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LOCAL COMPLEMENTS TO THE HAUSDORFF-YOUNG THEOREM FOR AMALGAMS

BY

MARIA L. TORRES DE SQUIRE

In memoriam Graciela Salicrup

ABSTRACT. Let G be a locally compact abelian group. An amalgam space $(L^p, \ell^q)(G)$ $(1 \leqq p, q \leqq \infty)$ is a Banach space of functions which belong locally to $L^p(G)$ and globally to ℓ^q. In this paper we present non-inclusion results related to the Hausdorff-Young theorem for amalgams.

§1. **Introduction**. Let G be a locally compact abelian group with dual group $\hat{G}$. An amalgam space $(L^p, \ell^q)(G)$ $(1 \leqq p, q \leqq \infty)$ is a Banach space of (equivalence classes of) functions on G which belong locally to L^p and globally to ℓ^q (a precise definition will be given in §2). For a historical background on these spaces see [8].

The Hausdorff-Young theorem for amalgams [10, Theorem 8], [1, Theorem II], states that, for $1 \leqq p, q \leqq 2$, the Fourier transform of a function in $(L^p, \ell^q)(G)$ belongs to $(L^{q'}, \ell^{p'})(\hat{G})$. J. J. F. Fournier [5] studied the possibility that for $1 \leqq p \leqq 2$ and a measurable subset E of $\hat{G}$, $L^{p\wedge}|E \subset L^r(E)$ for $r \neq p'$, where $L^{p\wedge}|E$ is the set of Fourier transforms of functions in L^p restricted to E. In this paper we deal with the corresponding problem for amalgams, that is, we want to know if $(L^p, \ell^q)^\wedge|E \subset (L^r, \ell^s)(E)$ for $r > q'$ and $s < p'$, $((L^{q'}, \ell^{p'}) \subset (L^r, \ell^s)$ whenever $q' \geqq r$ and $p' \leqq s)$.

Our main Theorems are Theorem 3.2, Theorem 4.3 and Theorem 6.2. These theorems are extensions of [5, Theorem 1, Theorem 2, Theorem 3]. We will conclude from them the following.

(i) If $\hat{G}$ is nondiscrete, E is not locally null, $1 \leqq p \leqq 2$ and $1 < q \leqq 2$, then for all $1 \leqq s \leqq \infty$, $(L^p, \ell^q)^\wedge|E \not\subset \bigcup_{r>q'} (L^r, \ell^s)(E)$.

(ii) If $\hat{G}$ is noncompact and $1 \leqq p, q \leqq 2$, then for all $1 \leqq r \leqq \infty$,

$$(L^p, \ell^q)^\wedge \not\subset \bigcup_{s<p'} (L^r, \ell^s)(\hat{G}).$$

(iii) If $\hat{G}$ is noncompact, then there exists an open set E of infinite measure such that for $1 < p < 2$ and $1 \leq q \leq 2$, $(L^p, \ell^q)^\wedge|E \subset (L^{q'}, \ell^2)(E)$.

§2. **Definition and properties of $(L^p, \ell^q)(G)$**. We denote by L^p_{loc} $(1 \leqq p \leqq \infty)$ the space of (equivalence classes of) functions f on G such that f restricted to any compact subset of G belongs to $L^p(G)$. The following definition of $(L^p, \ell^q)(G)$ is due to J. Stewart [12]; for a definition of amalgams on a locally compact not necessarily abelian group see [2] and [4].

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DEFINITION 2.1. *By the structure theorem [9, Theorem 24.30] G is topologically isomorphic to $R^a \times G_1$ where **a** is a nonnegative integer and G_1 is a locally compact abelian group which contains an open compact subgroup H. Let $L = [0, 1)^a \times H$ and $J = Z^a \times T$, where T is a transversal of H in G_1, i.e. $G_1 = \{t + H | t \in T\}$ is a coset decomposition of G_1. For $\alpha \in J$ we define $L_\alpha = \alpha + L$, and therefore G is equal to the disjoint union of relatively compact sets L_α.*

