J. Appl. Probab. **61,** 767–780 (2024)
*doi:*10.1017/jpr.2023.71

AGING NOTIONS, STOCHASTIC ORDERS, AND EXPECTED UTILITIES

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Abstract

There are some connections between aging notions, stochastic orders, and expected utilities. It is known that the DRHR (decreasing reversed hazard rate) aging notion can be characterized via the comparative statics result of risk aversion, and that the location-independent riskier order preserves monotonicity between risk premium and the Arrow–Pratt measure of risk aversion, and that the dispersive order preserves this monotonicity for the larger class of increasing utilities. Here, the aging notions ILR (increasing likelihood ratio), IFR (increasing failure rate), IGLR (increasing generalized likelihood ratio), and IGFR (increasing generalized failure rate) are characterized in terms of expected utilities. Based on these observations, we recover the closure properties of ILR, IFR, and DRHR under convolution, and of IGLR and IGFR under product, and investigate the closure properties of the dispersive order, location-independent riskier order, excess wealth order, the total time on test transform order under convolution, and the star order under product. We have some new findings.

Keywords: Stochastic comparisons; risk aversion; increasing likelihood ratio; increasing failure rate; decreasing reversed hazard rate; increasing generalized likelihood ratio; increasing generalized failure rate

2020 Mathematics Subject Classification: Primary 60E15
Secondary 91B16; 90B25

1. Introduction and motivation

Aging distributions and their appealing properties have wide applications in probability, statistics, reliability, optimization, information, operations management, econometrics, and other fields of applied probability. For example, in reliability, concepts of aging describe how a component or system improves or deteriorates with age, and significantly affect the decisions people make with respect to maintenance, repair/replacement, and warranties [5, 27, 47]. In operations management, the generalized failure rate of a distribution has demonstrable importance in pricing, revenue, and supply chain management [13, 29, 31, 36, 51]. In the statistical literature, log-concavity has a wide range of applications to statistical modeling and estimation, MCMC algorithms, and machine learning [42, 49]. In econometrics, log-concavity (a strong

Received 4 November 2022; accepted 7 September 2023.
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aging notion) also plays a critical role [2, 3]. The most common aging notions used are ILR (increasing likelihood ratio, also termed log-concavity), IFR (increasing failure rate), DRHR (decreasing reversed hazard rate), and IFRA (increasing failure rate average). In this paper we focus on these aging notions as well as IGLR (increasing generalized likelihood ratio) and IGFR (increasing generalized failure rate). IGLR and IGFR, have already appeared in the reliability literature under the names IPLR (increasing proportional likelihood ratio) and IPHR (increasing proportional failure rate); IPLR was introduced in [41] and studied in [8, 37, 38], among others.

Stochastic orders offer more insights into the comparison of two random variables (vectors) than only through their means and variances, which may not exist. Stochastic orders have been successfully applied to reliability, actuarial science, operations research, and other related fields. Various types of stochastic orders and associated properties have developed rapidly over the years, resulting in a large body of literature; surveys can be found in [7, 32, 35, 44]. In this paper we consider the following stochastic orders: the *dispersive order*, *location-independent riskier order*, *excess wealth order*, *total time on test transform order*, and the *star order*. The formal definitions of these orders is given in Sections 2.2 and 4.6.

There are some connections between aging notions, stochastic orders, and expected utilities. [21] characterized DRHR aging notion via a comparative statics result of risk aversion by using the class of increasing and concave utilities (Proposition 2.1); [22] proved that the location-independent riskier order preserves monotonicity between risk premium and the Arrow–Pratt measure of risk aversion for the class of increasing and concave utilities (Proposition 2.3); and [28] proved that the dispersive order preserves this monotonicity for the larger class of increasing utilities (Proposition 2.2). It is known from [44] that the excess wealth order can also be characterized via the class of increasing and convex utilities and the concept of more risk seeking (Proposition 2.4).

The main purposes of this paper are to characterize the ILR, IFR, IGLR, and IGFR aging notions in terms of expected utilities, and to present several applications of these characterizations. After recalling the formal definitions of aging notions and stochastic orders in Section 2, we give our main results in Sections 3 and 4. Section 3 is devoted to characterizing ILR, IFR, IGLR, and IGFR. Applications of these characterizations are offered in Section 4, where we recover the closure properties of ILR, IFR, DRHR under convolution, and of IGLR and IGFR under product, and investigate the closure properties of the dispersive order, location-independent riskier order, excess wealth order, the total time on test transform order, and the star order under convolution. We have some new findings: see, for example, Theorems 4.3, 4.4, and 4.5. In Section 5 we raise an open problem: how to characterize the aging notion IFRA in terms of expected utilities. The proofs of two lemmas are relegated to the Appendix.

Throughout, 'increasing' and 'decreasing' mean 'nondecreasing' and 'nonincreasing', respectively. All integrals and expectations are implicitly assumed to exist whenever they are written. For a distribution function F, $\overline{F} = 1 - F$ denotes its survival function, and the inverse of F is taken to be the left-continuous version defined by $F^{-1}(p) = \inf\{x\colon F(x) \geq p\}$, $p \in (0, 1)$, with $F^{-1}(0) = \inf\{x\colon F(x) > 0\}$ and $F^{-1}(1) = \sup\{x\colon F(x) < 1\}$.

2. Aging notions, stochastic orders, and preliminaries

2.1. Aging notions

Let $f\colon I \to \mathbb{R}_+ := [0, \infty)$ be a function, where I is an interval of $\mathbb{R}$. f is said to be log-concave if, for all $x, y \in I$ and $\alpha \in (0, 1)$, $f(\alpha x + (1-\alpha)y) \geq [f(x)]^{\alpha}[f(y)]^{1-\alpha}$. When $I \neq \mathbb{R}$,

https://doi.org/10.1017/jpr.2023.71 Published online by Cambridge University Press

log-concavity of f on I can be interpreted as log-concavity over $\mathbb{R}$ by setting $f(x) = 0$ for all $x \notin I$. Log-convexity does not possess such a property.

