Nilpotent Fundamental Groups 17w5112

Ján Mináč (Western University),
Florian Pop (University of Pennsylvania),
Adam Topaz (Oxford),
Kirsten Wickelgren (Georgia Institute of Technology)

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1 Overview of the Field

One of the main guiding principles in modern Galois theory is the deep interaction between arithmetic and geometry. This principle is evident in Grothendieck's étale fundamental group of a geometrically connected variety, which is an extension of the absolute Galois group of the base field by the geometric fundamental group. Such groups have been studied via their nilpotent/unipotent completions for many decades, simply because such completions are more amenable to cohomological calculations. This point of view has seen a recent surge of activity, primarily revolving around new results in anabelian geometry, as well as new structural results about absolute Galois groups arising from the norm-residue isomorphism.

2 Recent Developments

2.1 Anabelian Geometry

Nilpotent and solvable quotients of fundamental/Galois groups play a central role in recent results in anabelian geometry. In relationship with the section conjecture, nilpotent quotients of geometric fundamental groups provide a plethora of obstructions for the existence of sections, and therefore also to the existence of rational points. As mentioned above, the main benefit of working with such obstructions is that they can be studied using cohomological methods. More precisely, these obstructions manifest themselves as Massey products in cohomology; such results have been pervasive in the work of Wickelgren. However, it is still unknown whether these obstructions are complete. For instance, there are known counter-examples due to Hoshi to the geometrically pro-p variant of the section conjecture, as well as counter-examples due to Wickelgren to the geometrically 2-step nilpotent section conjecture. But it is still unknown which minimal variants of the section conjecture are plausible.

Nilpotent quotients of Galois groups also play a key role in the birational setting. For instance, Pop proved a refinement of the birational p-adic section conjecture using mod-p meta-abelian quotients of absolute Galois groups. This strengthens Koenigsmann's original proof of the birational section conjecture which used absolute Galois groups. It's important to mention, however, that many of Pop's methods actually work with the smaller "abelian-by-central" (i.e. 2-step nilpotent) quotients, but it is still open whether or not the mod-p abelian-by-central variant of the birational p-adic section conjecture holds true.

Abelian-by-central quotients of Galois groups play a central role in Bogomolov's program in birational anabelian geometry, whose goal is to reconstruct higher-dimensional geometric function fields using the pro-ℓ

abelian-by-central quotient of their absolute Galois group. This program was formulated into a precise functorial conjecture by Pop, and this formulation is now commonly known as the Bogomolov-Pop conjecture. While the general statement of the conjecture is still wide open, it has been proven in several important cases by Bogomolov-Tschinkel, Pop and Silberstein. If successful, Bogomolov's program would go far beyond Grothendieck's original anabelian philosophy for two primary reasons. First, it considers purely geometric objects whereas Grothendieck's anabelian philosophy revolved around a strong interaction of arithmetic and geometry. And second, and it considers "almost-abelian" Galois groups, whereas Grothendieck's anabelian philosophy revolved around Galois groups which were highly non-abelian.

Some of the key steps in Bogomolov's Program are by now well-developed, but there are still many open questions. For instance, the "local theory" detects decomposition/inertia groups of so-called "quasi-divisorial valuations" using pro-ℓ abelian-by-central Galois groups; this theory was developed over the last several years by Bogomolov-Tschinkel, Pop, and Topaz. It is still a major open problem to give similar recipes for divisorial valuations. The "global theory" is the second major component of Bogomolov's Program, and Pop has some general results in this direction. In the global theory, it is still a major open problem to construct so-called "rational quotients" using group-theoretical methods. Finally, there is some recent work due to Topaz which suggests the plausibility of a mod-ℓ abelian-by-central variant of Bogomolov's Program, although there currently no known complete case of this mod-ℓ variant.

Lastly, we discuss the question of Ihara from the 1980's about finding a (non-tautological) combinatorial description of absolute Galois groups. Being motivated by Ihara's work on braid groups, this combinatorial description should arise from the Galois action of absolute Galois groups on geometric fundamental groups. This question was formulated into a precise conjecture by Oda-Matsumoto, based on motivic evidence, and this question/conjecture is now commonly referred to as the "I/OM." The original I/OM using the full geometric fundamental groups of algebraic varieties was proven by Pop (unpublished) in the 90's. Nevertheless, using anabelian techniques mentioned above, Pop recently proved a strengthening of this result which instead considers the Galois action on pro-ℓ abelian-by-central quotients of geometric fundamental groups. It is still a major open question whether similar results hold for the geometric fundamental group of certain small categories of varieties, such as the Teichmüller modular tower.

