Computer Algebra for Lattice Path Combinatorics

Alin Bostan

Inria — informatiques mathématiques

based on joint works with

F. Chyzak, M. Van Hoeij, M. Kauers, L. Pech, K. Raschel, B. Salvy

Séminaire de Combinatoire Philippe Flajolet,

Institut Henri Poincaré, March 28, 2013

Why Lattice Paths?

Applications in many areas of science

- probability theory (branching processes, games of chance, . . .)
- operations research (queueing theory, . . .)
- discrete mathematics (permutations, trees, words, urns, . . .)
- statistical physics (Ising model, . . .)

Journal of Statistical Planning and Inference 140 (2010) 2237–2254

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A history and a survey of lattice path enumeration

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ARTICLE INFO

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Keywords:
Lattice path
Reflection principle
Method of images

ABSTRACT

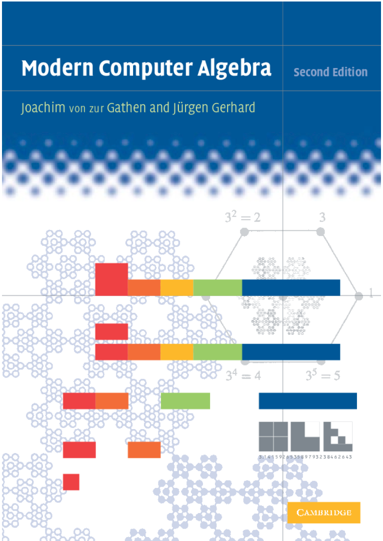
In celebration of the Sixth International Conference on Lattice Path Counting and Applications, it is befitting to review the history of lattice path enumeration and to survey how the topic has progressed thus far.

We start the history with early games of chance specifically the ruin problem which later appears as the ballot problem. We discuss André's Reflection Principle and its misnomer, its relation with the method of images and possible origins from physics and Brownian motion, and the earliest evidence of lattice path techniques and solutions.

In the survey, we give representative articles on lattice path enumeration found in the literature in the last 35 years by the lattice, step set, boundary, characteristics counted, and solution method. Some of this work appears in the author's 2005 dissertation.

Why Computer Algebra?

Because we like it!



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Discrete Mathematics 306 (2006) 992–1021

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Combinatorial aspects of continued fractions

P. Flajolet

IRIA, 78150 Rocquencourt, France

Abstract

We show that the universal continued fraction of the Stieltjes-Jacobi type is equivalent to the characteristic series of labelled paths in the plane. The equivalence holds in the set of series in non-commutative indeterminates. Using it, we derive direct combinatorial proofs of continued fraction expansions for series involving known combinatorial quantities: the Catalan numbers, the Bell and Stirling numbers, the tangent and secant numbers, the Euler and Eulerian numbers. . . . We also show combinatorial interpretations for the coefficients of the elliptic functions, the coefficients of inverses of the Tchebycheff, Charlier, Hermite, Laguerre and Meixner polynomials. Other applications include cycles of binomial coefficients and inversion formulae. Most of the proofs follow from direct geometrical correspondences between objects.

THE EVOLUTION OF TWO STACKS IN BOUNDED SPACE AND RANDOM WALKS IN A TRIANGLE

Philippe FLAJOLET

INRIA
Rocquencourt
78150 Le Chesnay (France)

ABSTRACT

We analyse a simple storage allocation scheme in which two stacks grow and shrink inside a shared memory area. To that purpose, we provide analytic expressions for the number of 2-dimensional random walks in a triangle with two reflecting barriers and one absorbing barrier.

We obtain probability distributions and expectations of characteristic parameters of that shared memory scheme, namely the sizes of the stacks and the time until the system runs out of memory.

This provides a complete solution to an open problem posed by Knuth in "The Art of Computer Programming", Vol. 1, 1968 [Ex. 2.2.2.13].

Theoretical Computer Science 49 (1987) 283–309 283
North-Holland

ANALYTIC MODELS AND AMBIGUITY OF CONTEXT-FREE LANGUAGES*

Philippe FLAJOLET
INRIA, Rocquencourt, 78150 Le Chesnay Cedex, France

Abstract. We establish that several classical context-free languages are inherently ambiguous by proving that their counting generating functions, when considered as analytic functions, exhibit some characteristic form of transcendental behaviour. To that purpose, we survey some general results on elementary analytic properties and enumerative uses of algebraic functions in relation to formal languages. In particular, the paper contains a general density theorem for unambiguous context-free languages.

Adv. Appl. Prob. **32,** 750–778 (2000)
Printed in Northern Ireland

THE FORMAL THEORY OF BIRTH-AND-DEATH PROCESSES, LATTICE PATH COMBINATORICS AND CONTINUED FRACTIONS

PHILIPPE FLAJOLET,* *INRIA*
FABRICE GUILLEMIN,** *France Telecom*

Abstract

Classic works of Karlin and McGregor and Jones and Magnus have established a general correspondence between continuous-time birth-and-death processes and continued fractions of the Stieltjes–Jacobi type together with their associated orthogonal polynomials. This fundamental correspondence is revisited here in the light of the basic relation between weighted lattice paths and continued fractions otherwise known from combinatorial theory. Given that sample paths of the embedded Markov chain of a birth-and-death process are lattice paths, Laplace transforms of a number of transient characteristics can be obtained systematically in terms of a fundamental continued fraction and its family of convergent polynomials. Applications include the analysis of evolutions in a strip, upcrossing and downcrossing times under flooring and ceiling conditions, as well as time, area, or number of transitions while a geometric condition is satisfied.

Keywords: Lattice path combinatorics; continued fractions; orthogonal polynomials; birth-and-death process; first passage time; excursions; transient characteristics.

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Theoretical Computer Science 281 (2002) 37–80

Theoretical Computer Science

www.elsevier.com/locate/tcs

Basic analytic combinatorics of directed lattice paths

Cyril Banderier*, Philippe Flajolet

Algorithms Project, INRIA, Rocquencourt, 78150 Le Chesnay, France

Abstract

This paper develops a unified enumerative and asymptotic theory of *directed two-dimensional lattice paths* in half-planes and quarter-planes. The lattice paths are specified by a finite set of rules that are both time and space homogeneous, and have a privileged direction of increase. (They are then essentially one-dimensional objects.) The theory relies on a specific "kernel method" that provides an important decomposition of the algebraic generating functions involved, as well as on a generic study of singularities of an associated algebraic curve. Consequences are precise computable estimates for the number of lattice paths of a given length under various constraints (bridges, excursions, meanders) as well as a characterization of the limit laws associated to several basic parameters of paths.

Keywords: Lattice path; Analytic combinatorics; Kernel method; Singularity analysis; Generalized ballot problem; Catalan numbers

Séminaire Lotharingien de Combinatoire **54** (2006), Article B54g

THE FERMAT CUBIC, ELLIPTIC FUNCTIONS, CONTINUED FRACTIONS, AND A COMBINATORIAL EXCURSION

ERIC VAN FOSSEN CONRAD AND PHILIPPE FLAJOLET

Kindly dedicated to Gérard···Xavier Viennot on the occasion of his sixtieth birthday.

ABSTRACT. Elliptic functions considered by Dixon in the nineteenth century and related to Fermat's cubic, $x^3+y^3=1$, lead to a new set of continued fraction expansions with sextic numerators and cubic denominators. The functions and the fractions are pregnant with interesting combinatorics, including a special Pólya urn, a continuous-time branching process of the Yule type, as well as permutations satisfying various constraints that involve either parity of levels of elements or a repetitive pattern of order three. The combinatorial models are related to but different from models of elliptic functions earlier introduced by Viennot, Flajolet, Dumont, and Françon.