The *amalgam space* $(L^p, \ell^q)(G) = (L^p, \ell^q)$ $(1 \leqq p, q \leqq \infty)$ is the linear space

$$\left\{ f \in L^p_{\text{loc}} \,\middle|\, \|f\|_{p,q} = \left[\sum_\alpha \left[\int_{L_\alpha} |f|^p \right]^{q/p} \right]^{1/q} < \infty \right\},$$

endowed with the norm $\|\cdot\|_{p,q}$. We make the appropriate changes for p, q infinite. For a definition of this space on $\hat{G}$ see [12, pp. 1283]. We define for a subset E of G, the space $(L^p, \ell^q)(E)$ to be the space of functions $f \in L^p_{\text{loc}}$ such that

$$\left[\sum_\alpha \|f\|^q_{L^p(L_\alpha \cap E)} \right]^{1/q} < \infty.$$

The amalgam $(C_0, \ell^q)(G) = (C_0, \ell^q)$ $(1 \leqq q \leqq \infty)$ is the intersection of the space C_0, the space of continuous functions which vanish at infinity, and $(L^\infty, \ell^q)(G)$. Note that $C_c(G) = C_c$, the space of continuous functions with compact support, is included in all amalgam spaces and that $(L^p, \ell^p) = L^p$. The Banach spaces (L^p, ℓ^q) $(1 \leqq p, q \leqq \infty)$ satisfy the following inclusion relations [12, p. 1284].

$$(2.1) \qquad (L^p, \ell^{q_1}) \subset (L^p, \ell^{q_2}) \qquad \text{if} \quad q_1 \leqq q_2$$

$$(2.2) \qquad (L^{p_2}, \ell^q) \subset (L^{p_1}, \ell^q) \qquad \text{if} \quad p_2 \geqq p_1$$

$$(2.3) \qquad (L^p, \ell^q) \subset L^p \cap L^q \qquad \text{if} \quad p > q$$

$$(2.4) \qquad L^q \subset (L^q, \ell^\infty) \cap (L^1, \ell^q) \qquad \text{if} \quad 1 < q < \infty.$$

REMARK 2.2. The inclusions (2.1) and (2.2) are strict if G is noncompact, nondiscrete, respectively, while (2.3) and (2.4) are strict if G is neither compact nor discrete.

REMARK 2.3. If G is compact (discrete), then $(L^p, \ell^q) = L^p((L^p, \ell^q) = \ell^q)$ for $1 \leqq p, q \leqq \infty$.

As usual $(\check{f})\hat{f}$ will denote the (inverse of) Fourier transform of f. We will denote by $(L^p, \ell^q)\hat{}$ the set of Fourier transforms of functions in (L^p, ℓ^q) and $(L^p, \ell^q)\hat{}|E$ will be the set of functions in $(L^p, \ell^q)\hat{}$ restricted to the subset E of $\hat{G}$.

§3. **The case where $\hat{G}$ is nondiscrete**. Theorem 1 of [5] is as follows.

THEOREM 3.1. *If $\hat{G}$ is nondiscrete and E is not locally null, then for* $1 < q \leqq 2$,

$$(3.1) \qquad L^q\hat{}|E \not\subset \bigcup_{r > q'} L^r(E).$$

We generalize this result by proving the next theorem. We observe that for the

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particular case $1 < q \leqq 2$, Theorem 3.2 implies Bloom's Theorem 1 in [3].

THEOREM 3.2. *If $\hat{G}$ is nondiscrete and $E \subset \hat{G}$ is not locally null then for $1 < q \leqq 2$,*

$$(L^\infty, \ell^q)\hat{}|E \not\subset \bigcup_{r>q'} (L^r, \ell^\infty)(E). \tag{3.2}$$

NOTE. If G is discrete, then (3.2) becomes (3.1) by Remark 2.3. Hence (3.2) extends (3.1) when G is neither compact nor discrete because in this case (L^∞, ℓ^q) and L^r $(1 \leqq q, r < \infty)$ are proper subspaces of L^q and (L^r, ℓ^∞) respectively (Remark 2.2).