2.2 Galois Groups and the Norm-Residue Isomorphism

In another direction, one has the subject of determining explicit properties about the structure of large Galois groups of fields, specifically absolute Galois groups and maximal pro-ℓ Galois groups. Most such results rely on the recent proof of the norm-residue isomorphism theorem due to Voevodsky-Rost et al. This highly-celebrated theorem imposes strict restrictions on the Galois cohomology ring, and determining the precise implications of these restrictions on the structure of absolute Galois groups is an active research area. In the same vein, there is also the open question of making some aspects of the norm-residue isomorphism explicit by tying it to explicit constructions of certain pro-ℓ Galois extensions.

In the last few years, there has been a surge of interest in this area, specifically concentrated around the vanishing of higher Massey products in Galois cohomology. This is particularly highlighted in the works of Hopkins-Wickelgren, Mináč-Tân and Efrat-Matzri, which show that certain higher Massey products vanish over essentially all fields. Still, at this point it remains open whether all higher Massey products vanish when they are defined. In any case, such results have important group-theoretical consequences which are summarized in the so-called Kernel-Unipotent Conjecture. Some partial results towards this conjecture were recently obtained by Efrat-Mináč and Mináč-Tân, but the general conjecture remains open.

A related topic is about formality and 1-formality in Galois cohomology. Massey products are a basic obstruction to the formality and 1-formailty of the differential graded algebra (DGA) underlying Galois cohomology. Positselski has recently provided examples where absolute Galois groups are not formal. This implies that there is more information in the DGA than is available from the Norm Residue theorem. The work discussed above gives information about this DGA, which could potentially lead to a refinement of the Norm Residue theorem itself. There are also interesting questions worth exploring in connection with the cyclotomic character and formality of Galois cohomology in certain cases, and some striking connections, due to Positselski, between Koszulity and Bogomolov's conjecture about the freeness of the commutator subgroup of p-Sylow subgroups of absolute Galois groups.

A second facet in this subject comes from the Elementary Type Conjecture, which aims to determine the structure of finitely-generated maximal pro-ℓ Galois groups of fields. These results are closely related to the "local-theory" in anabelian geometry which was mentioned above. In the last several years, it has become apparent that abelian-by-central quotients of absolute Galois groups completely control much of the local theory, in "tame" situations. On the other hand, the very recent work of Koenigsmann-Strommen suggests that similar results might also hold in "wild" situations. A complete picture of the local theory in both the tame and wild situations could potentially lead to the resolution of the Elementary-Type conjecture.

2.3 Unipotent Fundamental Groups and Hyperplane Arrangements

The pro-nilpotent completion of the geometric fundamental group has a natural algebraic analogue in the pro-unipotent fundamental group. The pro-unipotent fundamental group has been an object of great interest ever since the work of Deligne, Sullivan, Morgan and Griffiths, who studied the de Rham fundamental groups of compact Kahler manifolds using Hodge theory. More generally, Deligne's theory of mixed hodge structures and the weight filtration has been extensively used to study unipotent fundamental groups of possibly non-proper varieties. By now, unipotent fundamental groups are fairly ubiquitous, and they are particularly pervasive throughout the work of Hain-Matsumoto, Deligne-Goncharov, as well as many others.

Such ideas have also taken center stage in the theory of hyperplane arrangements, as appears in the work of Cohen, Dimca, Denham, Papadima, Suciu, Wang and others. In this context, the connection between the (unipotent) fundamental group of the complement of a hyperplane arrangement and the combinatorial structure (matroid) of the hyperplane arrangement is of particular interest, and determining the precise connection is a major research area. A very similar question comes up in higher-dimensional (birational) anabelian geometry, where one must understand the precise connection between the geometry of a divisor and the group-theoretical structure of inertia elements in the fundamental group of its complement. In connection with section (2), formality, 1-formality, graded-formality, etc. for fundamental groups of complements of a hyperplane arrangements are again questions which are of great interest in the theory, where such questions can be studied using the weight filtration in de-Rham and/or ℓ-adic cohomology.