The Annals of Probability
2005, Vol. 33, No. 3, 1200–1233
DOI 10.1214/009117905000000026

ANALYTIC URNS

BY PHILIPPE FLAJOLET, JOAQUIM GABARRÓ AND HELMUT PEKARI

INRIA Rocquencourt, Universitat Politècnica de Catalunya and Universitat Politècnica de Catalunya

This article describes a purely analytic approach to urn models of the generalized or extended Pólya–Eggenberger type, in the case of *two* types of balls and constant "balance," that is, constant row sum. The treatment starts from a quasilinear first-order partial differential equation associated with a combinatorial renormalization of the model and bases itself on elementary conformal mapping arguments coupled with singularity analysis techniques. Probabilistic consequences in the case of "subtractive" urns are new representations for the probability distribution of the urn's composition at any time n, structural information on the shape of moments of all orders, estimates of the speed of convergence to the Gaussian limit and an explicit determination of the associated large deviation function. In the general case, analytic solutions involve Abelian integrals over the Fermat curve $x^h + y^h = 1$. Several urn models, including a classical one associated with balanced trees (2–3 trees and fringe-balanced search trees) and related to a previous study of Panholzer and Prodinger, as well as all urns of balance 1 or 2 and a sporadic urn of balance 3, are shown to admit of explicit representations in terms of Weierstraß elliptic functions: these elliptic models appear precisely to correspond to regular tessellations of the Euclidean plane.

Fourth Colloquium on Mathematics and Computer Science DMTCS proc. **AG**, 2006, 59–118

Some exactly solvable models of urn process theory

Philippe Flajolet, Philippe Dumas, and Vincent Puyhaubert

Algorithms Project, INRIA, F-78153 Le Chesnay (France)

We establish a fundamental isomorphism between discrete-time balanced urn processes and certain ordinary differential systems, which are nonlinear, autonomous, and of a simple monomial form. As a consequence, all balanced urn processes with balls of two colours are proved to be analytically solvable in finite terms. The corresponding generating functions are expressed in terms of certain Abelian integrals over curves of the Fermat type (which are also hypergeometric functions), together with their inverses. A consequence is the unification of the analyses of many classical models, including those related to the coupon collector's problem, particle transfer (the Ehrenfest model), Friedman's "adverse campaign" and Pólya's contagion model, as well as the OK Corral model (a basic case of Lanchester's theory of conflicts). In each case, it is possible to quantify very precisely the probable composition of the urn at any discrete instant. We study here in detail "semi-sacrificial" urns, for which the following are obtained: a Gaussian limiting distribution with speed of convergence estimates as well as a characterization of the large and extreme large deviation regimes. We also work out explicitly the case of 2-dimensional triangular models, where local limit laws of the stable type are obtained. A few models of dimension three or greater, e.g., "autistic" (generalized Pólya), cyclic chambers (generalized Ehrenfest), generalized coupon-collector, and triangular urns, are also shown to be exactly solvable.

J. Symbolic Computation (1995) **20**, 653–671

Computer Algebra Libraries for Combinatorial Structures

PHILIPPE FLAJOLET AND BRUNO SALVY

Algorithms Project, INRIA, 78153 Le Chesnay, France

(Received 29 December 1994)

This paper introduces the framework of decomposable combinatorial structures and their traversal algorithms. A combinatorial type is decomposable if it admits a specification in terms of unions, products, sequences, sets, and cycles, either in the labelled or in the unlabelled context. Many properties of decomposable structures are decidable. Generating function equations, counting sequences, and random generation algorithms can be compiled from specifications. Asymptotic properties can be determined automatically for a reasonably large subclass. Maple libraries that implement such decision procedures are briefly surveyed (`LUO`, `combstruct`, `equivalent`). In addition, libraries for manipulating holonomic sequences and functions are presented (`gfun`, `Mgfun`).

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Journal of Symbolic Computation 41 (2006) 1–29

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Fast computation of special resultants

Alin Bostan[a,*], Philippe Flajolet[a], Bruno Salvy[a], Éric Schost[b]

[a] *Algorithms Project, Inria Rocquencourt, 78153 Le Chesnay, France*
[b] *LIX, École polytechnique, 91128 Palaiseau, France*

Received 3 September 2003; accepted 9 July 2005
Available online 25 October 2005

Abstract

We propose fast algorithms for computing *composed products* and *composed sums*, as well as *diamond products* of univariate polynomials. These operations correspond to special multivariate resultants, that we compute using power sums of roots of polynomials, by means of their generating series.

Keywords: Diamond product; Composed product; Composed sum; Complexity; Tellegen's principle

Context: enumeration of lattice walks

▷ *Nearest-neighbor walks in the quarter plane* $\mathbb{N}^2$; admissible steps

$$\mathfrak{S} \subseteq \{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ $\mathfrak{S}$-*walks* = walks in $\mathbb{N}^2$ starting at $(0, 0)$ and using steps in $\mathfrak{S}$.

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$$F_{\mathfrak{S}}(t; x, y) = \sum_{n=0}^{\infty} \Big(\sum_{i,j=0}^{\infty} f_{\mathfrak{S}}(n; i, j) x^i y^j \Big) t^n \ \in \mathbb{Q}[x, y][[t]].$$

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Questions: Given $\mathfrak{S}$, what can be said about $F_{\mathfrak{S}}(t; x, y)$?
Structure? (algebraic / holonomic) *Explicit form*? *Asymptotics*?

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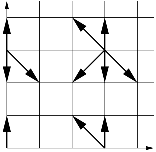
$F_{\mathfrak{S}}(t; 0, 0) \rightsquigarrow$ counts $\mathfrak{S}$-walks returning to the origin (excursions);
$F_{\mathfrak{S}}(t; 1, 1) \rightsquigarrow$ counts $\mathfrak{S}$-walks with prescribed length;
$F_{\mathfrak{S}}(t; 1, 0) \rightsquigarrow$ counts $\mathfrak{S}$-walks ending on the horizontal axis.

Small step sets in the quarter plane

$\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$  There are 2^8 such sets.

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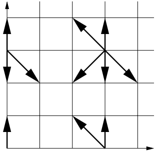
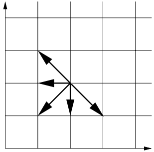
$\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$  There are 2^8 such sets.

Some of these $2^8 = 256$ step sets are:



trivial,

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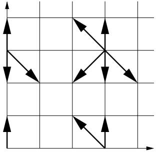
$\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$  There are 2^8 such sets.

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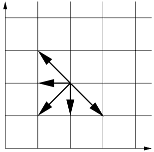
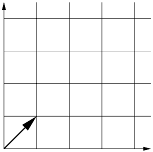
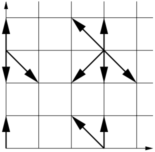
 

trivial, simple,

Small step sets in the quarter plane

$\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$  There are 2^8 such sets.

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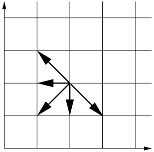

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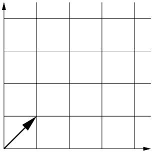

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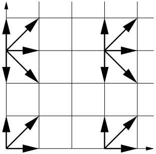
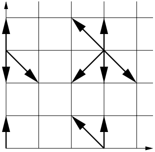


intrinsic to the half plane,

Small step sets in the quarter plane

$\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0,0)\}$  There are 2^8 such sets.