Let Z be a random variable with distribution function H. Then Z or H is said to be ILR (*increasing likelihood ratio*) or PF_2 (*Pólya frequency of order 2*) if Z has a log-concave density or probability function, IFR (*increasing failure rate*) if $\overline{H}(x)$ is log-concave in $x \in \mathbb{R}$, and DRHR (*decreasing reversed hazard rate*) if $H(x)$ is log-concave in $x \in \mathbb{R}$. It is well known [5] that ILR $\Longrightarrow$ DRHR and IFR. If Z has a probability density function h, then Z is IFR if and only if the failure rate function $\lambda(t) := h(t)/\overline{H}(t)$ is increasing on $\{t\colon H(t) < 1\}$, and Z is DRHR if and only if the reversed hazard rate function $\mu(t) := h(t)/H(t)$ is decreasing on $\{t\colon H(t) > 0\}$. For more on ILR or log-concavity and its applications, see [10, 17, 34, 48, 50].

Let Z be a positive random variable with distribution function H and density function h. Then Z or H is said to be IGLR (*increasing generalized likelihood ratio*) if $h(\mathrm{e}^x)$ is log-concave in $x \in \mathbb{R}$, and IGFR (*increasing generalized failure rate*) if $x\lambda(x)$ is increasing in $x \in \{t\colon H(t) < 1\}$. IGFR distributions were introduced as a tool in many supply chain models and also in stochastic models of applied probability. [39] investigated closure properties of IGFR under common transformations of random variables, and compared the class of IGFR distributions with the class of IFR distribution. [30] provided alternative characterizations of IGFR distributions that lead to simplified verification of whether the IGFR condition holds. [4] investigated the closure property of IGFR under left and right truncations, and extended the IGFR notion to discrete distributions. Many commonly used families of random variables do possess the IGLR [19] and IGFR properties [4]. Further properties and characterizations of IGLR and IGFR can be found in [8, 19, 37, 41]. For example,

- Z is IGFR if and only if $Z \le_{\mathrm{hr}} aZ$ for all $a > 1$;
- Z is IGLR if and only if $Z \le_{\mathrm{lr}} aZ$ for all $a > 1$,

where $\le_{\mathrm{hr}}$ and $\le_{\mathrm{lr}}$ denote the hazard rate and likelihood ratio orders, respectively. Their formal definitions can be found in [44]. Obviously, both IGLR and IFR imply IGFR.

Proposition 2.1 gives a characterization of the aging notion of DRHR in terms of expected utilities. To present the proposition, we recall the Arrow–Pratt definition of 'more risk averse' from [40]. Let u and v be two increasing utility functions. u is said to be more risk averse than v if there exists an increasing and concave function κ such that $u(x) = \kappa(v(x))$ for all x or, equivalently, $r_u(x) \ge r_v(x)$ for all x, where $r_u(x)$ is the Arrow–Pratt absolute local measure of risk aversion defined by $r_u(x) = -u''(x)/u'(x)$ whenever u and v are twice differentiable.

Proposition 2.1 ([21].) *Let random variables X and Y admit a comparative statics result in the sense that*

$$\mathbb{E}[v(X)] \le \mathbb{E}[v(Y)] \Longrightarrow \mathbb{E}[u(X)] \le \mathbb{E}[u(Y)] \tag{2.1}$$

whenever u, v are increasing concave with u more risk averse than v. Let Z be a random variable. Then Z is DRHR if and only if

$$\mathbb{E}[v(X+Z)] \le \mathbb{E}[v(Y+Z)] \Longrightarrow \mathbb{E}[u(X+Z)] \le \mathbb{E}[u(Y+Z)]$$

whenever X and Y satisfy (2.1)*, independent of Z, and u, v are increasing concave with u more risk averse than v.*

2.2. Stochastic orders

We first recall the definitions of the dispersive order, excess wealth order, location-independent riskier order, and total time on test transform order. For the last three orders we use different but equivalent definitions to those in the literature. This will help us to understand the connections among these three orders.

Definition 2.1. Let X and Y be two random variables with respective distribution functions F and G.

(i) X is said to be smaller than Y in the dispersive order, denoted by $X \le_{\mathrm{disp}} Y$, if $F^{-1}(\beta) - F^{-1}(\alpha) \le G^{-1}(\beta) - G^{-1}(\alpha)$ for all $0 < \alpha < \beta < 1$ [9].

(ii) X is said to be smaller than Y in the excess wealth order, denoted by $X \le_{\mathrm{ew}} Y$, if $(1/(1-p)) \int_p^1 [G^{-1}(u) - F^{-1}(u)]\,\mathrm{d}u$ is increasing in $p \in (0, 1)$ [16, 43].

(iii) X is said to be smaller than Y in the location-independent riskier order, denoted by $X \le_{\mathrm{lir}} Y$, if $(1/p) \int_0^p [G^{-1}(u) - F^{-1}(u)]\,\mathrm{d}u$ is increasing in $p \in (0, 1)$ [22].

(iv) X is said to be smaller than Y in the total time on test transform order, denoted by $X \le_{\mathrm{ttt}} Y$, if $(1/(1-p)) \int_0^p [G^{-1}(u) - F^{-1}(u)]\,\mathrm{d}u$ is increasing in $p \in (0, 1)$ [20, 25].

