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Multiple Zeta Values

and

Euler-Zagier Numbers

by

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http://www.math.jussieu.fr/∼miw/articles/ps/MZV.ps

1. Introduction

For $s_1, \ldots, s_k$ in $\mathbb{Z}$ with $s_1 \geq 2$,

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1 > \cdots > n_k \geq 1} n_1^{-s_1} \cdots n_k^{-s_k}.$$

$\boxed{k=1}$ integer values of *Riemann zeta function* $\zeta(s)$.

Euler: $\zeta(s)\pi^{-s} \in \mathbb{Q}$ for s even ≥ 2.

Fact: No known other algebraic relations between values of Riemann zeta function at positive integers.

Expected: there is no further relation :

Are the numbers

$$\pi,\ \zeta(3),\ \zeta(5), \ldots, \zeta(2n+1), \ldots$$

algebraically independent?

Means:
For $n \geq 0$ *and* $P \in \mathbb{Q}[X_0, X_1, \ldots, X_n] \setminus \{0\}$,

$$P\big(\pi, \zeta(3), \zeta(5), \ldots, \zeta(2n+1)\big) \neq 0\ ?$$

F. Lindemann (1882): π is transcendental.

R. Apéry (1978): $\zeta(3)$ is irrational.

T. Rivoal (2000): infinitely many irrational numbers among $\zeta(3),\ \zeta(5), \ldots, \zeta(2n+1), \ldots$

Theorem (T. Rivoal). *Let $\epsilon > 0$. For any sufficiently large n, the $\mathbb{Q}$-vector space spanned by the n numbers*

$$\zeta(3),\ \zeta(5), \ldots, \zeta(2n+1)$$

has dimension

$$\geq \frac{1-\epsilon}{1+\log 2} \cdot \log n.$$

The proof also yields:

There exists on odd integer j with $5 \leq j \leq 169$ such that the three numbers

$$1, \zeta(3),\ \zeta(j)$$

are linearly independent over $\mathbb{Q}$.

2. Sketch of Proof of Rivoal's Theorem

Goal: *Given a sufficiently large odd integer a, construct a sequence of linear forms in $(a+1)/2$ variables, with integer coefficients, such that the numbers*

$$\ell_n = p_{0n} + \sum_{i=1}^{(a-1)/2} p_{in}\zeta(2i+1)$$

satisfy, for $n \to \infty$,

$$|\ell_n| = \alpha^{-n+o(n)}$$

and

$$|p_{in}| \leq \beta^{n+o(n)}$$

with

$$\alpha \simeq a^{2a} \quad \text{and} \quad \beta \simeq (2e)^{2a}.$$

It will follow that the $(a+1)/2$ numbers

$$1,\ \zeta(3),\ \zeta(5),\ \dots, \zeta(a)$$

span a $\mathbb{Q}$-vector space of dimension at least

$$1 + \frac{\log \alpha}{\log \beta} \simeq \frac{\log a}{1 + \log 2}$$

(Nesterenko's Criterion). $\square$

Explicit construction of the linear forms

Previous works of R. Apéry, F. Beukers, E. Nikishin, K. Ball, D. Vasilyev, ...

Pochammer symbol: $(m)_0 = 1$ and, for $k \geq 1$,

$$(m)_k = m(m+1)\dots(m+k-1).$$

Set $r = \big[a(\log a)^{-2}\big]$. Define

$$d_m = \text{l.c.m. of } \{1, 2, \dots, m\},$$

$$R_n(t) = n!^{a-2r}\, \frac{(t-rn+1)_{rn}(t+n+2)_{rn}}{(t+1)_{n+1}^a},$$

$$S_n = \sum_{k=0}^{\infty} R_n(k), \qquad \ell_n = d_{2n}^a S_{2n}.$$

Write the partial fraction expansion

$$R_n(t) = \sum_{i=1}^{a} \sum_{j=0}^{n} \frac{c_{ijn}}{(t+j+1)^i}$$

with

$$c_{ijn} = \frac{1}{(a-i)!} \left(\frac{d}{dt}\right)^{a-i} \big(R_n(t)(t+j+1)^a\big)_{|t=-j-1}.$$