PROOF. Since E is not locally null, it contains a subset of positive measure. By the inner regularity of the Haar measure this subset contains a compact set of positive measure. Therefore it is enough to prove the theorem for compact sets E of positive measure. Suppose that

$$(L^\infty, \ell^q)\hat{}|E \subset (L^r, \ell^\infty)(E) \quad \text{for some} \quad r \in (q', \infty).$$

Take $f \in L^q(G)$ and let $\phi \in C_c(\hat{G})$ such that $\phi \equiv 1$ on E and $\check{\phi} \in (L^{q'}, \ell^1)(G)$ [12, Theorem 3.1]. By (2.3) we have that $(L^{q'}, \ell^1) \subset L^1$, so ϕ is equal to the Fourier transform of $\check{\phi}$. By [3, §7 i)] the function $f * \check{\phi}$ belongs to $(L^\infty, \ell^q)(G)$, and hence by our assumption $(f * \check{\phi})\hat{} = \hat{f}\phi$ restricted to E belongs to $(L^r, \ell^\infty)(E) = L^r(E)$ (E is a compact set!). So $\hat{f}|E \in L^r(E)$ and this contradicts Theorem 3.1. Thus

$$(L^\infty, \ell^q)\hat{}|E \not\subset (L^r, \ell^\infty)(E) \quad \text{for all} \quad q' < r < \infty. \tag{3.3}$$

The rest of the proof is similar to [5, Theorem 1]. For $q' < r < \infty$, define the function F_r on $(L^\infty, \ell^q)(G)$ by

$$F_r(f) = \|\hat{f}|E\|_{r,\infty}. \tag{3.4}$$

The function F_r takes the value infinity by (3.3). Also $F_r(\alpha f) = \alpha F_r(f)$ and $F_r(f + g) \leqq F_r(f) + F_r(g)$ for all nonnegative real α and all f, g in $(L^\infty, \ell^q)(G)$. These properties of F_r imply that the set $V_\alpha = \{f \in (L^\infty, \ell^q)(G) | F_r(f) > \alpha\}$ is dense in $(L^\infty, \ell^q)(G)$. Moreover F_r is lower semicontinuous because $F_r = \sup\{F_g | g \in \mathcal{U}\}$, where $\mathcal{U} = \{g \in (L^q, \ell^1) | \, \|g\|_{r',1} \leqq 1\}$ (see (2.2)) and F_g is a continuous function on $(L^\infty, \ell^q)(G)$ defined by $F_g(f) = |\int_E \hat{f} g|$. Hence, by the Baire theorem the set $\{f \in (L^\infty, \ell^q)(G) | F_r(f) = \infty\}$ is of type G_δ. Choose a strictly decreasing sequence $\{r_n\}$ converging to q'. Again by Baire's theorem (as in [11, Corollary of Theorem 5.6]) the set $\{f \in (L^\infty, \ell^q)(G) | \, \|\hat{f}|E\|_{r_n,\infty} = \infty$ for all $n \in N\}$ is a dense G_δ set. Take f in this set; since $(L^\infty, \ell^q) \subset (L^1, \ell^q)$ we have by the Hausdorff-Young Theorem that $\hat{f}|E \in (L^{q'}, \ell^\infty)(E)$. If also $\hat{f}|E \in (L^r, \ell^\infty)(E)$, then for all sufficiently large n, we have that $\hat{f}|E \in (L^{r_n}, \ell^\infty)(E)$ by (2.2), and this contradicts the choice of f. Hence $\hat{f}|E \notin (L^r, \ell^\infty)(E)$ for all $r > q'$ and the proof is complete. □

COROLLARY 3.3. *If $\hat{G}$ is nondiscrete and $1 < q \leqq 2$, then*

$$(L^\infty, \ell^q)\hat{} \not\subset \bigcup_{r>q'} (L^r, \ell^\infty)(\hat{G}).$$

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By (2.1), (2.2) and Theorem 3.2 we have the following result.