[43] proved that $X \le_{\mathrm{disp}} Y \Longrightarrow X \le_{\mathrm{lir}} Y$ and $X \le_{\mathrm{ew}} Y$, and [15] observed that

$$X \le_{\mathrm{ew}} Y \Longleftrightarrow -X \le_{\mathrm{lir}} -Y. \tag{2.2}$$

Therefore, all properties of the order $\le_{\mathrm{lir}}$ can be substituted for the order $\le_{\mathrm{ew}}$, and vice versa. When X and Y have an equal finite mean, it follows from the definitions that $X \le_{\mathrm{ttt}} Y \Longleftrightarrow X \ge_{\mathrm{ew}} Y$.

As pointed out in [25], the orders $\le_{\mathrm{disp}}$, $\le_{\mathrm{lir}}$, and $\le_{\mathrm{ew}}$ are location independent while the order $\le_{\mathrm{ttt}}$ is location dependent: $X \le_{\mathrm{disp}} [\le_{\mathrm{lir}}, \le_{\mathrm{ew}}]\, Y \Longrightarrow X \le_{\mathrm{disp}} [\le_{\mathrm{lir}}, \le_{\mathrm{ew}}]\, Y + c$ for any $c \in \mathbb{R}$, and $X \le_{\mathrm{ttt}} Y \Longrightarrow X \le_{\mathrm{ttt}} Y + c$ for any $c > 0$, where the latter implication does not hold for $c < 0$. Each of the orders $\le_{\mathrm{disp}}$, $\le_{\mathrm{lir}}$, and $\le_{\mathrm{ew}}$ compares the variability of the underlying random variables, while the order $\le_{\mathrm{ttt}}$ combines comparison of location with comparison of variation.

The above four orders can be characterized in different ways; we now list three characterizations of the orders $\le_{\mathrm{disp}}$, $\le_{\mathrm{lir}}$, and $\le_{\mathrm{ew}}$ in terms of expected utilities. It is still unknown whether the order $\le_{\mathrm{ttt}}$ can be characterized by utility functions.

Proposition 2.2. ([28].) *$X \le_{\mathrm{disp}} Y$ if and only if, for all increasing functions u and v with u being more risk averse than v, and for every real number $c \in \mathbb{R}$,*

$$\mathbb{E}[v(X-c)] \ge \mathbb{E}[v(Y)] \Longrightarrow \mathbb{E}[u(X-c)] \ge \mathbb{E}[u(Y)].$$

Proposition 2.3. ([22].) *$X \le_{\mathrm{lir}} Y$ if and only if, for all increasing concave functions u and v with u being more risk averse than v, and for every real number $c \in \mathbb{R}$,*

$$\mathbb{E}[v(X-c)] \ge \mathbb{E}[v(Y)] \Longrightarrow \mathbb{E}[u(X-c)] \ge \mathbb{E}[u(Y)].$$

Proposition 2.4. ([44, Theorem 3.C.2].) *$X \le_{\mathrm{ew}} Y$ if and only if, for all increasing convex functions v and κ such that $u(\cdot) = \kappa(v(\cdot))$, and for every real number $c \in \mathbb{R}$,*

$$\mathbb{E}[v(X-c)] \ge \mathbb{E}[v(Y)] \Longrightarrow \mathbb{E}[u(X-c)] \ge \mathbb{E}[u(Y)].$$

Proposition 2.3 has the following interpretation in economics [15]: Let v be the utility function of a risk averse agent (that is, v is increasing and concave), and let X and Y be two random

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assets. If the agent is willing to pay c (not necessarily nonnegative) for the replacement of the asset Y by the asset X, then so does another agent with utility function u, which is Arrow–Pratt more risk averse than the first. Similar interpretations of Propositions 2.2 and 2.4 can be given in economics.

For further properties and applications of these orders, see [6, 11, 12, 18, 23, 24, 26, 28, 33, 45, 46], among others.

3. Main results

3.1. Characterization of IFR

Observe that a random variable X is IFR if and only if $-X$ is DRHR. A characterization of IFR can be deduced from Proposition 2.1 as follows, which was also mentioned implicitly in [21, Note 16].

Theorem 3.1. *Let the random variables X and Y admit a comparative statics result in the sense that*

$$\mathbb{E}[v(X)] \le \mathbb{E}[v(Y)] \Longrightarrow \mathbb{E}[u(X)] \le \mathbb{E}[u(Y)] \tag{3.1}$$

whenever v and κ are increasing and convex such that $u(\cdot) = \kappa(v(\cdot))$. Let Z be a random variable. Then Z is IFR if and only if

$$\mathbb{E}[v(X+Z)] \le \mathbb{E}[v(Y+Z)] \Longrightarrow \mathbb{E}[u(X+Z)] \le \mathbb{E}[u(Y+Z)]$$

whenever X and Y satisfy (3.1), independent of Z, and v and κ are increasing convex such that $u(\cdot) = \kappa(v(\cdot))$.

3.2. Characterization of ILR

Before we present the characterization of ILR in terms of expected utilities, we need two useful lemmas. Lemma 3.1 characterizes the unique crossing property of two distribution functions in terms of expected utilities, and Lemma 3.2 states that the unique crossing property is preserved under the convolution with an ILR distribution. The proof of Lemma 3.1 is given in Appendix A.1.

The following notation will be used [44, (1.A.18)]: Let $a(x)$ be defined on I, where I is a subset of the real line. The number of sign changes of $a(x)$ on I is defined by $S^-(a) = \sup S^-[a(x_1), a(x_2), \ldots, a(x_m)]$, where $S^-[y_1, y_2, \ldots, y_m]$ is the number of sign changes of the indicated sequence, zero terms being discarded, and the supremum is extended over all sets $x_1 < x_2 < \cdots < x_m$ such that $x_i \in I$ and $2 \le m < \infty$.