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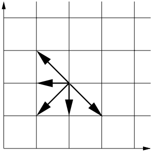

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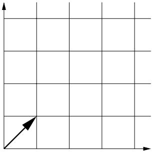

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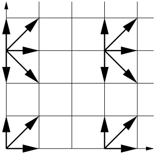

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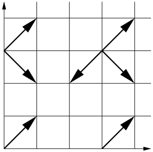

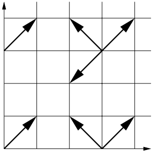

symmetrical.

Small step sets in the quarter plane

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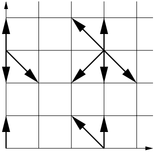

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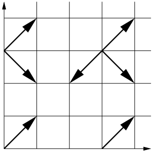

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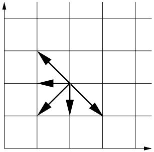

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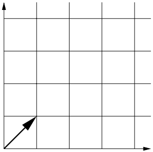

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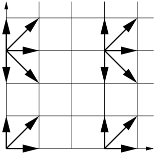

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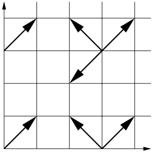


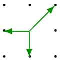
symmetrical.

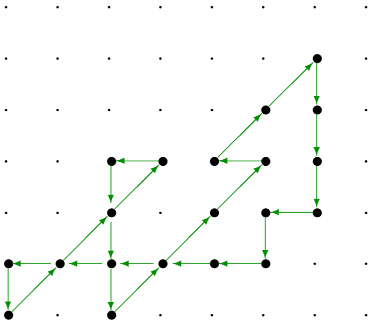
Finally, there remain 79 inherently different cases!

Two important cases: Kreweras and Gessel walks

$\mathfrak{S} = \{\downarrow, \leftarrow, \nearrow\}$ $\quad F_{\mathfrak{S}}(t; x, y) \equiv K(t; x, y)$

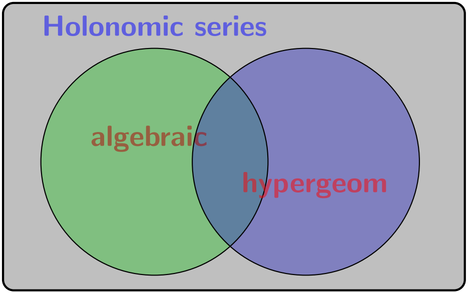
$\mathfrak{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$ $\quad F_{\mathfrak{S}}(t; x, y) \equiv G(t; x, y)$



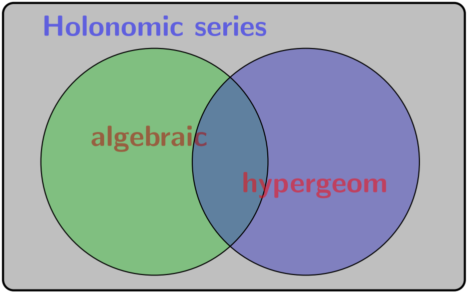


Example: A Kreweras excursion.

Important classes of univariate power series

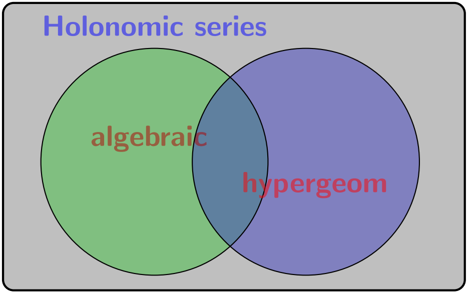


Important classes of univariate power series



Holonomic: $S(t) \in \mathbb{Q}[[t]]$ satisfying a linear differential equation with polynomial coefficients $c_r(t)S^{(r)}(t) + \cdots + c_0(t)S(t) = 0$.

Important classes of univariate power series



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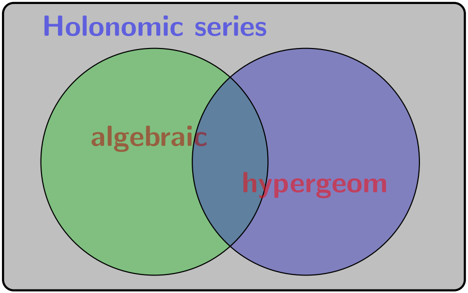
Algebraic: $S(t) \in \mathbb{Q}[[t]]$ root of a polynomial $P \in \mathbb{Q}[t, T]$.

Important classes of univariate power series



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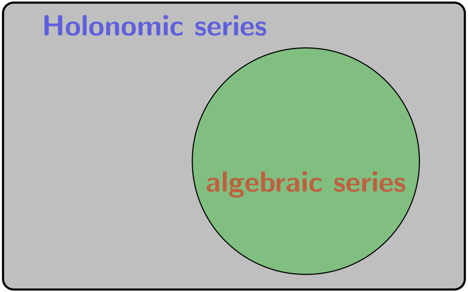
Algebraic: $S(t) \in \mathbb{Q}[[t]]$ root of a polynomial $P \in \mathbb{Q}[t, T]$.

Hypergeometric: $S(t) = \sum_n s_n t^n$ such that $\frac{s_{n+1}}{s_n} \in \mathbb{Q}(n)$. E.g.

$$ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \middle| \, t\right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1)\cdots(a+n-1). $$

Important classes of multivariate power series



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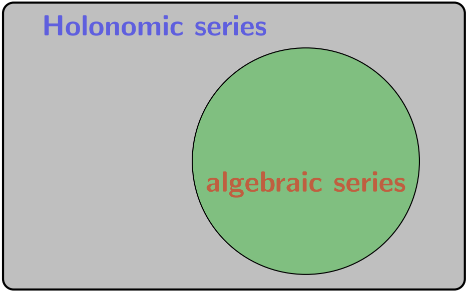

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$S \in \mathbb{Q}[[x, y, t]]$ is *algebraic* if it is the root of a $P \in \mathbb{Q}[x, y, t, T]$.

Main results (I): algebraicity of Gessel walks

Theorem [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\begin{matrix} 1/3 & 2/3 & 1 \\ & 3/2 & 2 \end{matrix} \,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

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▷ Fresh news: human proof just announced [B., Kurkova, Raschel].

Main results (II): Explicit form for $G(t;x,y)$

Theorem Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \cdots$ be a root of

$$(V-1)(1+3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2+83)t^6 + \cdots$ be a root of

$$x(V-1)(V+1)U^3 - 2V(3x+5xV-8Vt)U^2$$
$$-xV(V^2-24V-9)U + 2V^2(xV-9x-8Vt) = 0,$$

let $W = t^2 + (y+8)t^4 + 2(y^2+8y+41)t^6 + \cdots$ be a root of

$$y(1-V)W^3 + y(V+3)W^2 - (V+3)W + V - 1 = 0.$$

Then $G(t;x,y)$ is equal to

$$\frac{\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t-xy} - \frac{1}{tx(y+1)}.$$

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De : **Philippe Flajolet**
Objet : Rép : ca y est...
Date : 7 août 2008 09:49:21 HAEC
À : Alin Bostan
Cc : Bruno Salvy , Philippe Flajolet

La courbe P(x,y)=0 est de genre 0 et possede une merveilleuse
parametrisation rationnelle:

x = -(t-1)*(t^2-t+1)^3/(t^2*(t-2)^6), y = t*(t^3-2)*(t-2)^3/((t-1)*(t^2-t+1)^3);

Y a de la structure....