Set $p_{in} = d_{2n} q_{i,2n}$ where

$$q_{0,n} = -\sum_{i=1}^{a} \sum_{j=1}^{n} c_{ijn} \sum_{k=0}^{j-1} \frac{1}{(k+1)^i}$$

and

$$q_{in} = \sum_{j=0}^{n} c_{ijn} \quad (1 \le i \le a).$$

Estimate for $|p_{in}|$:

$$c_{ijn} = \frac{1}{2\pi i} \int_{|t+j+1|=1/2} R_n(t)(t+j+1)^{i-1} dt.$$

Estimate for $|\ell_n|$:

$$S_n = \frac{\big((2r+1)n+1\big)!}{n!^{2r+1}} \cdot I_n,$$

$$I_n = \int_{[0,1]^{a+1}} F(\underline{x}) \cdot \frac{dx_1 dx_2 \ldots dx_{a+1}}{(1 - x_1 x_2 \cdots x_{a+1})^2},$$

$$F(x_1, x_2, \ldots, x_{a+1}) = \left(\frac{\prod_{i=1}^{a+1} x_i^r (1 - x_i)}{(1 - x_1 x_2 \cdots x_{a+1})^{2r+1}} \right)^n.$$

3. Shuffle Product for Series

Reflexion Formula:

$$\begin{aligned}\zeta(s)\zeta(s') &= \sum_{n\geq 1} n^{-s} \cdot \sum_{n'\geq 1} (n')^{-s'} \\ &= \sum_{n>n'\geq 1} n^{-s}(n')^{-s'} + \sum_{n'>n\geq 1} n^{-s}\ (n')^{-s'} \\ &\qquad\qquad\qquad\qquad + \sum_{n\geq 1} n^{-s-s'} \\ &= \zeta(s,s') + \zeta(s',s) + \zeta(s+s').\end{aligned}$$

Example:

$$\zeta(s)^2 = 2\zeta(s,s) + \zeta(2s).$$

For $s = 2$: $\quad \zeta(2) = \pi^2/6,\ \zeta(4) = \pi^4/90,$

$$\zeta(2,2) = \sum_{m>n\geq 1} (mn)^{-2} = \frac{\pi^4}{120}.$$

Other example:

$$\zeta(2)\zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5).$$

Shuffle relations arising from the series representation.

$$\boxed{\zeta(\underline{s})\zeta(\underline{s}') = \sum_{\underline{\sigma}} \zeta(\underline{\sigma}),}$$

where $\underline{\sigma} = (\sigma_1, \ldots, \sigma_h)$ ranges over the tuples obtained as follows:

$$\begin{array}{llccccl} \underline{s} \rightarrow & (& s_1 & 0 & s_2 & \cdots & s_k &) \\ \underline{s}' \rightarrow & (& 0 & s'_1 & s'_2 & \cdots & 0 &) \\ & & & & & & & \\ \underline{\sigma} = & (& s_1 & s'_1 & s_2 + s'_2 & \cdots & s_k &) \end{array}$$

Hence $\max\{k, k'\} \leq h \leq k + k'$.

<u>Example</u>: $k = k' = 1$, $\underline{s} = s$, $\underline{s}' = s'$, then

$$\begin{array}{llll} s \rightarrow & (s \quad 0) & (0 \quad s) & s \\ s' \rightarrow & (0 \quad s) & (s' \quad 0) & s' \\ & & & \\ \sigma = & (s \quad s') & (s' \quad s) & s + s' \end{array}$$

so that

$$\{\underline{\sigma}_1, \underline{\sigma}_2, \underline{\sigma}_3\} = \{(s, s'),\ (s', s),\ s + s'\}.$$

Other Description:

Alphabet with two letters $X = \{x_0, x_1\}$.

Words: $$X^* = \{x_0^{a_1} x_1^{b_1} \cdots x_0^{a_h} x_1^{b_h}\}.$$

Non-commutative polynomials: $\mathbb{Q}\langle X\rangle$.

For $s \geq 1$ set $y_s = x_0^{s-1} x_1$.