Lemma 3.1. *Let X and Y be two random variables with respective distribution functions F\break and G. Then $S^-(G-F) \le 1$, with the sign sequence being $-, +$ in the case of equality, if and only if*

$$\mathbb{E}[v(X)] \le \mathbb{E}[v(Y)] \Longrightarrow \mathbb{E}[u(X)] \le \mathbb{E}[u(Y)] \tag{3.2}$$

whenever u and v are both increasing with u more risk averse than v.

Now, let Z_1 and Z_2 be two risky prospects, and let W be the initial wealth, independent of Z_1 and Z_2. If W has a log-concave density function, and if one decision maker v prefers Z_2 to Z_1, then so does the other more risk averse decision maker u, as stated in the next lemma.

Lemma 3.2. ([21], Theorem 5].) *Let W, X, and Y be random variables with W independent of X and Y, and suppose that $S^-(G-F) \le 1$ and the sign sequence is $-, +$ in the case of equality.*

Then $\mathbb{E}[v(W+X)] \le \mathbb{E}[v(W+Y)]$ *implies that* $\mathbb{E}[u(W+X)] \le \mathbb{E}[u(W+Y)]$ *whenever (i) u and v are increasing functions with u more risk averse than v, and (ii) W has a log-concave density function.*

Theorem 3.2. *Let random variables X and Y admit a comparative statics result in the sense that*

$$\mathbb{E}[v(X)] \le \mathbb{E}[v(Y)] \Longrightarrow \mathbb{E}[u(X)] \le \mathbb{E}[u(Y)] \tag{3.3}$$

whenever u, v are increasing functions with u more risk averse than v. Let Z be a random variable with density function h that does not vanish on an interval I. Then Z is ILR if and only if

$$\mathbb{E}[v(X+Z)] \le \mathbb{E}[v(Y+Z)] \Longrightarrow \mathbb{E}[u(X+Z)] \le \mathbb{E}[u(Y+Z)] \tag{3.4}$$

whenever X and Y satisfy (3.3)*, independent of Z, and u, v are increasing with u more risk averse than v.*

Proof. To prove necessity, assume that Z is ILR, and that X and Y satisfy (3.3), and denote by F and G the distribution functions of X and Y, respectively. Then, from Lemma 3.1, it follows that $S^-(G-F) \le 1$ with the sign sequence being $-$, $+$ in the case of equality. Therefore, by Lemma 3.2, (3.4) holds.

To prove sufficiency, assume that (3.4) holds. Consider two special random variables X and Y as follows: $Y=0$ and X takes values $y-x$ and $y-z$ with respective probabilities p and $1-p$, $p \in (0, 1)$, and $y, z, x \in I$ are arbitrary. Denote by F_W the distribution function of any random variable W. It is easy to see that $S^-(F_Y - F_X) \le 1$ with the sign sequence being $-$, $+$ in the case of equality. By Lemma 3.1, X and Y satisfy (3.3). Again by Lemma 3.1, we conclude that $S^-(F_{Y+Z} - F_{X+Z}) \le 1$ with the sign sequence being $-$, $+$ in the case of equality, i.e., for any $\delta > 0$,

$$H(y) > pH(x) + (1-p)H(z) \Longrightarrow H(y+\delta) \ge pH(x+\delta) + (1-p)H(z+\delta), \tag{3.5}$$

where H is the distribution function of Z. Since H is strictly increasing and continuous, the inverse H^{-1} of H exits on (0,1). Making a change of variables in the above inequalities, from (3.5) we obtain that, for any r,s,t in (0,1) and any $\delta > 0$,

$$s > pr + (1-p)t \Longrightarrow H(H^{-1}(s)+\delta) \ge pH(H^{-1}(r)+\delta) + (1-p)H(H^{-1}(t)+\delta). \tag{3.6}$$

For given r and t, choose $s = pr + (1-p)t + \varepsilon$ such that $s \in (0, 1)$ and $\varepsilon > 0$ satisfies the first inequality in (3.6). So it follows that

$$H(H^{-1}(pr + (1-p)t + \varepsilon) + \delta) \ge pH(H^{-1}(r)+\delta) + (1-p)H(H^{-1}(t)+\delta).$$

Since H and H^{-1} are continuous, upon taking the limit as $\varepsilon \to 0$ we obtain that, for all $\delta > 0$, $g_\delta(\alpha) = H(H^{-1}(\alpha) + \delta)$ is concave in $\alpha \in (0, 1)$. In other words, $g_\delta \circ H(x) = H(x+\delta)$ for all x with g_δ concave. Differentiating on both sides of this, we obtain that $g_\delta{}' \circ H(x)h(x) = h(x+\delta)$ for all $\delta > 0$. Thus, $h(x+\delta)/h(x) = g'_\delta \circ H(x)$, which is decreasing in x. Therefore, $h(x)$ is log-concave. This completes the proof. □

3.3. Characterizations of IGFR and IGLR

Let Z be a positive random variable. Observe that

- Z is IGFR if and only if $\log Z$ is IFR;
- Z is IGLR if and only if $\log Z$ is ILR.

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Based on this observation, applying Theorems 3.1 and 3.2 yields the following two results characterizing the IGFR and IGLR properties of a positive random variable.

Theorem 3.3. *Let X and Y be two positive random variables admitting a comparative statics result in the sense that*

$$\mathbb{E}[v(X)] \leq \mathbb{E}[v(Y)] \Longrightarrow \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)] \tag{3.7}$$

whenever v and κ are increasing convex such that $u(\cdot) = \kappa(v(\cdot))$. Let Z be a positive random variable. Then Z is IGFR if and only if

$$\mathbb{E}[v(XZ)] \leq \mathbb{E}[v(YZ)] \Longrightarrow \mathbb{E}[u(XZ)] \leq \mathbb{E}[u(YZ)]$$

whenever X and Y satisfy (3.7), independent of Z, and v and κ are increasing convex such that $u(\cdot) = \kappa(v(\cdot))$.