Phil

Main results (III): Experimental classification of walks with holonomic $F_{\mathfrak{S}}(t;1,1)$ [B. & Kauers 2009]

OEIS Tag	Sample step set	Equation sizes			OEIS Tag	Sample step set	Equation sizes		
A000012		1, 0	1, 1	1, 1	A000079		1, 0	1, 1	1, 1
A001405		2, 1	2, 3	2, 2	A000244		1, 0	1, 1	1, 1
A001006		2, 1	2, 3	2, 2	A005773		2, 1	2, 3	2, 2
A126087		3, 1	2, 5	2, 2	A151255		6, 8	4, 16	–
A151265		6, 4	4, 9	6, 8	A151266		7, 10	5, 16	–
A151278		7, 4	4, 12	6, 8	A151281		3, 1	2, 5	2, 2
A005558		2, 3	3, 5	–	A005566		2, 2	3, 4	–
A018224		2, 3	3, 5	–	A060899		2, 1	2, 3	2, 2
A060900		2, 3	3, 5	8, 9	A128386		3, 1	2, 5	2, 2
A129637		3, 1	2, 5	2, 2	A151261		5, 8	4, 15	–
A151282		3, 1	2, 5	2, 2	A151291		6, 10	5, 15	–
A151275		9, 18	5, 24	–	A151287		7, 11	5, 19	–
A151292		3, 1	2, 5	2, 2	A151302		9, 18	5, 24	–
A151307		8, 15	5, 20	–	A151318		2, 1	2, 3	2, 2
A129400		2, 1	2, 3	2, 2	A151297		7, 11	5, 18	–
A151312		4, 5	3, 8	–	A151323		2, 1	2, 3	4, 4
A151326		7, 14	5, 18	–	A151314		9, 18	5, 24	–
A151329		9, 18	5, 24	–	A151331		3, 4	3, 6	–

Equation sizes = {order, degree}(rec, diffeq, algeq).

▷ Computer-driven; confirmed by human proofs in [Bousquet-Mélou & Mishna, 2010].

Experimental classification of walks with holonomic $F_{\mathfrak{S}}(t;1,1)$

OEIS Tag	Steps	Equation sizes			Asymptotics	OEIS Tag	Steps	Equation sizes			Asymptotics
A000012		1, 0	1, 1	1, 1	1	A000079		1, 0	1, 1	1, 1	2^n
A001405		2, 1	2, 3	2, 2	$\frac{\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{2^n}{\sqrt{n}}$	A000244		1, 0	1, 1	1, 1	3^n
A001006		2, 1	2, 3	2, 2	$\frac{3\sqrt{3}}{2\Gamma(\frac{1}{2})}\frac{3^n}{n^{3/2}}$	A005773		2, 1	2, 3	2, 2	$\frac{\sqrt{3}}{\Gamma(\frac{1}{2})}\frac{3^n}{\sqrt{n}}$
A126087		3, 1	2, 5	2, 2	$\frac{12\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{2^{3n/2}}{n^{3/2}}$	A151255		6, 8	4, 16	–	$\frac{24\sqrt{2}}{\pi}\frac{2^{3n/2}}{n^2}$
A151265		6, 4	4, 9	6, 8	$\frac{2\sqrt{2}}{\Gamma(\frac{1}{4})}\frac{3^n}{n^{3/4}}$	A151266		7, 10	5, 16	–	$\frac{\sqrt{3}}{2\Gamma(\frac{1}{2})}\frac{3^n}{\sqrt{n}}$
A151278		7, 4	4, 12	6, 8	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(\frac{1}{4})}\frac{3^n}{n^{3/4}}$	A151281		3, 1	2, 5	2, 2	$\frac{1}{2}3^n$
A005558		2, 3	3, 5	–	$\frac{8}{\pi}\frac{4^n}{n^2}$	A005566		2, 2	3, 4	–	$\frac{4}{\pi}\frac{4^n}{n}$
A018224		2, 3	3, 5	–	$\frac{2}{\pi}\frac{4^n}{n}$	A060899		2, 1	2, 3	2, 2	$\frac{\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{4^n}{\sqrt{n}}$
A060900		2, 3	3, 5	8, 9	$\frac{4\sqrt{3}}{3\Gamma(\frac{1}{3})}\frac{4^n}{n^{2/3}}$	A128386		3, 1	2, 5	2, 2	$\frac{6\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{2^n3^{n/2}}{n^{3/2}}$
A129637		3, 1	2, 5	2, 2	$\frac{1}{2}4^n$	A151261		5, 8	4, 15	–	$\frac{12\sqrt{3}}{\pi}\frac{2^n3^{n/2}}{n^2}$
A151282		3, 1	2, 5	2, 2	$\frac{A^2B^{3/2}}{2^{3/4}\Gamma(\frac{1}{2})}\frac{B^n}{n^{3/2}}$	A151291		6, 10	5, 15	–	$\frac{4}{3\Gamma(\frac{1}{2})}\frac{4^n}{\sqrt{n}}$
A151275		9, 18	5, 24	–	$\frac{12\sqrt{30}}{\pi}\frac{(\sqrt{24})^n}{n^2}$	A151287		7, 11	5, 19	–	$\frac{\sqrt{8}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
A151292		3, 1	2, 5	2, 2	$\frac{\sqrt[4]{3}C^2D^{3/2}}{8\Gamma(\frac{1}{2})}\frac{D^n}{n^{3/2}}$	A151302		9, 18	5, 24	–	$\frac{\sqrt{5}}{3\sqrt{2}\Gamma(\frac{1}{2})}\frac{5^n}{\sqrt{n}}$
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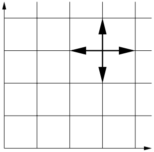
A151312		4, 5	3, 8	–	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	A151323		2, 1	2, 3	4, 4	$\frac{\sqrt{2}\,3^{3/4}}{\Gamma(\frac{1}{4})}\frac{6^n}{n^{3/4}}$
A151326		7, 14	5, 18	–	$\frac{2\sqrt{3}}{3\Gamma(\frac{1}{2})}\frac{6^n}{\sqrt{n}}$	A151314		9, 18	5, 24	–	$\frac{EF^{7/2}}{5\sqrt{95}\pi}\frac{(2F)^n}{n^2}$
A151329		9, 18	5, 24	–	$\frac{\sqrt{7/3}}{3\Gamma(\frac{1}{2})}\frac{7^n}{\sqrt{n}}$	A151331		3, 4	3, 6	–	$\frac{8}{3\pi}\frac{8^n}{n}$

▷ Computer-driven; recent human proofs of asymptotics by [Fayolle & Raschel, 2012].