For $\underline{s} = (s_1, \ldots, s_k)$ with $s_i \geq 1$, set

$$\begin{aligned} x_{\underline{s}} &= y_{s_1} \cdots y_{s_k} \\ &= x_0^{s_1 - 1} x_1 x_0^{s_2 - 1} x_1 \cdots x_0^{s_k - 1} x_1. \end{aligned}$$

The number k of factors x_1 is the *depth* of the word $x_{\underline{s}}$ and of the tuple $\underline{s}$.
The number $p = s_1 + \cdots + s_k$ of letters is the *weight*.

The set of such $x_{\underline{s}}$'s is $X^* x_1$ togeether with the null word e (corresponds to $\emptyset$ with $k = 0$).

Convergent words: $x_0 X^* x_1 \bigcup \{e\}$.

Set $\boxed{\zeta(w) = \zeta(\underline{s}) \text{ for } w = x_{\underline{s}}}$ with $\zeta(\emptyset) = \zeta(e) = 1$.

Convergent polynomials: $\mathbb{Q}\langle X\rangle_{\text{conv}} \subset \mathbb{Q}\langle X\rangle$.
Extend ζ by linearity to $\mathbb{Q}\langle X\rangle_{\text{conv}}$.

Law $*$ on $\mathbb{Q}\langle X\rangle_{\text{conv}}$:

$$e * w = w \quad \text{for} \quad w \in X^* x_1$$

and, for $s \geq 1$ and $t \geq 1$, w and w' in $X^* x_1$,

$$(y_s w) * (y_t w') = \\ y_s(w * y_t w') + y_t(y_s w * w') + y_{s+t}(w * w').$$

Then

$$x_{\underline{s}} * x_{\underline{s}'} = \sum_{\underline{\sigma}} x_{\underline{\sigma}}.$$

Proposition. *For w and w' in $x_0 X^* x_1$,*

$$\zeta(w)\zeta(w') = \zeta(w * w').$$

<u>Connection with quasi-symmetric functions</u>

Commutative infinite alphabet: $\underline{t} = \{t_1, t_2, \ldots\}$

Formal power series: $\mathbb{Q}[[\underline{t}]]$.

To $w = x_{\underline{s}} \in X^* x_1$ associate

$$F_w(\underline{t}) = \sum_{n_1 > \cdots > n_k \geq 1} t_{n_1}^{s_1} \cdots t_{n_k}^{s_k}.$$

Then for w and w' in $X^* x_1$ we have

$$F_w(\underline{t}) F_{w'}(\underline{t}) = F_{w * w'}(\underline{t}).$$

For $w \in x_0 X^* x_1$, $\zeta(w)$ is the value of $F_w(\underline{t})$ with $t_n = 1/n$, $(n \geq 1)$.

4. Shuffle Product for Integrals

Let $\underline{s} = (s_1, \ldots, s_k)$ with $s_i \geq 1$; set $p = s_1 + \cdots + s_k$. Define $\epsilon_i \in \{0, 1\}$ for $1 \leq i \leq p$ by

$$x_{\underline{s}} = x_{\epsilon_1} \cdots x_{\epsilon_p}.$$

For instance for $\underline{s} = (2, 3)$ with $p = 5$:

$$x_{(2,3)} = x_0 x_1 x_0^2 x_1, \quad (\epsilon_1, \ldots, \epsilon_5) = (0, 1, 0, 0, 1).$$

Define differential forms:

$$\omega_0(t) = \frac{dt}{t} \quad \text{et} \quad \omega_1(t) = \frac{dt}{1-t}\cdot$$

Let Δ_p be the simplex in $\mathbb{R}^p$:

$$\Delta_p = \{\underline{t} \in \mathbb{R}^p \; ; \; 1 > t_1 > \cdots > t_p > 0\}.$$

Proposition. *For* $s_1 \geq 2$,

$$\zeta(\underline{s}) = \int_{\Delta_p} \omega_{\epsilon_1}(t_1) \cdots \omega_{\epsilon_p}(t_p).$$

Example:

$$\zeta(2,3) = \int_{\Delta_5} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2} \cdot \frac{dt_3}{t_3} \cdot \frac{dt_4}{t_4} \cdot \frac{dt_5}{1-t_5}\cdot$$