The theory of unipotent fundamental groups and hyperplane arrangements can therefore be seen as a bridge between subjects (1) and (2) above. Moreover, there is a clear overlap between the three subjects, and some similarity between the techniques they use. But there are also some significant differences between their methods, as they strive for different goals. Therefore, we believe that increased interaction between these three subjects could lead to new collaborations and new results, by rethinking current research areas from new points of view. In this respect, the primary goal of this workshop was to facilitate such new interaction between the three subjects mentioned above.

3 Open Problems

During two open problem sessions on Thursday and Friday, workshop participants presented and discussed open problems in the field. Other open problems were discussed during talks and breaks. Here is a list of these open problems.

3.1 Thursday

1. (K. Wickelgren, presenting an open problem posed by Micheal J. Hopkins and K. Wickelgren). Let k be a field. The nth Milnor K-theory group $K_n^M(k)$ is defined by tensoring the underlying multiplicative group k^* of k with itself n-times, and quotienting out by the subgroup generated by elements $a_1 \otimes a_2 \otimes \ldots \otimes a_n$ where $a_i + a_j = 1$ for some $i \neq j$, i.e.,

$$K_n^M(k) = (k^*)^{\otimes n} / \langle a_1 \otimes a_2 \otimes \ldots \otimes a_n : a_i + a_j = 1 \rangle. \tag{1}$$

   Milnor K-theory thus combines the multiplication on k with the addition in some interesting manner. View the presentation (1) as a *field arithmetic* description.

   Let Gal_k denote the absolute Galois group of k. The Milnor/Bloch-Kato Conjecture proved by Voevodsky and Rost provides the following description of the cohomology ring $H^*(\mathrm{Gal}_k)$ of absolute

Galois groups. Let N be prime to the chracteristic of k. Applying $H^*(\mathrm{Gal}_k, -)$ to the Kummer exact sequence

$$1 \to \mu_N \to \mathcal{G}_m \stackrel{z \mapsto z^N}{\to} \mathcal{G}_m \to 1$$

produces the sequence

$$k^* \stackrel{z \mapsto z^N}{\to} k^* \to H^1(\mathrm{Gal}_k, \mu_N) \to 0,$$

and therefore the Kummer isomorphism $k^*/(k^*)^N \cong H^1(\mathrm{Gal}_k, \mu_N) = H^1(\mathrm{Gal}_k, \mathbb{Z}/N(1))$. The cup product then produces a map $(k^*)^{\otimes n} \to H^n(\mathrm{Gal}_k, \mathbb{Z}/N(n))$. The Steinberg relation implies that this map factors through a map from mod N Milnor K-theory

$$K_n^M(k)/N \cong H^n(\mathrm{Gal}_k, \mathbb{Z}/N(n)),$$

and the Milnor/Bloch-Kato Conjecture is that this map is an isomorphism. For example, we have a ring isomorphism

$$H^*(\mathrm{Gal}_k, \mathbb{Z}/2) \cong \oplus_n (k^*)^{\otimes n}/\langle x \otimes (1-x), 2\rangle,$$

(the Milnor conjecture). The Milnor/Bloch-Kato Conjecture is also called the Norm-Residue isomorphism theorem.

There is a differential graded algebra DGA of cochains $C^*(\mathrm{Gal}_k, \mathbb{Z}/2)$ computing the cohomology with $\mathbb{Z}/2$-coefficients, and the coefficient group can be generalized as well, but with $\mathbb{Z}/2$-coefficients, the complication from the twists is absent. One could alternatively assume that k contains roots of unity, or perhaps use all twists. Let $C^*(\mathrm{Gal}_k)$ denote a DGA arising in this way, but not specifying the coefficients.

Question: Is there a field arithmetic description of the DGA of cochains $C^*(\mathrm{Gal}_k)$ refining the Milnor/Bloch-Kato description of Galois cohomology?