The group of a walk: an example



The characteristic polynomial $\chi_{\mathfrak{S}} = x + \frac{1}{x} + y + \frac{1}{y}$

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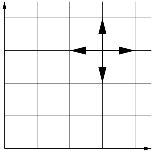

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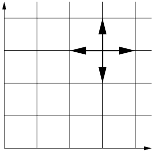

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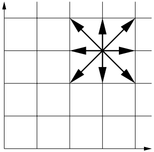
and thus under any element of the group

$$\langle \psi, \phi \rangle = \left\{ (x, y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

The group of a walk: the general case



The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j)\in\mathfrak{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$

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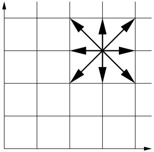

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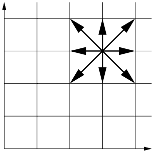

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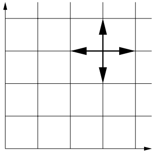
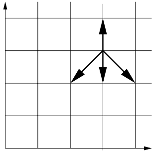
and thus under any element of the group

$$\mathcal{G}_{\mathfrak{S}} := \langle \psi, \phi \rangle.$$

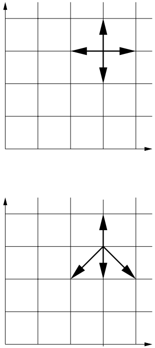
Examples of groups of walks

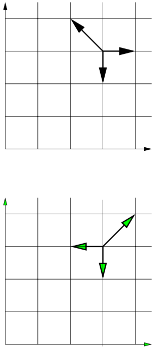




Order 4,

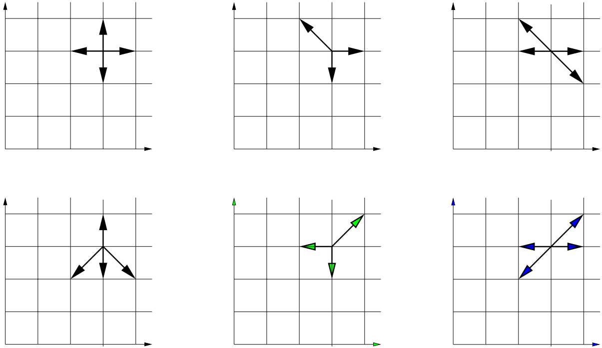
Examples of groups of walks





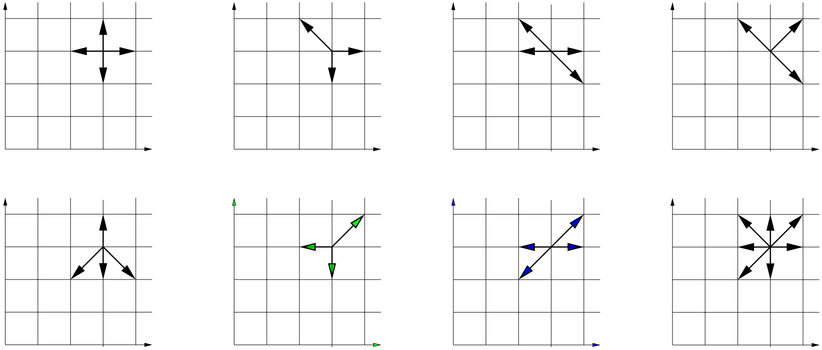
Order 4, order 6,

Examples of groups of walks



Order 4, order 6, order 8,

Examples of groups of walks



Order 4, order 6, order 8, order ∞.

The 79 cases: finite and infinite groups

79 step sets

The 79 cases: finite and infinite groups

79 step sets

→ 23 admit a finite group

[Mishna'07]

→ 56 have an infinite group

[Bousquet-Mélou-Mishna'10]

The 79 cases: finite and infinite groups

79 step sets

23 admit a finite group
[Mishna'07]

all are holonomic
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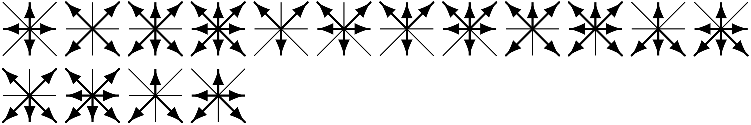
→ 56 have an infinite group [Bousquet-Mélou-Mishna'10] ⟶ all non-holonomic

- [Mishna-Rechnitzer'07] and [Melczer-Mishna'12] for 5 singular cases
- [Kurkova-Raschel'12] and [B.-Raschel-Salvy'12] for all others

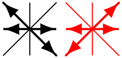
The 23 cases with a finite group

(i) 16 with a *vertical symmetry*, and group isomorphic to D_2



(ii) 5 with a *diagonal* or an *anti-diagonal symmetry*, and group isomorphic to D_3

(iii) 2 with group isomorphic to D_4



In red, cases with algebraic generating functions

(ii)+(iii): zero drift $\sum_{s \in \mathfrak{S}} s$

Main results (IV): explicit expressions for the 19 holonomic transcendental cases

Theorem [B.-Chyzak-Van Hoeij-Kauers-Pech, 2011]

Let $\mathfrak{S}$ be one of the 19 step sets with finite group $\mathcal{G}_{\mathfrak{S}}$, and such that the generating series $F = F_{\mathfrak{S}}(t; x, y)$ is not algebraic.
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Example (King walks in the quarter plane, A025595)

F  $(t; 1, 1) = \dfrac{1}{t} \displaystyle\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix} \frac{3}{2} & \frac{3}{2} \\ & 2 \end{matrix} \,\middle|\, \frac{16x(1+x)}{(1+4x)^2} \right) dx$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

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▷ Computer-driven discovery and proof; no human proof yet.
▷ Proof uses creative telescoping, ODE factorization, ODE solving.

Main results (V): algorithmic proof of non-holonomy for the 51 non-singular cases with infinite group

Theorem [B.-Rachel-Salvy, 2012]

Let $\mathfrak{S}$ be one of the 51 step sets with infinite group $\mathcal{G}_{\mathfrak{S}}$, and such that the excursions series $F_{\mathfrak{S}}(t;0,0)$ is not equal to 1.
Then $F_{\mathfrak{S}}(t;0,0)$, and in particular $F_{\mathfrak{S}}(t;x,y)$, are non-holonomic.

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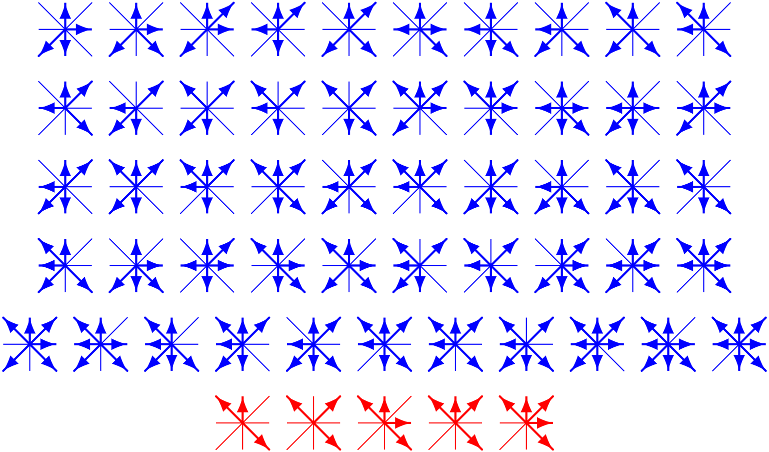
▷ *Algorithmic, computer-driven, proof*. Uses Gröbner basis computations, polynomial factorization, cyclotomy testing.
▷ Based on *two ingredients*: asymptotics + irrationality.

▷ [Kurkova & Raschel 2012] Alternative proof of $F_{\mathfrak{S}}(t;x,y)$ is non-holonomic. No human proof yet for $F_{\mathfrak{S}}(t;0,0)$ *non-holonomic*.