Theorem 3.4. *Let X and Y be two positive random variables admitting a comparative statics result in the sense that*

$$\mathbb{E}[v(X)] \leq \mathbb{E}[v(Y)] \Longrightarrow \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)] \tag{3.8}$$

whenever u, v are increasing functions with u more risk averse than v. Let Z be a positive random variable with density function h that does not vanish on an interval I. Then Z is IGLR if and only if

$$\mathbb{E}[v(XZ)] \leq \mathbb{E}[v(YZ)] \Longrightarrow \mathbb{E}[u(XZ)] \leq \mathbb{E}[u(YZ)]$$

whenever X and Y satisfy (3.8), independent of Z, and u, v are increasing with u more risk averse than v.

In Theorems 3.3 and 3.4, comparative statics results are preserved after multiplication of the choice variables by another independent nonnegative random variable Z whenever Z is IGFR and IGLR, respectively. These results would be appropriate, for instance, to the case of portfolio choice where returns are all nominal and the rate of inflation is a random variable [21].

4. Applications

4.1. Closure property of aging notions under convolution

It is well known that

- ILR is closed under convolution [14, p. 17];
- IFR is closed under convolution [44, Corollary 1.B.39];
- DRHR is closed under convolution [44, Corollary 1.B.63].

These closure properties of aging notions ILR, IFR, and DRHR under convolutions can be obtained directly from Theorems 3.2 and 3.1 and Proposition 2.1, respectively. For example, to prove the closure property of ILR under convolution, let Z_1 and Z_2 be two independent ILR random variables, and let X and Y be two other random variables, independent of Z_1 and Z_2, satisfying (3.3) whenever u, v are increasing functions with u more risk averse than v. Since Z_1 and Z_2 are ILR, applying Theorem 2.2 twice yields

$$\mathbb{E}[v(X+Z_1)] \leq \mathbb{E}[v(Y+Z_1)] \Longrightarrow \mathbb{E}[u(X+Z_1)] \leq \mathbb{E}[u(Y+Z_1)],$$
$$\mathbb{E}[v(X+Z_1+Z_2)] \leq \mathbb{E}[v(Y+Z_1+Z_2)] \Longrightarrow \mathbb{E}[u(X+Z_1+Z_2)] \leq \mathbb{E}[u(Y+Z_1+Z_2)].$$

Again applying Theorem 3.2 to the second line, we conclude that $Z_1 + Z_2$ is ILR.

Similarly, we can recover that IGFR and IGLR properties of positive random variables are preserved under product by Theorems 3.3 and 3.4.

4.2. Closure property of the dispersive order under convolution

By Theorem 3.2 and Proposition 2.2, we can recover the next two known results. We give the proof of the first one; the proof of the second one is similar.

Theorem 4.1. ([44, Theorem 3.B.7].) *Z is ILR if and only if $Z \leq_{\text{disp}} Z+Y$ for any random variable Y independent of Z.*

Proof. Suppose that Z is ILR. For any random variable Y, $0 \leq_{\text{disp}} Y$. By Proposition 2.2, for any constant $c \in \mathbb{R}$, $X' = -c$ and Y satisfy (3.3). From Theorem 3.2, it follows that

$$\mathbb{E}[v(Z-c)] \leq \mathbb{E}[v(Y+Z)] \Longrightarrow \mathbb{E}[u(Z-c)] \leq \mathbb{E}[u(Y+Z)]$$

whenever u, v are increasing with u more risk averse than v. Again by Proposition 2.2, we conclude that $Z \leq_{\text{disp}} Z+Y$.

The sufficiency is the same as in the proof of Theorem 3.2. □

Theorem 4.2. ([44, Theorem 3.B.8].) *Z is ILR if and only if $X+Z \leq_{\text{disp}} Y+Z$ whenever $X \leq_{\text{disp}} Y$ and Z is independent of X and Y.*

Note that ILR is closed under convolution. Repeated application of Theorem 4.2 yields [44, Theorem 3.B.9]. That is, let (X_i, Y_i), $i = 1, 2, \ldots, n$, be independent pairs of random variables such that $X_i \leq_{\text{disp}} Y_i$ for $i = 1, 2, \ldots, n$. If the X_i and Y_i are all ILR, then $\sum_{i=1}^n X_i \leq_{\text{disp}} \sum_{i=1}^n Y_i$.

4.3. Closure property of the location-independent riskier order under convolution

[18] proved that Z is DRHR if and only if $Z \leq_{\text{lir}} Z+Y$ for any random variable Y, independent of Z. Based on this fact and Propositions 2.1 and 2.3, we conclude the following result.

Theorem 4.3. *Let Z be a random variable. Then Z is DRHR if and only if $X+Z \leq_{\text{lir}} Y+Z$ whenever $X \leq_{\text{lir}} Y$ and Z is independent of X and Y.*

Note that DRHR is closed under convolution. Repeated application of Theorem 4.3 yields the following corollary.

Corollary 4.1. *Let (X_i, Y_i), $i = 1, 2, \ldots, n$, be independent pairs of random variables such that $X_i \leq_{\text{lir}} Y_i$ for $i = 1, 2, \ldots, n$. If the X_i and Y_i are all DRHR, then $\sum_{i=1}^n X_i \leq_{\text{lir}} \sum_{i=1}^n Y_i$.*

[18] obtained Theorem 4.3 and Corollary 4.1 with $X \leq_{\text{lir}} Y$ and $X_i \leq_{\text{lir}} Y_i$ replaced by the stronger assumptions $X \leq_{\text{disp}} Y$ and $X_i \leq_{\text{disp}} Y_i$, respectively.

https://doi.org/10.1017/jpr.2023.71 Published online by Cambridge University Press

4.4. Closure property of the excess wealth order under convolution

Observe that a random variable X is IFR if and only if $-X$ is DRHR. In terms of (2.2), all results for the order $\leq_{\rm lir}$ in Section 4.3 can be translated for the order $\leq_{\rm ew}$. The first one is [44, Theorem 3.C.8], which states that Z is IFR if and only if $Z \leq_{\rm ew} Z + Y$ for any random variable Y, independent of Z.