COROLLARY 3.4. *If $\hat{G}$ is nondiscrete, E is not locally null, $1 \leqq p \leqq 2$ and $1 < q \leqq 2$, then for all $1 \leqq s \leqq \infty$,*

$$(L^p, \ell^q)\hat{}|E \not\subset \bigcup_{r>q'} (L^r, \ell^s)(E).$$

Hence

$$(L^p, \ell^q)\hat{} \not\subset \bigcup_{r>q'} (L^r, \ell^s)(\hat{G}).$$

§4. **The case where $\hat{G}$ is noncompact**. Theorem 2, b) of [5] is as follows:

THEOREM 4.1. *If $\hat{G}$ is noncompact and $1 \leqq p \leqq 2$, then*

$$L^{p\hat{}} \not\subset \bigcup_{s<p'} L^s(\hat{G}). \tag{4.1}$$

We prove that under the same conditions

$$(L^p, \ell^1)\hat{} \not\subset \bigcup_{s<p'} (L^1, \ell^s)(\hat{G}). \tag{4.2}$$

NOTE. If G is compact, then (4.2) becomes (4.1) by Remark 2.3, hence (4.2) extends (4.1) when G is neither compact nor discrete because in this case (L^p, ℓ^1) and L^s are proper subspaces of L^p and (L^1, ℓ^s) for $1 < p, s < \infty$ (Remark 2.2).

Theorem 4.3 is fully proved in [13, Theorem 10.2]. We present a short version of this proof which uses the following result [2, Theorem IV].

PROPOSITION 4.2. *Let $\Phi(G) = \{\phi \in C_c(G) | \hat{\phi} \in (C_0, \ell^1)(\hat{G})\}$, endowed with the norm $\phi \rightarrow \|\hat{\phi}\|_{p,q}$, $1 \leqq p, q \leqq \infty$. If $\mu \in M_\infty(G)$ and there exists a constant C such that for all $\phi \in \Phi$,*

$$\left|\int_G \phi(x) d\mu(x)\right| \leqq C\|\hat{\phi}\|_{p,q},$$

then $\hat{\mu} \in (L^{p'}, \ell^{q'})(\hat{G})$.

THEOREM 4.3. *If $\hat{G}$ is noncompact and $1 \leqq p \leqq 2$, then (4.2) holds.*

PROOF. Case 1) $p = 2$. Let E be a compact subset of G of positive measure with interior Ω, and $1 \leqq s < 2$. By [3, Theorem 1] there exists $f \in (L^\infty, \ell^2)(\hat{G})$ such that $\check{f} - \check{g}$ does not vanish identically on Ω for all g in $(L^1, \ell^s)(\hat{G})$. Take $\phi \in C_c(G)$ such that $\phi \equiv 1$ on E. Since $\check{f} \in L^2$ by (2.3) and $\phi \in (L^\infty, \ell^2)$ we have that $\check{f}\phi \in (L^2, \ell^1)(G)$ by [3, §7 h)], so $\check{f}\phi \in L^1 \cap L^2$ and this implies that the inverse of the Fourier transform of $(\check{f}\phi)\hat{}$ is equal to $\check{f}\phi$. Therefore $(\check{f}\phi)\hat{}$ is not in $(L^1, \ell^s)(\hat{G})$ because $\check{f} - \check{f}\phi$ does vanish on Ω. Hence there exists a function $\check{f}\phi \in (L^2, \ell^1)(G)$ such that $(\check{f}\phi)\hat{} \notin (L^1, \ell^s)(\hat{G})$. This means that

$$(L^2, \ell^1)\hat{} \not\subset (L^1, \ell^s)(\hat{G}) \quad \text{for all} \quad 1 \leqq s < 2. \tag{4.3}$$

A simple modification of the argument just given shows that

$$(4.4) \qquad (L^2, \ell^1)(E)^\wedge \not\subset (L^1, \ell^2)(G.)$$

(Take $\phi \equiv 1$ on some open subset Ω_0 of Ω such that $\overline{\Omega_0} \subset \Omega$ with support in E).

Case 2) $p < 2$. Similarly to (4.3) we want to prove that

$$(4.5) \qquad (L^p, \ell^1)^\wedge \not\subset (L^1, \ell^s)(\hat{G}) \quad \text{for all} \quad 1 \leqq s < p'.$$

If $1 \leqq s < 2$, we know by (4.3) that (4.5) holds because $p < 2$ and then $(L^2, \ell^1) \subset (L^p, \ell^1)$ by (2.2). So we consider the case when $2 \leqq s < p'$ and assume the contrary. By the Closed Graph Theorem the map $T: (L^p, \ell^1) \to (L^1, \ell^s)(\hat{G})$ given by $T(f) = \hat{f}$ is bounded. Indeed, let $\{f_n\}$ be a sequence in (L^p, ℓ^1) such that $\lim \|f_n\|_{p1} = 0$. We assume that $\lim \|\hat{f}_n - g\|_{1,s} = 0$ and take $\phi \in C_c(\hat{G})$. Since $\phi \in (L^\infty, \ell^{s'}) \cap (L^1, \ell^p)$ and $(L^\infty, \ell^{s'})$ is the dual of (L^1, ℓ^s) [2, §7 g)] we have by the Hölder and Hausdorff-Young inequalities [8, Theorem 2.2] that $\int g\phi = 0$. This implies that $g = 0$ by a standard measure theory argument.