The 56 cases with infinite group



In blue, non-singular cases, solved by [B., Raschel & Salvy, 2012]
In red, singular cases, solved by [Melczer & Mishna 2012]

Summary – classification of 2D non-singular walks

The Big Theorem Let $\mathfrak{S}$ be one of the 74 non-singular step sets. The following assertions are equivalent:

(1) The full generating series $F_{\mathfrak{S}}(t;x,y)$ is holonomic

(2) the excursions generating series $F_{\mathfrak{S}}(t;0,0)$ is holonomic

(3) the excursions seq. $[t^n]\, F_{\mathfrak{S}}(t;0,0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$

(4) the group $\mathcal{G}_{\mathfrak{S}}$ is finite (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^\star \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)

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In this case, $F_{\mathfrak{S}}(t; x, y)$ is expressible using nested radicals. If not, $F_{\mathfrak{S}}(t; x, y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

Main methods

(1) for proving non-holonomy
  - (1a) Infinite number of singularities, or lacunary
  - (1b) Asymptotics

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  - (2a) Diagonals, or positive parts, of rational functions
  - (2b) Guess'n'Prove

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▷ All methods are algorithmic.

Methodology for proving algebraicity

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Experimental mathematics –Guess'n'Prove– approach:

(S1) high order expansion of the generating series $F_{\mathfrak{S}}(t; x, y)$;

(S2) guessing candidates for minimal polynomials of $F_{\mathfrak{S}}(t; x, 0)$ and $F_{\mathfrak{S}}(t; 0, y)$, based on Hermite-Padé approximation;

(S3) rigorous certification of the minimal polynomials, based on (exact) polynomial computations.

Step (S1): high order series expansions

$f_{\mathfrak{S}}(n; i, j)$ satisfies the recurrence with constant coefficients

$$f_{\mathfrak{S}}(n+1; i, j) = \sum_{(u,v)\in\mathfrak{S}} f_{\mathfrak{S}}(n; i-u, j-v) \quad \text{for} \quad n, i, j \geq 0$$

$+$ init. cond. $f_{\mathfrak{S}}(0; i, j) = \delta_{0,i,j}$ and $f_{\mathfrak{S}}(n; -1, j) = f_{\mathfrak{S}}(n; i, -1) = 0$.

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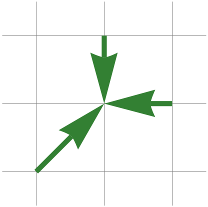
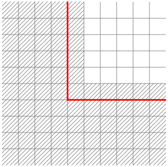
Example: for the Kreweras walks,

$$\begin{aligned} k(n+1; i, j) = {} & k(n; i+1, j) \\ & + k(n; i, j+1) \\ & + k(n; i-1, j-1) \end{aligned}$$





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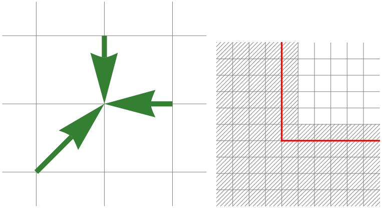

▷ Recurrence is used to compute $F_{\mathfrak{S}}(t;x,y) \bmod t^N$ for large N.

$$\begin{aligned} K(t;x,y) &= 1 + xyt + (x^2y^2+y+x)t^2 + (x^3y^3+2xy^2+2x^2y+2)t^3 \\ &+ (x^4y^4+3x^2y^3+3x^3y^2+2y^2+6xy+2x^2)t^4 \\ &+ (x^5y^5+4x^3y^4+4x^4y^3+5xy^3+12x^2y^2+5x^3y+8y+8x)t^5+\cdots \end{aligned}$$

Step (S2): guessing equations for $F_{\mathfrak{S}}(t;x,y)$, a first idea

In terms of generating series, the recurrence on $k(n;i,j)$ reads

$$\boxed{\begin{aligned}&\big(xy-(x+y+x^2y^2)t\big)K(t;x,y)\\ &\qquad = xy - xt\,K(t;x,0) - yt\,K(t;0,y)\end{aligned}} \qquad \text{(KerEq)}$$

▷ This *kernel equation* can be seen as a multivariate analogue of $(1-t-t^2)\cdot\sum_{n\geq 0}\ell_n t^n = 1,$ where ℓ_n are the Fibonacci numbers.

▷ A similar kernel equation holds for $F_{\mathfrak{S}}(t;x,y)$, for any $\mathfrak{S}$-walk.

Corollary. $F_{\mathfrak{S}}(t;x,y)$ is holonomic (resp. algebraic) if and only if $F_{\mathfrak{S}}(t;x,0)$ and $F_{\mathfrak{S}}(t;0,y)$ are both holonomic (resp. algebraic).

▷ Crucial simplification: equations for $G(t;x,y)$ are huge ($\approx$30Gb)

Step (S2): guessing equations for $F_{\mathfrak{S}}(t; x, 0)$ & $F_{\mathfrak{S}}(t; 0, y)$

Task 1: Given the first N terms of $S = F_{\mathfrak{S}}(t; x, 0) \in \mathbb{Q}[x][[t]]$, search for a *differential equation* satisfied by S at precision N:

$$\mathcal{L}_{x,0}(S) = c_r(x,t) \cdot \frac{\partial^r S}{\partial t^r} + \cdots + c_1(x,t) \cdot \frac{\partial S}{\partial t} + c_0(x,t) \cdot S = 0 \bmod t^N.$$

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- Both tasks amount to linear algebra in size N over $\mathbb{Q}(x)$.
- In practice, we use modular Hermite-Padé approximation (Beckermann-Labahn algorithm) combined with (rational) evaluation-interpolation and rational number reconstruction.
- Fast (FFT-based) arithmetic in $\mathbb{F}_p[t]$ and $\mathbb{F}_p[t]\langle\frac{t}{\partial t}\rangle$.

Step (S2): guessing equations for $G(t; x, 0)$ and $G(t; 0, y)$

Using $N = 1200$ terms of $G(t; x, y)$, we guessed candidates

- $\mathcal{P}_{x,0}$ in $\mathbb{Z}[x, t, T]$ of tridegree $(32, 43, 24)$, 21 digits coefficients
- $\mathcal{P}_{0,y}$ in $\mathbb{Z}[y, t, T]$ of tridegree $(40, 44, 24)$, 23 digits coefficients

such that

$$\mathcal{P}_{x,0}(x, t, G(t; x, 0)) = \mathcal{P}_{0,y}(y, t, G(t; 0, y)) = 0 \bmod t^{1200}.$$

[†] of order 11, and bidegree $(96, 78)$ for $G(t; x, 0)$, and $(68, 28)$ for $G(t; 0, y)$

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▹ We actually first guessed *differential equations*[†], then computed their *p-curvatures* to empirically certify them. This led to suspect the algebraicity of $G(t; x, 0)$ and $G(t; 0, y)$, using a conjecture of Grothendieck (on differential equations modulo p) as an oracle.

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▷ Guessing $\mathcal{P}_{x,0}$ by *undetermined coefficients* would require solving a dense linear system of size $\approx 100\,000$, and ≈ 1000 digits entries!

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Step (S3): warm-up – Gessel excursions are algebraic

Theorem $G(t;0,0) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}$ *is algebraic*.

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3. $r(t) = \sum_{n=0}^{\infty} r_n t^n$ being algebraic, it is holonomic, and so is (r_n):

$$(n+2)(3n+5)r_{n+1} - 4(6n+5)(2n+1)r_n = 0, \qquad r_0 = 1$$

$\Rightarrow$ solution $r_n = \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}4^{2n} = g_n$, thus $g(t) = r(t)$ is algebraic.