Theorem 4.4. *Let Z be a random variable. Then Z is IFR if and only if $X + Z \leq_{\rm ew} Y + Z$ whenever $X \leq_{\rm ew} Y$ and Z is independent of X and Y.*

Corollary 4.2. *Let (X_i, Y_i), $i = 1, 2, \ldots, n$, be independent pairs of random variables such that $X_i \leq_{\rm ew} Y_i$ for $i = 1, 2, \ldots, n$. If the X_i and Y_i are all IFR, then $\sum_{i=1}^{n} X_i \leq_{\rm ew} \sum_{i=1}^{n} Y_i$.*

[44, Theorems 3.C.9 and 3.C.10] are Theorem 4.4 and Corollary 4.2 with $X \leq_{\rm ew} Y$ and $X_i \leq_{\rm ew} Y_i$ replaced by the stronger assumptions $X \leq_{\rm disp} Y$ and $X_i \leq_{\rm disp} Y_i$, respectively.

4.5. Closure property of the total time on test transform order under convolution

[20] obtained the following connection between the total time on test transform and the excess wealth orders, which will be useful in establishing the closure property of the total time on test transform order under convolution. Recall that, for two random variables X and Y with respective distribution functions F and G, X is said to be smaller thant Y, denoted by $X \leq_{\rm st} Y$, if $F(x) \geq G(x)$ for all $x \in \mathbb{R}$.

Proposition 4.1 ([20].) *Let X and Y be two random variables with finite means. Then $X \leq_{\rm ttt} Y$ if and only if there exists a random variable W such that $X \geq_{\rm ew} W \leq_{\rm st} Y$ and $\mathbb{E}X = \mathbb{E}W$.*

Theorem 4.5. *Let X, Y, and Z be three random variables with finite means such that $X \leq_{\rm ttt} Y$. If Z is IFR and independent of X and Y, then $X + Z \leq_{\rm ttt} Y + Z$.*

Proof. Assume that $X \leq_{\rm ttt} Y$. By Proposition 4.1, there exists a random variable W such that $X \geq_{\rm ew} W \leq_{\rm st} Y$ and $\mathbb{E}X = \mathbb{E}W$, where W is independent of Z. From Theorem 4.4, it follows that $X + Z \geq_{\rm ew} W + Z$. Since $Z + W \leq_{\rm st} Z + Y$ and $\mathbb{E}(X + Z) = \mathbb{E}(W + Z)$, again applying Proposition 4.1 yields that $X + Z \leq_{\rm ttt} Y + Z$. □

Note that IFR is closed under convolution. Repeated application of Theorem 4.5 yields the following corollary.

Corollary 4.3. *Let (X_i, Y_i), $i = 1, 2, \ldots, n$, be independent pairs of random variables such that $X_i \leq_{\rm ttt} Y_i$ for $i = 1, \ldots, n$. If the X_i and Y_i are all IFR, then $\sum_{i=1}^{n} X_i \leq_{\rm ttt} \sum_{i=1}^{n} Y_i$.*

[20] obtained Theorem 4.5 and Corollary 4.3 with IFR replaced by a stronger aging notion ILR.

4.6. Closure property of the star order under product

Let X and Y be two nonnegative random variables with distribution functions F and G, respectively. Recall that X is smaller than Y in the star order, denoted by $X \leq_* Y$, if $G^{-1}F(x)$ is star-shaped in $x \in \mathbb{R}_+$, i.e. $G^{-1}F(x)/x$ is increasing in $x \in \mathbb{R}_+$.

Observe that $X \leq_{\rm disp} Y \Longleftrightarrow \mathrm{e}^X \leq_* \mathrm{e}^Y$. The next three results follow directly from Proposition 2.2 and Theorems 4.1 and 4.2, respectively.

Theorem 4.6. *$X \leq_* Y$ if and only if, for all increasing functions u and v with u being more risk averse than v, and for every real number $c > 0$,*

$$\mathbb{E}[v(cX)] \geq \mathbb{E}[v(Y)] \Longrightarrow \mathbb{E}[u(cX)] \geq \mathbb{E}[u(Y)].$$

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Theorem 4.7. *Z is IGLR if and only if $Z \le_* ZY$ for any positive random variable Y independent of Z.*

Theorem 4.8. *Z is IGLR if and only if $XZ \le_* YZ$ whenever $X \le_* Y$ and Z is independent of X and Y.*

Note that IGLR is closed under convolution. Repeated application of Theorem 4.8 yields the following corollary.

Corollary 4.4. *Let (X_i, Y_i), $i = 1, 2, \ldots, n$, be independent pairs of positive random variables such that $X_i \le_* Y_i$ for $i = 1, 2, \ldots, n$. If the X_i and Y_i are all IGLR, then $\prod_{i=1}^n X_i \le_* \prod_{i=1}^n Y_i$.*

5. Concluding remarks

We have characterized the ILR, IFR, IGLR, and IGFR aging notions in terms of expected utilities, and presented several applications of these characterizations. As we know, IFRA is an important aging notion. It is still an open problem which kind of expected utilities can be used to characterize IFRA. This challenging question requires future work. However, we have a conjecture (Conjecture 5.1) concerning the characterization of IFRA.