Proof. Expand $1/(1-t) = \sum_{s \geq 0} t^s$. □

Shuffle on X^*:

$$e ⧢ w = w ⧢ e = w,$$

and, for i and j in $\{0,1\}$, u and v in X^*,

$$(x_i u) ⧢ (x_j v) = x_i(u ⧢ x_j v) + x_j(x_i u ⧢ v)$$

Example. Computation of $y_2 ⧢ y_3 = x_0x_1 ⧢ x_0^2x_1$: get $x_0x_1x_0^2x_1$ once, $x_0^2x_1x_0x_1$ three times and $x_0^3x_1^2$ six times. Hence

$$y_2 ⧢ y_3 = y_2y_3 + 3y_3y_2 + 6y_4y_1.$$

Corollary.

$$\zeta(w)\zeta(w') = \zeta(w ⧢ w')$$

for w and w' in $x_0X^*x_1$.

Proof. The Cartesian product $\Delta_p \times \Delta_{p'}$ is the union of $(p+p')!/p!p'!$ simplices. $\square$

<u>Example</u>. From

$$y_2 ⧢ y_3 = y_2 y_3 + 3 y_3 y_2 + 6 y_4 y_1$$

we deduce

$$\zeta(2)\zeta(3) = \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1).$$

On the other hand the shuffle relation for series gives

$$\zeta(2)\zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5),$$

hence

$$\zeta(5) = 2\zeta(3,2) + 6\zeta(4,1).$$

<u>*There are further relations*</u>.

<u>Example</u>:

$$x_1 ⧢ x_0 x_1 = x_1 x_0 x_1 + 2 x_0 x_1^2$$

and

$$x_1 * x_0 x_1 = x_1 x_0 x_1 + x_0 x_1^2 + x_0^2 x_1,$$

hence

$$x_1 ⧢ x_0 x_1 - x_1 * x_0 x_1 = x_0 x_1^2 - x_0^2 x_1.$$

<u>Fact</u>: $$\zeta(x_0 x_1^2) = \zeta(x_0^2 x_1).$$

Euler : $$\zeta(2,1) = \zeta(3).$$

Proposition. *For w and w' in $x_0X^*x_1$,*

$$\zeta(w)\zeta(w') = \zeta(w * w'),$$

$$\zeta(w)\zeta(w') = \zeta(w ⧢ w')$$

and

$$\zeta(x_1 ⧢ w - x_1 * w) = 0.$$

5. Symbolic Multizeta

Define $\mathrm{Ze}(\underline{s})$ for each $\underline{s} = (s_1, \dots, s_k)$, with $k \geq 0$ and $s_i \geq 1$. Next define $\mathrm{Ze}(w)$ for w in X^*x_1 by $\mathrm{Ze}(x_{\underline{s}}) = \mathrm{Ze}(\underline{s})$. Convergent symbols: $\mathrm{Ze}(\underline{s})$ with $s_1 \geq 2$ or $k = 0$; these are the $\mathrm{Ze}(w)$ with w in $x_0X^*x_1$ together with $\mathrm{Ze}(e) = \mathrm{Ze}(\emptyset)$.

Algebra of Convergent MZV:
$\mathrm{MZV}_{\mathrm{conv}}$ is the commutative algebra over $\mathbb{Q}$ generated by the convergent symbols $\mathrm{Ze}(\underline{s})$ with the relations

$$\mathrm{Ze}(w)\mathrm{Ze}(w') = \mathrm{Ze}(w * w'),$$

$$\mathrm{Ze}(w)\mathrm{Ze}(w') = \mathrm{Ze}(w ⧢ w')$$

and

$$\mathrm{Ze}(x_1 ⧢ w - x_1 * w) = 0$$

for w and w' in $x_0X^*x_1$.

Main Diophantine Conjecture. *The specialization morphism from* $\mathrm{MZV}_{\mathrm{conv}}$ *into* $\mathbb{C}$ *which maps* $\mathrm{Ze}(\underline{s})$ *onto* $\zeta(\underline{s})$ *is injective.*

Algebras MZV^* *and* $\mathrm{MZV}^{\sqcup\!\sqcup}$: generators $\mathrm{Ze}(\underline{s})$ with $\underline{s} = (s_1, \ldots, s_k)$, $k \geq 0$, $s_j \geq 1$, and $*$ (resp. $\sqcup\!\sqcup$) defined by

$$\mathrm{Ze}(w) * \mathrm{Ze}(w') = \mathrm{Ze}(w * w')$$

resp.

$$\mathrm{Ze}(w) \sqcup\!\sqcup \mathrm{Ze}(w') = \mathrm{Ze}(w \sqcup\!\sqcup w')$$

for $w \in X^* x_1$.