Step (S3): rigorous proof for Kreweras walks

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation

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shows that $U = K(t; x, 0)$ satisfies the *reduced kernel equation*

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2. $U = K(t;x,0)$ is the *unique solution* in $\mathbb{Q}[[x,t]]$ of (RKerEq).

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$$\boxed{0 = x\cdot y_0 - x\cdot t\cdot U(t,x) - y_0\cdot t\cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

2. $U = K(t;x,0)$ is the *unique solution* in $\mathbb{Q}[[x,t]]$ of (RKerEq).

3. The guessed candidate $\mathcal{P}_{x,0}$ has one solution $H(t,x)$ in $\mathbb{Q}[[x,t]]$.

Step (S3): rigorous proof for Kreweras walks

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$\implies \quad U = H(t,x)$ also satisfies (RKerEq).

5. *Uniqueness*: $H(t,x) = K(t;x,0) \implies K(t;x,0)$ is algebraic!

Algebraicity of Kreweras walks: our Maple proof in a nutshell

```
[bostan@inria ~]$ maple
    |\^/|     Maple 17 (APPLE UNIVERSAL OSX)
._|\|   |/|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2013
 \  MAPLE  /  All rights reserved. Maple is a trademark of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.

# HIGH ORDER EXPANSION (S1)
> st,bu:=time(),kernelopts(bytesused):
> f:=proc(n,i,j)
  option remember;
    if i<0 or j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else f(n-1,i-1,j-1)+f(n-1,i,j+1)+f(n-1,i+1,j) fi
  end:
> S:=series(add(add(f(k,i,0)*x^i,i=0..k)*t^k,k=0..80),t,80):

# GUESSING (S2)
> libname:=".",libname:gfun:-version();
                                      3.62
> gfun:-seriestoalgeq(S,Fx(t)):
> P:=collect(numer(subs(Fx(t)=T,%[1])),T):

# RIGOROUS PROOF (S3)
> ker := (T,t,x) -> (x+T+x^2*T^2)*t-x*T:
> pol := unapply(P,T,t,x):
> p1 := resultant(pol(z-T,t,x),ker(t*z,t,x),z):
> p2 := subs(T=x*T,resultant(numer(pol(T/z,t,z)),ker(z,t,x),z)):
> normal(primpart(p1,T)/primpart(p2,T));
                                        1

# time (in sec) and memory consumption (in Mb)
> trunc(time()-st),trunc((kernelopts(bytesused)-bu)/1000^2);
                                     7, 618
```

Step (S3): rigorous proof for Gessel walks

Same philosophy, but several complications:

- stepset diagonal symmetry is lost: $G(t; x, y) \neq G(t; y, x)$;
- $G(t; 0, 0)$ occurs in (KerEq);
- equations are $\approx 5\,000$ times bigger.

$\longrightarrow$ replace (RKerEq) by a *system* of 2 reduced kernel equations.

$\longrightarrow$ fast algorithms needed (e.g., [B.-Flajolet-Salvy-Schost'06] for computations with algebraic series).

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Fast computation of special resultants

Alin Bostan[a,*], Philippe Flajolet[a], Bruno Salvy[a], Éric Schost[b]

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[b] LIX, École polytechnique, 91128 Palaiseau, France

Received 3 September 2003; accepted 9 July 2005
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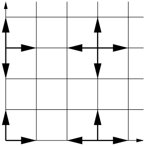
Holonomy via the finite group

Holonomy via the finite group: an example



The polynomial $\chi_{\mathfrak{S}} = \sum_{(i,j)\in\mathfrak{S}} x^i y^j = x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under $(x, y), (\frac{1}{x}, y), (\frac{1}{x}, \frac{1}{y}), (x, \frac{1}{y})$.

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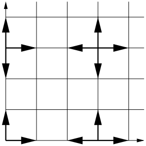

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$$J(t;x,y)xytF(t;x,y) = \quad txF(t;x,0) + tyF(t;0,y) - xy$$

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Summing up yields:

$$\sum_{\theta\in\mathcal{G}_{\mathfrak{S}}} (-1)^\theta \theta\left[xyt\,F(t;x,y)\right] = \frac{-xy + \frac{1}{x}y - \frac{1}{x}\frac{1}{y} + x\frac{1}{y}}{J(t;x,y)}$$

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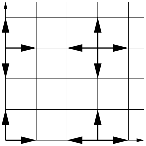

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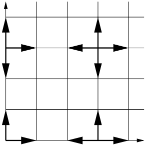

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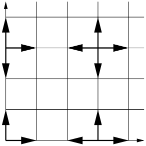

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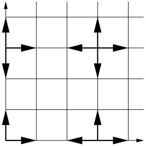

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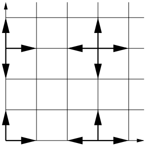

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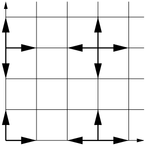

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The 19 transcendental holonomic cases

Theorem [Bousquet-Mélou & Mishna, 2009]
Let $\mathfrak{S}$ be one of the 19 step sets with finite group $\mathcal{G}_{\mathfrak{S}}$, and such that the generating series $F = F_{\mathfrak{S}}(t; x, y)$ is not algebraic. Then:

$$xyt\, F(t; x, y) = [x^{>}][y^{>}]\, \frac{\displaystyle\sum_{\theta \in \mathcal{G}_{\mathfrak{S}}} (-1)^{\theta} \cdot \theta(xy)}{J(t; x, y)}$$

In particular, $F(t; x, y)$ is holonomic.

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In particular, $F(t;x,y)$ is holonomic.

Proof: Use [Lipshitz'88] for positive parts of holonomic series.

If $F(t;x,y)$, as a formal series in t, has its coefficients in $\mathbb{Q}(x)[y,\frac{1}{y}]$, then $[y^{>}]F(t;x,y)$ is algebraic. If in addition $[y^{>}]F(t;x,y)$ as a formal series in t, has its coefficients in $\mathbb{Q}[x,\frac{1}{x},y]$, then $[x^{>}][y^{>}]F(t;x,y)$ is holonomic.

▷ Constructive proof, but it leads to a highly inefficient algorithm.

Explicit expressions for the 19 holonomic cases

Theorem [B.-Chyzak-Van Hoeij-Kauers-Pech, 2011]

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Sketch of the approach

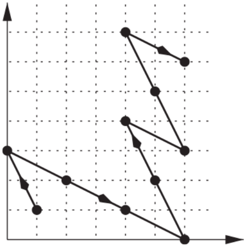
1. (BM&M) If $R = \sum_\theta \frac{(-1)^\theta \, \theta(xy)}{J(t;x,y)}$, then $F = \mathrm{Res}_u \left(\mathrm{Res}_v \, H\right)$, for $H = \frac{R(t;1/u,1/v)}{(1-xu)(1-yv)}$.
2. If $P \in \mathbb{Q}(x, y)[t]\langle\partial_t\rangle$ and $U, V \in \mathbb{Q}(x, y, u, v, t)$ such that $L(H) = \partial_u U + \partial_v V$, then $L(F(t; x, y)) = 0 \longrightarrow$ Chyzak's creative telescoping for finding L.
3. Factor L as $L^{(2)} \cdot L_1^{(1)} \cdots L_t^{(1)}$, then solve $L^{(2)}$ in terms of ${}_2F_1$s, and deduce F.