First, recall that a function $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\phi(0) = 0$ is star-shaped [antistar-shaped] if $\phi(x)/x$ is increasing [decreasing] in $x \in \mathbb{R}_+$. Any star-shaped function is increasing, while an antistar-shaped function may not be increasing. For example, $\phi(x) = x\mathbf{1}_{[0,t]}(x)$ is antistar-shaped but not increasing. Note that every increasing convex [concave] function ϕ on $\mathbb{R}_+$ with $\phi(0) = 0$ is star-shaped [antistar-shaped].

Conjecture 5.1. *Let X and Y be two nonnegative random variables admitting a comparative statics result in the sense that*

$$\mathbb{E}[v(X)] \le \mathbb{E}[v(Y)] \Longrightarrow \mathbb{E}[u(X)] \le \mathbb{E}[u(Y)] \tag{5.1}$$

whenever v and κ are star-shaped functions such that $u = \kappa(v)$. Let Z be a nonnegative random variable. Then Z is IFRA if and only if

$$\mathbb{E}[v(X+Z)] \le \mathbb{E}[v(Y+Z)] \Longrightarrow \mathbb{E}[u(X+Z)] \le \mathbb{E}[u(Y+Z)]$$

whenever X and Y satisfy (5.1), independent of Z, and v and κ are star-shaped with $u(\cdot) = \kappa(v(\cdot))$.

In Appendix A.2, Lemma A.1 is of independent interest, which is in the same vein as [22, Theorem 2]. We believe that Lemma A.1 will be useful in the proof of Conjecture 5.1.

Appendix A. Proofs of lemmas

A.1. Proof of Lemma 3.1

Proof. Necessity follows from [21, Theorem 1].

To prove sufficiency, consider the following two families of increasing functions:

$$\psi(x;\xi, y, z) = \begin{cases} 1, & x > z, \\ \xi, & y < x \le z, \\ 0 & x \le y; \end{cases} \qquad \varphi(x;\xi, y) = \begin{cases} \xi, & x > y, \\ 0, & x \le y, \end{cases}$$

where $0 \le \xi \le 1$ and $y < z$. Define $u(x) = \varphi(x;\xi, y)$ and $v(x) = \psi(x;\xi, y, z)$. It is easy to see that $u(x) = \kappa(v(x))$ for all x, where $\kappa(x) = \min\{x, \xi\}$ is increasing and concave. Obviously,

$$\mathbb{E}[u(Y) - u(X)] = -\xi[G(y) - F(y)], \tag{A.1}$$

$$\mathbb{E}[v(Y) - v(X)] = -\xi[G(y) - F(y)] - (1 - \xi)[G(z) - F(z)]. \tag{A.2}$$

It follows from (3.2), (A.1), and (A.2) that if $G(y) - F(y) > 0$ for some y, then $G(z) - F(z) \ge 0$ for all $z > y$. Otherwise, assume that $G(y) - F(y) > 0$ and $G(z) - F(z) < 0$ for some $y < z$. Then we can choose some $\xi \in (0, 1)$ such that $\mathbb{E}[v(Y)] \ge \mathbb{E}[v(X)]$. However, $\mathbb{E}[u(X)] > \mathbb{E}[u(Y)]$, which is contrary to (3.2). Therefore, $S^-(G - F) \le 1$ and the sign sequence is $-, +$ in the case of equality. This completes the proof of the lemma. □

A.2. Proof of Lemma A.1

[1] introduced an order called the upper length biased order between two nonnegative random variables based on the family of star-shaped functions. Here, we use the terminology 'star-shaped order' [44, Section 4.A]. Let X and Y be two nonnegative random variables with respective distribution functions F and G. X is said to be smaller than Y in the star-shaped [antistar-shaped] order, denoted by $X \le_{\text{ss}} [\le_{\text{ass}}] Y$, if $\mathbb{E}[\phi(X)] \le \mathbb{E}[\phi(Y)]$ for all star-shaped [antistar-shaped] functions $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$. It is known from [1, Theorems 3 and 6] that

$$X \le_{\text{ss}} Y \Longleftrightarrow \int_t^\infty x\,\mathrm{d}F(x) \le \int_t^\infty y\,\mathrm{d}G(y) \quad \text{for all } t \in \mathbb{R}_+,$$
$$X \le_{\text{ass}} Y \Longleftrightarrow \int_0^t x\,\mathrm{d}F(x) \le \int_0^t y\,\mathrm{d}G(y) \quad \text{for all } t \in \mathbb{R}_+.$$

The following lemma will be useful in the proof of Conjecture 5.1.

Lemma A.1. *Let X and Y be two nonnegative random variables with respective distribution functions F and G such that $Y \not\le_{\text{ss}} X$. Then*

$$\mathbb{E}[v(X)] \le \mathbb{E}[v(Y)] \Longrightarrow \mathbb{E}[u(X)] \le \mathbb{E}[u(Y)] \tag{A.3}$$

for all star-shaped functions v and κ such that $u(\cdot) = \kappa(v(\cdot))$ if and only if there exists some $z \in \mathbb{R}_+$ such that

$$\Delta(t) \text{ is increasing in } t \in [0, z], \tag{A.4}$$
$$\Delta(t) \le 0 \text{ for } t \in [0, z) \text{ and } \Delta(t) \ge 0 \text{ for } t \ge z, \tag{A.5}$$

where $\Delta(t) = \int_{(t,\infty)} x\,\mathrm{d}[G(x) - F(x)]$.