In [13], it was asked if $C^*(\mathrm{Gal}_k)$ is formal, i.e., quasi-isomorphic to $H^*(\mathrm{Gal}_k)$. If it were, then the field arithmetic description given in Milnor/Bloch-Kato would be the desired refinement. Interestingly, it is not, as shown by Positselski [30]. So there is information contained in $C^*(\mathrm{Gal}_k)$ which is lost on passage to cohomology. Is it possible to give a generator-relation description of $C^*(\mathrm{Gal}_k)$ in terms of the arithmetic of k?

In his paper [30, page 226] Positselski mentions that he is not aware of any counterexample to formality of $C^*(\mathrm{Gal}_k)$ if all roots of unity of powers l^n are in the field where $\mathbb{Z}/l$ are considered coefficients in Galois cohomology. Also it is interesting to notice that Example 6.3 can be considered as analogue of an example in [22]. This was discussed with J. Mináč and A. Topaz during the conference, and there is still a possibility that that Galois cohomology and cyclotomic character can contain information obtained from $C^*(\mathrm{Gal}_k)$. In fact there maybe possibly "enhanced notion of formality" taking into account also some Bockstein maps in Galois cohomology which could possibly explain Positselski's examples and could still lead to a deep formality-like property of $C^*(\mathrm{Gal}_k)$.

2. (F. Bogomolov) Let K be a function field over an algebraically closed field k of characteristic $\neq \ell$. Let G be a Sylow-ℓ subgroup of the absolute Galois group of K. It turns out that the isomorphism class of G (as a pro-ℓ group) only depends on $\mathrm{trdeg}(K|k)$ and k. Let $G^{(2)}$ denote the commutator subgroup of the profinite group G.

   Conjecture: *$G^{(2)}$ is a free pro-ℓ group.*

   This conjecture provides a strategy to give an alternative proof of the Bloch-Kato conjecture. Indeed, consider the spectral sequence associated to $G \twoheadrightarrow G^{\mathrm{ab}}$:

   $$\mathrm{H}^i(G^{\mathrm{ab}}, \mathrm{H}^j(G^{(2)}, \mathbb{Z}_\ell)) \Rightarrow \mathrm{H}^{i+j}(G, \mathbb{Z}_\ell)$$

   The conjecture implies that the $E_2^{i,j}$ terms of this spectral sequence all vanish for $j \geq 2$. The main point is that elements of $E_2^{i,0} = \mathrm{H}^i(G^{\mathrm{ab}}, \mathbb{Z}_\ell)$ can all be represented as sums of symbols.

   The cohomology of G^{ab} behaves similarly to the cohomology of an abelian variety. It might be the case that G^{ab} can be seen as a fundamental group of some sort of "universal Albanese variety" associated to k and $\mathrm{trdeg}(K|k)$. For a more detailed discussion, refer to [1].

3. (D. Litt) Let X be a smooth geometrically-connected curve over a finite field k of characteristic $\neq \ell$. After extending the base field, we may assume that the Frobenius acts trivially on $\mathrm{H}^1(\bar{X}, \mathbb{F}_\ell)$. Consider the action of Frobenius (Frob) on the $\mathbb{F}_\ell$ group-algebra

$$\mathbb{F}_\ell[[\pi_1(\bar{X})]].$$

Question: What are the sizes of the Jordan blocks of this action? More precisely, define:

$$r(n) = \min\{m \ : \ (\mathrm{Frob} - I)^m \mid \mathbb{F}_\ell[[\pi_1]]/\mathcal{I}^n = 0\}$$

(here $\mathcal{I}$ denotes the augmentation ideal.) Here are some interesting questions one might ask:

   (a) Can $r(n)$ be bounded non-trivially?

   (b) How does $r(n)$ change on passing to finite étale covers of X?

Satisfactory answers to these questions would lead to an alternate proof of a conjecture of de Jong (which has been proven by Gaitsgory). **Idea:** The above construction should give a notion of *weights* which should behave well when dealing with $\mathbb{F}_\ell$ coefficients.

4. (M. Florence) Let U be an open subscheme of $\operatorname{Spec}\mathbb{Z}$ and let $X \to U$ be a smooth projective (arithmetic) curve. For each closed point $x \in U$, consider the fibre X_x above x, as well as the number of rational points $X_x(k(x))$.

