90, 150, 420, 1890, 3570, 10206, 42966, 87318, 252252,

        1019304, 2172456, 6319170, 24810786, 54712086, 159906318,

        614406078, 1390381278, 4077926034, 15403838346, 35579546262,

        104633453340, 389788932240, 915500037120, 2698033909680,

        9934966920960, 23662402088400, 69853956443460,

        254691586091970, 613858758011010, 1814737822473510,

        6560221819413870, 15974712728822070, 47281700283443910,

        169643060817818490, 416825614497182130, 1234958443451187390
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