Proofs of non-holonomy

A historical example

Knight walks: $\mathfrak{S} = \{(-1,2),(2,-1)\}$

$$a_{m,n} = \begin{cases} 0 & \text{if } m < 0 \text{ or } n < 0 \\ 1 & \text{if } m = n = 1 \\ a_{m+1,n-2} + a_{m-2,n+1} & \text{otherwise} \end{cases}$$



↑													
12	24												
11			108										
10		24			312								
9	6			84			720						
8			24			204			1440				
7		6			60			408					
6	2			18			120			720			
5			6			36			204				
4		2			12			60			312		
3	1			4			18			84			
2			2			6			24			108	
1		1			2			6			24		
0				1			2			6			24
	0	1	2	3	4	5	6	7	8	9	10	11	12 →

Theorem [Bousquet-Mélou & Petkovsek'03] The generating series $F(x,y) = \sum_{m,n \geq 0} a_{m,n} x^m y^n$ is not holonomic.

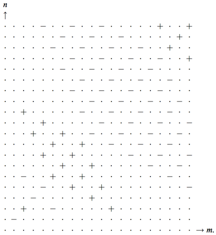
▷ Key argument: $F(x,0)$ has infinitely many singularities.

A historical example

$$a_{m,n} = \begin{cases} a_{m+1,n-2} + a_{m-2,n+1} - a_{m-1,n-1} & \text{if } m, n \geq 2 \\ -\delta_{(m,n),(1,1)} & \text{if } m \leq 1 \text{ or } n \leq 1 \end{cases}$$

n ↑

0	0	0	0	·	·	·	
0	0	−1	0	0	1	·	
0	0	0	0	−1	0	·	
0	0	0	−1	0	0	0	
0	0	1	0	0	−1	0	
0	−1	0	0	0	0	0	
0	0	0	0	0	0	0	→ m.



The series $F(x,y) = \sum_{m,n\geq 2} a_{m,n} x^{m-2} y^{n-2}$ satisfies

$$(x-y^2)(y-x^2)F(x,y) = xy - G(x) - G(y), \text{ for } G(x) = \sum_{m\geq 2} a_{m,2} x^{m+1}$$

$$\Longrightarrow \quad x^3 - G(x) - G(x^2) = 0 \Longrightarrow \quad G(x) = \sum_{i\geq 0} (-1)^i x^{3\cdot 2^i}$$

▷ G is lacunary, thus it is not holonomic, and so is F.

Algorithmic proof of non-holonomy
[B., Raschel & Salvy 2012]

Two ingredients:

1. Asymptotics of excursions:

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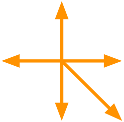
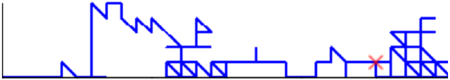
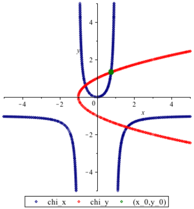
2. If $F_{\mathfrak{S}}(t;0,0)$ is holonomic, then it is a G-function, and $\alpha \in \mathbb{Q}$
(implied by deep number theory results [Chudnovsky-André-Katz])

Algorithmic proof of non-holonomy
[B., Raschel & Salvy 2012]

Irrationality of $\arccos(c)/\pi$ is proven *algorithmically* in two steps:

(S1) determine the minimal polynomial, μ_c, of c

(S2) prove that the numerator of $\mu_c\left(\frac{x^2+1}{2x}\right)$ contains no cyclotomic polynomial factor.

▷ The algorithm proves that $F(t;0,0)$ is non holonomic (and thus so is $F(t;x,y)$) for the 51 non-singular walks with infinite group.

The algorithm on the example

$\mathfrak{S} =$ 

```
> S:=[[-1,0],[0,1],[1,0],[1,-1],[0,-1]]:
> chi:=add(x^s[1]*y^s[2],s=S);
```

$$\chi := \frac{1}{x} + \frac{1}{y} + x + y + \frac{x}{y}$$

```
> chi_x:=numer(diff(chi,x));chi_y:=numer(diff(chi,y));
```

$$\chi_x := x^2 + x^2 y - y, \qquad \chi_y := y^2 - x - 1.$$

```
> G:=Groebner[Basis]([chi_x,chi_y, numer(t^2-
    diff(chi,x,y)^2/diff(chi,x,x)/diff(chi,y,y))],lexdeg([x,y],[t])):
> p:=factor(op(remove(has,G,{x,y})));
```

$$p := (4t^2 + 1)(8t^3 + 8t^2 + 6t + 1)(8t^3 - 8t^2 + 6t - 1).$$

The polynomial p has only two real roots, $\pm c$. Numerical evaluation of c identifies its minimal polynomial as $\mu_c = 8t^3 + 8t^2 + 6t + 1$

```
mu_c:=8*t^3+8*t^2+6*t+1:
R:=expand(x^3*subs(t=(x^2+1)/x/2, mu_c),sort);
```

$$R(x) = x^6 + 2x^5 + 6x^4 + 5x^3 + 6x^2 + 2x + 1.$$

```
> irreduc(R),numtheory[iscyclotomic](R,x);
```

true, false

Summary

- ☺ 2D classification of $F(t;0,0)$ and $F(t;x,y)$ is fully completed
- ☺ robust algorithmic methods:
  - Guess'n'Prove approach based on modern CA algorithms
  - Creative Telescoping for integration of rational functions
- ☹ Brute-force and/or use of naive algorithms = hopeless.
  E.g. size of algebraic equations for $G(t;x,y) \approx 30$Gb.
- ☺ Remarkable properties discovered experimentally. E.g.:
  all algebraic cases have solvable Galois groups

$$G(t;1,1) = -\frac{3}{6t} + \frac{\sqrt{3}}{6t}\sqrt{U(t) + \sqrt{\frac{16t(2t+3)+2}{(1-4t)^2 U(t)} - U(t)^2 + 3}}$$

where $\quad U(t) = \sqrt{1 + 4t^{1/3}(4t+1)^{1/3}/(4t-1)^{4/3}}.$

Summary

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$$\text{where} \quad U(t) = \sqrt{1 + 4t^{1/3}(4t+1)^{1/3}/(4t-1)^{4/3}}.$$

☹ lack of "purely human" proofs for many results. E.g.:
non-holonomy of $F(t;0,0)$ and ${}_2F_1$s expressions for $F(t;x,y)$

☹ still missing a unified proof of: finite group $\leftrightarrow$ holonomic

☹ open: is $F(t;1,1)$ non-holonomic in the 51 non-singular cases with infinite group?

Extensions

1. Longer 2D steps [B., Bousquet-Mélou & Melczer, in progress]
   - 680 step sets with one large step, 643 proven non holonomic, 32 of 37 have differential equations guessed.
   - 5910 step sets with two large steps, 5754 proven non holonomic, 69 of 156 have differential equations guessed.

2. 3D walks [B., Bousquet-Mélou, Kauers & Melczer, in progress]
   - 83 682 with 5 steps or less: B. and Kauers (2009) conjectured (up to equivalence) 35 holonomic steps. Now proved.
   - With 6 steps, 96 new holonomic cases: guessed, then proved.
   - New phenomenon (empirically discovered, no proof yet): $\exists$ step sets (3D Kreweras) with finite group and non-holonomic GF?!

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Thanks for your attention!