Question: Is there an algorithmic way to describe $\#(X_x(k(x)))$, as x varies, using the polynomials defining X?
Comment: Since we expect equidistribution of Frobenius elements, the number of points should behave randomly.

5. (D. Harbater) Let X be a smooth affine hyperbolic curve over an algebraically closed field k of characteristic $p > 0$.

Question: Does $\pi_1^{\text{ét}}(X)$ determine X up-to isomorphism?

If X is the affine line over the algebraic closure of a finite field, then the answer is *yes* (this is a theorem of Tamagawa). It was pointed out by Hoshi during the discussions, that the answer is also yes for a once-punctured elliptic curve.

In general, it is known that $\pi_1^{\text{ét}}(X)$ (in the above context) determines the genus of X, the number of punctures, the cardinality and the characteristic of the base field. (This is also due to Tamagawa.)

Remark (Bogomolov): In the cases that we know, it suffices to use certain small quotients of π_1.

6. (M. Florence) Let G be a profinite group endowed with a character $\chi : G \to \mathbb{Z}_p^\times$. Let $\mathbb{Z}/p^n(1)$ denote the G-module whose underlying abelian group is $\mathbb{Z}/p^n$, and such that G acts on $\mathbb{Z}/p^n(1)$ via χ. Assume that (G, χ) satisfies a *formal version of Hilbert 90.* In other words, for all open subgroups U of G, the canonical map

$$\mathrm{H}^1(U, \mathbb{Z}/p^n(1)) \to \mathrm{H}^1(U, \mathbb{Z}/p(1))$$

is surjective. Some questions that one could ask about such (G, χ):

   (a) Which groups satisfy this property? (Examples include free profinite groups, Demuškin groups.)

   (b) Do such groups always arise as Galois groups? I.e., does there exist a field F of characteristic $\neq p$, and an extension E of F which is p-closed, such that $G = \mathrm{Gal}(E|F)$ and χ is the cyclotomic character?

Related topics were discussed in C. Quadrelli's talk. See [7, 16, 21] for related papers.

3.2 Friday

1. (A. Topaz) Let g be a positive integer, and consider the (discrete) group:

$$\Sigma_g := \langle a_1, b_1, \ldots, a_g, b_g \ : \ \prod_i [a_i, b_i] = 1 \rangle.$$

   Let $\widehat{\Sigma}_g$ denote the pro-ℓ completion of Σ_g. Does there exist a field F of characteristic $\neq \ell$ which contains μ_{ℓ^∞}, such that $\widehat{\Sigma}_g$ is isomorphic to $\mathrm{Gal}(F(\ell)|F)$? Does such a field have any "distinguished" valuations? **Note:** This is a special case of question (6)(b) above.

2. (J. Mináč, continuing with (1)) More generally, do all pro-p Demuškin groups arise as maximal pro-p Galois groups of fields (of characteristic $\neq p$)? See [15, 16, 25, 26].

3. (J. Mináč) For a field F of characteristic $\neq 2$, let $W(F)$ denote the Witt ring of quadratic forms of F. Does there exists a field F such that $W(F)$ is isomorphic to $W(\mathbb{Q}_2)$, and such that 2 is a square in F?

4. (J. Mináč) Let F be a field of characteristic $\neq p$ which contains μ_p, and let $\mathrm{Gal}_F^{(n)}$ denote the p-Zassenhaus filtration on Gal_F. Can one describe the cohomology rings $\mathrm{H}^*(\mathrm{Gal}_F / \mathrm{Gal}_F^{(n)}, \mathbb{Z}/p)$ in terms of generators and relations? (Note, the answer is "yes" for $n = 2$ and $n = \infty$).

5. (A. Topaz) Let X be a smooth variety over $\mathbb{C}$, and let U be a (Zariski) open subvariety of X. Does there exist a smaller (Zariski) open subvariety V of U such that V is (rationally) 1-formal?

6. (F. Pop, J. Mináč) Does there exist a proof of the Bloch-Kato conjecture which is purely formal in nature? In other words, what *group-theoretical* properties of the absolute Galois group of a field (endowed with a character mimicking the cyclotomic character) ensure that the Bloch-Kato conjecture holds true?