<u>Remark</u>.

$$x_1 * x_1 = 2x_1^2 + x_0 x_1 \quad \text{and} \quad x_1 \sqcup\!\sqcup x_1 = 2x_1^2,$$

hence

$$\mathrm{Ze}(x_1) * \mathrm{Ze}(x_1) = 2\mathrm{Ze}(x_1^2) + \mathrm{Ze}(x_0 x_1)$$

while

$$\mathrm{Ze}(x_1) \sqcup\!\sqcup \mathrm{Ze}(x_1) = 2\mathrm{Ze}(x_1^2)$$

and $\zeta(x_0 x_1) = \zeta(2) \neq 0$.

Conjecture of

Zagier, Drinfeld, Kontsevich and Goncharov.

For $p \geq 2$ let d_p denote the dimension of the $\mathbb{Z}$-module in $\mathrm{MZV}_{\mathrm{conv}}$ spanned by the 2^{p-2} elements $\mathrm{Ze}(\underline{s})$ for $\underline{s} = (s_1, \dots, s_k)$ of length p and $s_1 \geq 2$.

Conjecture. *We have*

$$d_1 = 0, \quad d_2 = d_3 = d_4 = 1$$

and

$$d_p = d_{p-2} + d_{p-3} \quad \textit{for} \quad p \geq 4.$$

For each $p \geq 1$, define $\mathcal{Z}_p$ as the $\mathbb{Q}$-vector space spanned by the $\mathrm{Ze}(\underline{s})$ with $\underline{s}$ convergent of weight p; set $\mathcal{Z}_0 = \mathbb{Q}$. Then the sum of $\mathcal{Z}_p$ $(p \geq 0)$ is direct, and the conjecture means

$$\sum_{p \geq 0} q^p \dim_{\mathbb{Q}} \mathcal{Z}_p = \frac{1}{1 - q^2 - q^3}.$$

Remark: $d_p \to \infty$ by Rivoal's result.

6. Further Results

Écalle: for weight ≤ 10, independent generators are

$$\mathrm{Ze}(2),\ \mathrm{Ze}(3),\ \mathrm{Ze}(5),\ \mathrm{Ze}(7),\ \mathrm{Ze}(9),$$

$$\mathrm{Ze}(6,2),\ \mathrm{Ze}(8,2).$$

Polylogarithms
Classical: for $s \geq 1$ and $|z| < 1$,

$$\mathrm{Li}_s(z) = \sum_{n\geq 1} \frac{z^n}{n^s}.$$

Higher dimension: for $\underline{s} = (s_1, \ldots, s_k)$ with $s_i \geq 1$,

$$\mathrm{Li}_{\underline{s}}(z) = \sum_{n_1 > \cdots > n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} \cdots n_k^{s_k}}$$

Then

$$\mathrm{Li}_{u \sqcup\!\sqcup v}(z) = \mathrm{Li}_u(z)\mathrm{Li}_v(z)$$

If $s_1 \geq 2$, then $\mathrm{Li}_{\underline{s}}(1) = \zeta(\underline{s})$.

Work of Petitot (Lille): the functions Li_w, with w in X^*, are linearly independent over $\mathbb{C}$.

7. Related Topics

Further connections with:

Combinatoric (theory of quasisymmetric functions, Radford's Theorem and Lyndon words)

Lie and Hopf algebras

Resurgent series (Écalle's theory)

Mixed Tate motives on Spec$\mathbb{Z}$ (Goncharov's work)

Monodromy of differential equations

Fundamental group of the projective line minus three points and Belyi's Theorem

Absolute Galois group of $\mathbb{Q}$

Group of Grothendieck-Teichmüller

Knots theory and Vassiliev invariants

K-theory

Feynman diagrams and quantum field theory

Quasi-triangular quasi-Hopf algebras

Drinfeld's associator Φ_{KZ} (connexion of Knizhnik-Zamolodchikov).