Now, let $g \in (L^\infty, \ell^{s'})(\hat{G})$ and $\phi \in \Phi(\hat{G})$. Then

$$|\int g(\hat{x})\phi(\hat{x})d\hat{x}| = |\int g(\hat{x})T(\check{\phi})(\hat{x})d\hat{x}| \leqq \|g\|_{\infty,s'}\|T(\check{\phi})\|_{1,s} \leqq \|g\|_{\infty,s'}\|T\|\|\check{\phi}\|_{p,1}.$$

By Proposition 4.2, $\check{g} \in (L^{p'}, \ell^\infty)(G)$, and therefore $(L^\infty, \ell^{s'})(\hat{G})^\vee \subset (L^{p'}, \ell^\infty)(G)$ (this last inclusion can also be proved using Fournier's argument in [5, p. 169]).

On the other hand by Corollary 3.3 we have that $(L^\infty, \ell^{s'})(\hat{G})^\vee \not\subset (L^{p'}, \ell^\infty)$ because G is nondiscrete, $s' \leq 2$, and $p' > s = (s')'$. This contradiction shows (4.5).

From cases 1) and 2) we conclude that for $1 \leqq p \leqq 2$,

$$(4.6) \qquad (L^p, \ell^1)(G)^\wedge \not\subset (L^1, \ell^s)(\hat{G}) \quad \text{for all} \quad 1 \leqq s < p'.$$

For $s \in [1, p')$ define the function F_s on $(L^p, \ell^1)(G)$ by $F_s(f) = \|\hat{f}\|_{1,s}$. As in the proof of Theorem 3.2 the set $V_\alpha = \{f \in (L^p, \ell^1)(G) | F_s(f) > \alpha\}$ is dense in $(L^p, \ell^1)(G)$ for all real α. Also, F_s is lower semicontinuous and therefore the set $\{f \in (L^p, \ell^1) | F_s(f) = \infty\}$ is a G_δ set. If $\{s_n\}$ is a strictly increasing sequence converging to p', then by Baire's theorem the set $\{f \in (L^p, \ell^1)(G) | \|\hat{f}\|_{1,s_n} = \infty$ for all $n \in N\}$ is a dense set of type G_δ. Take f in this set; let $s \in [1, p')$. Since $f \in (L^p, \ell^1)$ its Fourier transform $\hat{f}$ belongs to $(L^\infty, \ell^{p'})$ by [8, Theorem 2.8], then by (2.1) we have that $\hat{f} \in (L^1, \ell^{s_n})(\hat{G})$ for all sufficiently large n if $\hat{f} \in (L^1, \ell^s)(\hat{G})$. From this we conclude that $\hat{f} \in (L^1, \ell^s)(\hat{G})$ for all $1 \leqq s < p'$ and this proves the theorem. □

The next corollary extends [5, Remark 2] if G is neither compact nor discrete.

Corollary 4.4. *If $E \subset G$ is not locally null, G is noncompact and $1 \leqq p \leqq 2$, then*

$$(4.7) \qquad (L^p, \ell^1)(E)^\wedge \not\subset \bigcup_{s<p'} (L^1, \ell^s)(\hat{G}).$$

Proof. As in Theorem 3.2 it is enough to prove the corollary for compact sets E of positive measure. It follows from (4.4) and an argument like that for the case 2) of Theorem 4.3 that $(L^p, \ell^1)(E)^\wedge \not\subset (L^1, \ell^s)(\hat{G})$ for all $s < 2$. The case where $2 < s < p'$ follows from Theorem 3.3 and a duality argument like that given in the last part of the proof of Theorem 4.3. The remainder of the argument uses the Baire Category Theorem as before.