Proof. To prove necessity, consider the following three-parameter family of star-shaped functions:

$$\psi(x;\xi, z, y) = \begin{cases} 0, & x \le z, \\ \xi x, & x \in (z, y], \\ x, & x > y, \end{cases}$$

where $0 < \xi \le 1$ and $0 \le z < y < \infty$. Choose $z_1 < z_2 < y$. Then $\psi(x;\xi, z_2, y) = \kappa(\psi(x;\xi, z_1, y))$, where

$$\kappa(x) = \begin{cases} 0, & x \le \xi z_2, \\ x, & x > \xi z_2 \end{cases}$$

https://doi.org/10.1017/jpr.2023.71 Published online by Cambridge University Press

is star-shaped. Hence, define $v(x) \equiv \psi(x, \xi, z_1, y)$ and $u(x) = \psi(x, \xi, z_2, y)$. Note that

$$\mathbb{E}[v(Y) - v(X)] = \xi\Delta(z_1) + (1-\xi)\Delta(y),$$
$$\mathbb{E}[u(Y) - u(X)] = \xi\Delta(z_2) + (1-\xi)\Delta(y).$$

It follows directly that if there are y, z_1, z_2 such that $0 \le z_1 < z_2 < y$ and $\Delta(y) > 0$, $0 > \Delta(z_1) > \Delta(z_2)$, then it is possible to choose some $\xi \in (0, 1)$ such that $\mathbb{E}[v(X)] = \mathbb{E}[v(Y)]$ but $\mathbb{E}[u(X)] > \mathbb{E}[u(Y)]$, which is contrary to what is required.

Choosing $\xi = 1$, we obtain that if $\Delta(z_1) \ge 0$ then $\Delta(z_2) \ge 0$ for $0 \le z_1 < z_2$. Therefore, $\Delta(\cdot)$ must not have a sign change from positive to negative. The only other cases to consider are those where $\Delta(\cdot)$ has a uniform sign, but these are the trivial cases $X \le_{ss} Y$ and $Y \le_{ss} X$. $Y \le_{ss} X$ is ruled out by the assumption, and $X \le_{ss} Y$ is admitted by the conditions of the theorem (choosing $z = 0$). This proves the necessity.

To prove sufficiency, suppose that (A.4) and (A.5) hold for some $z \in \mathbb{R}$. Then, for any $(s, t] \subset (0, z]$, $\int_{(s,t]} x\, \mathrm{d}[G(x) - F(x)] \le 0$. Since $v(x)/x$ is increasing on $\mathbb{R}_+$, by Lemma A.2(i) we have

$$\int_{(0,t]} v(x)\, \mathrm{d}[G(x) - F(x)] = \int_{(0,t]} \frac{v(x)}{x} \cdot x\, \mathrm{d}[G(x) - F(x)] \le 0.$$

On the other hand, note that $\int_{\mathbb{R}_+} v(x)\, \mathrm{d}[G(x) - F(x)] = \mathbb{E}[v(Y) - v(X)] \ge 0$. Therefore, for any $t \le z$, $\int_{(t,\infty)} v(x)\, \mathrm{d}[G(x) - F(x)] \ge 0$. Note that this also holds for $t > z$ by applying Lemma A.2(i) and using the fact $\Delta(t) \ge 0$ for all $t \ge z$. Since $\kappa(x)$ is star-shaped, $u(x)/v(x) = \kappa(v(x))/v(x)$ is increasing on $\mathbb{R}_+$. Then, again by Lemma A.2(i),

$$\mathbb{E}[u(Y) - u(X)] = \int_{\mathbb{R}_+} \frac{u(x)}{v(x)} \cdot v(x)\, \mathrm{d}[G(x) - F(x)] \ge 0.$$

This proves the sufficiency. □

Remark A.1. In Lemma A.1, assume additionally that X and Y have respective probability density functions f and g, which are continuous on $\mathbb{R}_+$. Then (A.3), (A.4), and (A.5) imply that, for any $t \in \mathbb{R}_+$,

$$\Delta(t) < 0 \Longrightarrow g(t) \le f(t). \tag{A.6}$$

Conversely, under the assumption $Y \not\le_{ss} X$, (A.6) implies (A.3), (A.4), and (A.5).

We outline the proof of this last assertion. Under the assumption $Y \not\le_{ss} X$, there exists an $x_0 > 0$ such that $\Delta(x_0) > 0$. If $\Delta(x) \ge 0$ for all x, then (A.3), (A.4), and (A.5) hold with $z = 0$. Then assume that there exists $y_0 > 0$ such that $\Delta(y_0) < 0$. We turn to prove that $\Delta(x)$ is increasing on $[0, y_0]$. If this is not true, there exists an an interval $[a, b] \subset [0, y_0]$ such that $\Delta(x) < 0$ for all $x \in [a, b]$, and $\Delta(a) > \Delta(b)$. So we can find $d \in (a, b)$ such that $\Delta'(d) = -d(g(d) - f(d)) < 0$, i.e. $g(d) > f(d)$. However, by (A.6), $\Delta(d) < 0$ implies $g(d) \le f(d)$, which is a contradiction. Therefore, (A.3), (A.4), and (A.5) follow.

Lemma A.2. ([5, p. 120].) *Let $W(x)$ be a Lebesgue–Stieltjes measure, not necessarily positive, and $-\infty \le a < b \le +\infty$.*

(i) *If $\int_{(t,b]} \mathrm{d}W(x) \ge 0$ for all $t \in [a, b]$, and if ψ is nonnegative and increasing, then $\int_{(a,b]} \psi(x)\, \mathrm{d}W(x) \ge 0$.*

(ii) *If $\int_{(a,t]} \mathrm{d}W(x) \ge 0$ for all $t \in [a, b]$, and if ψ is nonnegative and decreasing, then $\int_{(a,b]} \psi(x)\, \mathrm{d}W(x) \ge 0$.*

Acknowledgements

The authors are grateful to the Associate Editor and two anonymous referees for their comprehensive reviews of an earlier version of this paper.

Funding Information

W. Zhuang acknowledges the financial support from National Natural Science Foundation of China (No. 71971204). T. Hu is grateful for the financial support from National Natural Science Foundation of China (No. 71871208) and Anhui Center for Applied Mathematics.

Competing Interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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