7. (P. Guillot, J. Mináč, A. Topaz, N. D. Tân) Let F be a field of characteristic $\neq \ell$ such that $\mu_\ell \subset F$, and let $x_1, \ldots, x_n \in \mathrm{H}^1(F, \mathbb{Z}/\ell)$ be given. Assume that the n-fold Massey product $\langle x_1, \ldots, x_n \rangle$ is non-empty. Does $\langle x_1, \ldots, x_n \rangle$ contain 0? This is the so-called *n-Massey vanishing conjecture* in Galois cohomology, which was formulated by Mináč-Tân [24, 23], and which was strongly influenced by the previous work of Hopkins-Wickelgren [13]. For recent results towards this conjecture and beyond, refer to [11, 12, 19]. One can also consider a weaker version of the above question, where one additionally requires that all the cup-products $x_i \cup x_j$ vanish; this weaker version seems easier, but it is still far from trivial.

4 Presentation Highlights

The presentations in the workshop consisted of five survey talks (60 minutes) and 14 research talks (45-50 minutes). The survey talks were certainly among the highlights, while the shorter research talks presented the state of the art on the various topics covered in the workshop. All of the talks in the workshop were extremely interesting and added to the success of the workshop. Several speakers provided supplemental materials, all of which has been made available on the workshop web-page. Finally, almost all of the talks were recorded and can be viewed on the workshop web-page.

4.1 Survey Talks

The survey talks were some of the major highlights of the workshop. They helped considerably to introduce some of the main topics and results of the workshop, and they were successful in stimulating many further discussions throughout the week. Here is a brief summary of each of these survey talks.

A. Topaz gave a survey of *almost-abelian anabelian geometry*. As described in this talk, the goal of almost abelian anabelian geometry is to recover arithmetic and geometric information from two-step nilpotent

Galois-theoretical data. The talk focused primarily on two topics: Bogomolov's Programme in birational anabelian geometry, and the Ihara/Oda-Matsumoto conjecture. Aspects from both the pro-ℓ and mod-ℓ variants of the theory were discussed. Some relevant references include [3, 4, 28, 29, 27, 33].

T. Szamuely gave a survey of *fundamental groups and their variants*. This was an introduction to several variants of fundamental groups that arise in algebraic and arithmetic geometry. In addition to the usual topological fundamental group and Grothendieck's étale fundamental group, this survey included a description of Nori's fundamental group scheme, and Deligne's pro-algebraic fundamental group. Wherever possible, this talk highlighted both the Tannakian point of view, and the point of view using torsors. The speaker graciously provided typeset notes from his talk, which are available on the workshop webpage.

J. Stix gave a survey of *the section conjecture*. This is a major open conjecture, which he explained and then discussed examples and analogues, including a proof for abelian varieties over finite fields, and birational and minimalistic analogues. The speaker has written an excellent book on the subject [31].

K. Wickelgren gave a survey of *Massey products in Galois cohomology*, discussing vanishing results, notably the major advance to 4-fold Massey products in [12], as well as applications to the structure of Galois groups and automatic realization results. The first problem listed in the Open Problems section was also discussed.

A. Suciu gave a survey of *formality notions for spaces and groups*, and the slides from his talk are available on the workshop webpage. His talk provides many open problems: namely, he gave results on cdga's over fields of characteristic 0 and explained to the audience that many of these results may hold (and may not hold) in finite characteristic, and suggested the possibility of understanding the analogous structure in the case of finite characteristic. He discussed resonance varieties, characteristic varieties, and the Tangent cone Theorem as an alternative to Massey products for disproving the formality of a space.

4.2 Research Talks

The first day of the workshop had two research talks which focused on birational anabelian geometry and the Galois action on fundamental groups. M. Lüdtke spoke about his recent results [18] concerning anabelian geometry for one-dimensional function fields over algebraically closed fields. D. Litt spoke about his recent results [17] concerning the Galois action on ℓ-adic unipotent completions of geometric fundamental groups.

One primary focus of the second day was the section conjecture and its variants. I. Dan-Cohen spoke about his recent preprint [9], joint with T. Schlank, about rational motivic homotopy theory and its connections with Kim's unipotent variant of the section conjecture. A. Betts spoke about his recent preprint [5] which studies local hights on abelian varieties from a motivic/anabelian point of view; a handout from this lecture is available on the workshop web-page. F. Bogomolov spoke about a rational, almost-abelian variant of the section conjecture for higher-dimensional function fields over algebraically closed fields, which appears in his joint work with M. Rovinsky and Y. Tschinkel [2].