The next result is a direct consequence of (2.1) and (2.2).

COROLLARY 4.5. *If $\hat{G}$ is noncompact and $1 \leqq p, q \leqq 2$, then for $1 \leqq r \leqq \infty$, $(L^p, \ell^q)\hat{} \not\subset \bigcup_{s<p'} (L^r, \ell^s)(\hat{G})$.*

§5. **The case where $\hat{G}$ is neither compact nor discrete**. Theorem 2, c) of [5] is as follows.

THEOREM 5.1. *If $\hat{G}$ is neither compact nor discrete and $1 < p \leqq 2$, then*

$$L^{p\hat{}} \not\subset \bigcup_{r \neq p'} L^r(\hat{G}). \tag{5.1}$$

The next result generalizes Theorem 5.1.

THEOREM 5.2. *If $\hat{G}$ is neither compact nor discrete and $1 < p \leqq q \leqq 2$ or $1 \leqq p < q \leqq 2$, then*

$$(L^p, \ell^q)\hat{} \not\subset \bigcup_{r \neq p', q'} [(L^r, \ell^\infty) \cap (L^1, \ell^r)]. \tag{5.2}$$

NOTE. If $p = q$, then Theorem 5.2 says that for $1 < p \leqq 2$,

$$L^{p\hat{}} \not\subset \bigcup_{r \neq p'} (L^r, \ell^\infty) \cap (L^1, \ell^r).$$

This improves the right side of (5.1) because by (2.4) the space L^r is a proper subspace of $(L^r, \ell^\infty) \cap (L^1, \ell^r)$ $(1 < r < \infty)$.

PROOF. By Corollary 3.3 there exists $f \in (L^\infty, \ell^q)$ such that

$$\hat{f} \notin \bigcup_{r > q'} (L^r, \ell^\infty). \tag{5.3}$$

By Theorem 4.3 there exists $h \in (L^p, \ell^1)$ such that

$$\hat{h} \notin \bigcup_{r < p'} (L^1, \ell^r). \tag{5.4}$$

We shall see that one of the three functions $\hat{f}$, $\hat{h}$, $\hat{f} + \hat{h}$ in $(L^p, \ell^q)\hat{}$ (remember that $(L^\infty, \ell^q) \subset (L^p, \ell^q)$ and $(L^p, \ell^1) \subset (L^p, \ell^q)$) does not belong to $\bigcup_{r \neq p', q'} (L^r, \ell^\infty) \cup (L^1, \ell^r)$.

Suppose $1 \leqq p < q \leqq 2$ and assume that $\hat{f} \in (L^{r_1}, \ell^\infty) \cap (L^1, \ell^{r_1})$, $\hat{h} \in (L^{r_2}, \ell^\infty) \cap (L^1, \ell^{r_2})$, and $\hat{f} + \hat{h} \in (L^{r_0}, \ell^\infty) \cap (L^1, \ell^{r_0})$ for some r_1, r_2, r_0 distinct from both p' and q'.

Since $\hat{f} \notin (L^{r_1}, \ell^\infty)$ if $r_1 > q'$ and $\hat{h} \notin (L^1, \ell^{r_2})$ if $r_2 < p'$, we conclude that $r_1 < q' < p' < r_2$. So by the inclusion properties (2.1) and (2.2) we have that

a) If $r_1 < q' < p' < r_2 \leqq r_0$, then $\hat{f} + \hat{h} \in (L^{r_2}, \ell^\infty)$. Hence, $\hat{f} = (\hat{f} + \hat{h}) - \hat{h} \in (L^{r_2}, \ell^\infty)$ and this contradicts (5.3) as $r_2 > q'$.

b) If $r_1 < q' < p' < r_0 < r_2$ or $r_1 < q' < r_0 < p' < r_2$, then $\hat{h} \in (L^{r_0}, \ell^\infty)$. Hence, $\hat{f} = (\hat{f} + \hat{h}) - \hat{h} \in (L^{r_0}, \ell^\infty)$ and again this contradicts (5.3) as $r_0 > q'$.

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c) If $r_1 < r_0 < q' < p' < r_2$, then $\hat{f} \in (L^1, \ell^{r_0})$. Hence, $\hat{h} = (\hat{f} + \hat{h}) - \hat{f} \in (L^1, \ell^{r_0})$ and this contradicts (5.4) as $r_0 < p'$.

d) If $r_0 < r_1 < q' < p' < r_2$, then $\hat{f} + \hat{h} \in (L^1, \ell^{r_1})$. Hence, $\hat{h} = (\hat{f} + \hat{h}) - \hat{f} \in (L^1, \ell^{r_1})$ and again this contradicts (5.4) as $r_1 < p'$. From a) − d) we conclude that $r_0 = p'$ or $r_0 = q'$. This contradiction shows (5.2). The proof for $1 < p \leqq q \leqq 2$ is the same. □

REMARK 5.3. Theorem 5.2 is no longer true if $1 \leqq q < p \leqq 2$ because in this case for any $p' < r < q'$ we have that

$$(L^p, \ell^q)^\wedge \subset (L^{q'}, \ell^{p'}) \subset (L^r, \ell^\infty) \cap (L^1, \ell^r).$$

§6. $\Lambda(q)$-sets in nondiscrete groups.

DEFINITION 6.1. [7, §2] *Let $E \subset \hat{G}$. A function f in $L^p(G)$ $(1 \leqq p \leqq 2)$ is an E-function if $\hat{f}$ is essentially supported by E. A set E is a* $\Lambda(q)$-set *for $1 < q < \infty$ if any E-function in $L^1(G)$ also belongs to $L^q(G)$.*

Fournier proved [5, Theorem 3] that any noncompact group G contains open sets E of infinite measure which are $\Lambda(r)$-sets for any $r \in (2, \infty)$ and have the property that for $1 < p < 2$,

$$L^{p\wedge}|E \subset \bigcap_{2<r<p'} L^r(E). \tag{6.1}$$

The generalization of (6.1) is as follows (c.f. [6, Theorem 4.1]).

THEOREM 6.2. *Let $\hat{G}$ be noncompact and E be a set as in [5, Theorem 3]. If $1 < p \leqq 2$, and $1 \leqq q \leqq 2$, then*

$$(L^p, \ell^q)^\wedge|E \subset (L^{q'}, \ell^2)(E) = \bigcap_{1 \leqq r \leqq q'} (L^r, \ell^2)(E). \tag{6.2}$$

NOTE. If $p = q$, then Theorem 6.2 says that

$$L^{p\wedge}|E \subset \bigcap_{1 \leqq r \leqq p'} (L^r, \ell^2)(E)$$

and this improves the right side of (6.1) because by (2.2) we have that

$$\bigcap_{1 \leqq r \leqq p'} (L^r, \ell^2) \subset \bigcap_{2 \leqq r \leqq p'} (L^r, \ell^2) \subset \bigcap_{2 \leqq r \leqq p'} L^r.$$

PROOF. If $p = 2$, then (6.2) is just the Hausdorff-Young theorem. So we assume that $1 < p < 2$ and $1 \leqq q \leqq 2$. Consider the inverse of the Fourier transform $\vee : (L^q, \ell^2)(E) \to (L^2, \ell^{q'})(G)$ [8, Theorem 2.8]. If $f \in (L^q, \ell^2)(E)$, then $\check{f}$ is an E-function [7, §2], [8, §8], and therefore by [7, Theorem 1] (see also [5, Remark 4]) the function actually maps $(L^q, \ell^2)(E)$ into $(L^{p'}, \ell^{q'})(G)$. Then its adjoint transform $\vee^* : (L^p, \ell^q)(G) \to (L^{q'}, \ell^2)(E)$ is such that

$$\vee^*(\bar{g})(f) = \int_G \bar{g}(x) \check{f}(x) dx \quad \text{for all} \quad g \in (L^p, \ell^q)(G)$$