There were two additional talks on the second day whose topic was unrelated to the section conjecture. E. Bayer-Fluckiger spoke about cohomological invariants of G-Galois algebras, and recent results establishing the Hasse principle for their self-dual normal bases; the slides from this talk are available on the workshop web-page. D. Neftin spoke about local approximation and specialization, emphasizing recent new results in the context of the Grunwald problem.

The third day of the workshop focused on recent developments in the context of Massey products in Galois cohomology. E. Matzri spoke about his recent work [19] concerning the vanishing of triple Massey products *with higher weight* in Galois cohomology. P. Guillot spoke about his recent joint work with J. Mináč, A. Topaz, N. D. Tân and O. Wittenberg [11, 12], concerning vanishing of quadruple Massey products in the Galois cohomology of number fields.

The morning session on the fourth day of the workshop saw two talks related to Galois cohomology. A. Schultz spoke about the parametrizing space of bi-cyclic elementary p-extensions and its Galois-module structure, in analogy with Kummer theory. S. Chebolu spoke about his joint work with J. Carlson and J. Mináč [6], concerning the finite-generation problem in Tate cohomology of finite groups.

The afternoon session on the fourth day saw two additional talks with different focuses. Y. Hoshi spoke about his recent joint work with S. Mochizuki and A. Minamide [14], which gives a new simple group-theoretical characterization of the Grothendieck-Teichmüller group. D. Harbater spoke about the Galois theory of arithmetic function fields, especially his work on local-to-global principles using patching techniques, which is part of his long-term ongoing project joint with J. Hartmann, D. Krashen, J.-L. Colliot-Thélène, Parimala and Suresh.

The last day of the workshop saw a single research talk by C. Quadrelli, who spoke about his ongoing joint work with I. Efrat, J. Mináč and T. Weigel, which studies the group-theoretical structure of absolute Galois groups, endowed with their cyclotomic character, from a cohomological point of view.

4.3 Informal Discussions

In addition to the survey and research talks, many participants of the workshop also engaged in several informal discussions and explanations of recent research. Here are three which we are aware of. T. Feng explained his recent paper [8] using cohomology operations to study a question of Tate. M. Florence explained some ideas behind his recent joint work with C. De Clercq [10], which aims to give a new simple and purely algebraic proof of the norm-residue isomorphism theorem. A. Topaz explained his recent work [32] concerning reconstruction of higher dimensional function fields from their generic cohomology endowed with their infinite-dimensional mixed Hodge structure.

5 Scientific Progress Made

J. Mináč and A. Topaz discussed ideas related to the differential graded algebra of étale cochains on $\operatorname{Spec} k$. R. Davis and K. Wickelgren made progress on a project on the 2-nilpotent mod p quotient of the étale fundamental group, which is joint with R. Pries. F. Bogomolov, F. Pop, A. Suciu and A. Topaz had some discussions around the open problem concerning formality (c.f. §3.2 (5)) and its connection with structure of cohomology and anabelian geometry. P. Guillot, J. Mináč, A. Topaz and O. Wittenberg had some discussions concerning Massey products in Galois cohomology, which could lead to further progress in their work. Further discussions between the team above, E. Matzri, and D. Neftin also took place, which could possibly lead to new advances towards the n-Massey vanishing conjecture.

These are just a few of the research discussions that we are aware of, but certainly many more took place. Especially important were several informal discussions between young researchers, including graduate students and postdocs, and more senior researchers, which will lead to new research exchanges. We expect that many of these discussions will eventually lead to new collaborations and new results, which combine ideas and tool from different areas.

6 Outcome of the Meeting

The primary goal of this workshop was to increase interaction between the three subjects mentioned in the overview: anabelian geometry; the group theory of pro-ℓ and absolute Galois groups via the norm-residue isomorphism; and (unipotent) pro-algebraic fundamental groups. From this point of view, the workshop was extremely successful, and this is backed up by comments we received from several participants. We hope to organize a similar workshop again in the near future.

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