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and $f \in (L^q, \ell^2)(E)$. Since $L^q \subset (L^q, \ell^2)$ and either $(L^p, \ell^q) \subset L^q$ or $L^q \subset (L^p, \ell^q)$ according to whether $p \geqq q$ or $q < p$, we can apply the Parseval identity (as in [9, 31.48 a)]) and we have that for all $f \in L^q(E)$ and either $g \in (L^p, \ell^q)(G)$ if $p \geqq q$ or $g \in L^q(G)$ if $q > p$; $\bigvee^*(\bar{g})(f) = \int_E f(\hat{x})\bar{\hat{g}}(\hat{x})d\hat{x}$. Since $\bigvee^*(\bar{g}) \in L^{q'}$ (see (2.3)) we conclude that $\bigvee^*(g) = \hat{g}|E$ for all $g \in (L^p, \ell^q)(G)$ if $p \geqq q$ or for all $g \in L^q$ if $q > p$. Hence $\hat{g}|E \in (L^{q'}, \ell^2)(E)$ for all $g \in (L^p, \ell^q)$ if $p \geqq q$ and $\hat{g}|E \in (L^{q'}, \ell^2)(E)$ for all $g \in L^q$ if $q > p$. In this last case we have that

$$\|\hat{g}|E\|_{q',2} \leqq \|\bigvee^*\|\|g\|_{p,q} \quad \text{for all} \quad g \in L^q(G). \tag{6.3}$$

Since L^q is dense in (L^p, ℓ^q) by [2, §7 c)] we conclude that $\hat{g}|E \in (L^{q'}, \ell^2)(E)$ for all $g \in (L^p, \ell^q)(G)$ and the proof is complete. □

The next corollary is a generalization of [5, Corollary of Theorem 3] and its proof is (mutatis mutandis) the same.

COROLLARY 6.3. *If G is infinite and* $1 < p < 2$, $1 \leqq q \leqq 2$, *then*

$$(L^p, \ell^q)(G)\hat{} \nsubseteq (L^{q'}, \ell^{p'})(\hat{G}).$$

§7. **Final remarks**. By the Hausdorff-Young theorem we have that if $1 \leqq p \leqq 2$, then $(L^p, \ell^1) \subset (C_0, \ell^{p'})$. So the only possible improvement is global, i.e. for a not locally null set E

$$(L^p, \ell^1)\hat{}|E \subset (C_0, \ell^s)(E) \quad \text{for some} \quad s < p'.$$

If $\hat{G}$ is compact and E is not locally null, then for $1 \leqq p \leqq 2$, $(L^p, \ell^1) = \ell^1$ and we have that for $s < p'$,

$$(L^p, \ell^1)\hat{}|E = \ell^{1\hat{}}|E \subset C_0(E) = (C_0, \ell^s)(E).$$

The next result follows from Theorems 4.3 and 6.2 together with (2.2).

PROPOSITION 7.1. *Let* $\hat{G}$ *be noncompact. Then*

i) *If* $1 \leqq p \leqq 2$, *then* $(L^p, \ell^1)\hat{} \not\subset \bigcup_{s<p'} (C_0, \ell^s)$.

ii) *If* $1 < p < 2$, *then there exists an open set E of infinite measure such that* $(L^p, \ell^1)\hat{}|E \subset (C_0, \ell^2)(E)$.

Now, by (2.2) if $2 < p \leqq \infty$, then $(L^p, \ell^q) \subset (L^2, \ell^q)$ and by the Hausdorff-Young theorem we have that $(L^p, \ell^q)\hat{} \subset (L^{q'}, \ell^2)(\hat{G})$ for $1 \leqq q \leqq 2$. Then we consider the possibility that for a not locally null set E the inclusion $(L^p, \ell^q)\hat{}|E \subset (L^r, \ell^s)(E)$ holds for some $r > q'$ and $s < 2$. If $\hat{G}$ is discrete, then $(L^p, \ell^q) = L^p$ and for E a not locally null set, $2 < p \leqq \infty$ and $1 \leqq q \leqq 2$, we have that

$$(L^p, \ell^q)\hat{}|E = L^{p\hat{}}|E \subset \ell^2|E = (L^r, \ell^2)(E) \quad \text{for} \quad r > q'.$$

Finally, Theorem 3.2 implies the following result.

PROPOSITION 7.2. *If* $\hat{G}$ *is nondiscrete and E is not locally null, then for* $2 < p \leqq \infty$ *and* $1 < q \leqq 2$,

$$(L^p, \ell^q)^\wedge|E \not\subset \bigcup_{r>q', s<2} (L^r, \ell^s)(E).$$

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UNIVERSITY OF REGINA
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