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A CLOSURE THEOREM FOR ANALYTIC SUBGROUPS OF REAL LIE GROUPS

BY
D. Ž. DJOKOVIĆ

Introduction. Let G be a real Lie group, A a closed subgroup of G and B an analytic subgroup of G. Assume that B normalizes A and that AB is closed in G. Then our main result (Theorem 1) asserts that $\bar{B}=\overline{A\cap B}\cdot B$.

This result generalizes Lemma 2 in the paper [4]. G. Hochschild has pointed out to me that the proof of that lemma given in [4] is not complete but that it can be easily completed.

In the first section we state and prove Theorem 1 and in the second section we give several applications of Theorem 1. The results of the second section are not new except, perhaps, Proposition 1 which is a slight generalization of Theorem 2 of M. Gotô [1]. Propositions 2 and 5 are well-known results of M. Gotô [1], [2]. All the references in the paper are to Hochschild's book [3].

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The closure theorem.

THEOREM 1. *Let G be a real Lie group, A a closed subgroup of G and B an analytic subgroup of G. We assume that B normalizes A and that AB is closed in G. Then we have*

$$\bar{B}=\overline{A\cap B}\cdot B, \tag{1}$$

where $^{-}$ *denotes the closure of subsets in G. In particular, B is closed in G if and only if $A\cap B$ is closed in G.*

Proof. In this proof we shall say that a triple (G, A, B), satisfying the conditions of this theorem, is *good* if (1) is valid for that triple. Next we shall make several reductions. $L(A)$ will be the Lie algebra of A, etc.

First reduction. The identity component $(AB)_1=S$ of AB is also closed in G. If A_1 is the identity component of A then we have $A_1\subset A\cap S\subset S$. Since the topology of S has a countable base it follows that $A\cap S$ consists of countably many cosets of A_1. Since B is connected we have $B\subset S$ and $S=S\cap AB=(A\cap S)B$. Thus S consists of countably many cosets of A_1B and consequently

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S and A_1B have the same dimension. Since $A_1B \subset S$ we must have $S = A_1B$.

If the triple (S, A_1, B) is good then it is clear that the triple (G, A, B) is also good. This means that from now on we can assume that A is connected and that $G = AB$.

Second reduction. Let C be the analytic subgroup of G such that $L(C) = L(A) \cap L(B)$. Since $\bar{A} = A$ and $C \subset A$ we have $\bar{C} \subset A$. Since B normalizes C it also normalizes $\bar{C}$ and $B\bar{C}$ is an analytic subgroup of G. Assume that the triple $(G, A, B\bar{C})$ is good. Since the closure of $B\bar{C}$ is $\bar{B}$ and the closure of $A \cap B\bar{C} = (A \cap B)\bar{C}$ is $\overline{A \cap B}$ it follows that the triple (G, A, B) is also good.

Hence it suffices to prove that the triple $(G, A, B\bar{C})$ is good. Note that we have $L(B\bar{C}) = L(B) + L(\bar{C})$ and $L(\bar{C}) \subset L(A)$ so that $L(A) \cap L(B\bar{C}) = L(\bar{C})$. This means that in the sequel we can assume, in addition, that the group C defined above is closed in G.

Third reduction. We claim that it suffices to prove

$$(A/C) \cap \overline{(B/C)} \subset \overline{(A/C) \cap (B/C)} \tag{2}$$

where A/C and B/C are considered as subsets of the homogeneous space G/C. Note that the opposite inclusion of (2) is valid because A is closed in G. Hence if (2) is valid then in fact we have

$$(A/C) \cap \overline{(B/C)} = \overline{(A/C) \cap (B/C)}.$$

This implies that $A \cap \bar{B} = \overline{A \cap B}$ and consequently

$$\bar{B} = \bar{B} \cap (AB) = (A \cap \bar{B})B = \overline{A \cap B} \cdot B.$$

Hence (2) implies (1) as claimed.

Proof of (2). Since B normalizes A we have an action $B \to \text{Aut}(A)$ of B on A by conjugation. Let $A \propto B$ be the semi-direct product of A and B corresponding to this action. Let $f: A \propto B \to G$ be the canonical continuous homomorphism which is characterized by $f((a, 1)) = a$ for $a \in A$ and $f((1, b)) = b$ for $b \in B$. Since f is surjective we can consider G/C as a homogeneous space of the group $A \propto B$. The fixer F of the point $C \in G/C$ in $A \propto B$ is $F = \{(x, y) \in A \propto B \mid xy \in C\}$. We can identify $(A \propto B)/F$ and G/C as homogeneous spaces of the group $A \propto B$ by the canonical map which sends the coset $(a, b)F$ to the coset abC. We see that $C \propto C \subset F$. On the other hand it is easy to check that they have the same dimension. Thus $C \propto C$ is the identity component of F. Hence, the canonical map $p: (A \propto B)/(C \propto C) \to (A \propto B)/F = G/C$ is a covering.

Let (b_nC) be a sequence in B/C which converges to a point $aC \in A/C$. Let W be a nbd of the point $(a, 1)(C \propto C)$ in $(A \propto B)/(C \propto C)$ such that p maps W

homeomorphically onto a nbd of aC in G/C. We may assume that $b_nC \in p(W)$ for all n. Let $(x_n, y_n)(C \propto C)$ be the unique point of W such that

$$p((x_n, y_n)(C \propto C)) = x_n y_n C = b_n C = (1, b_n)F.$$

Since $b_nC \to aC$ as $n \to \infty$ and p induces a homeomorphism $W \to p(W)$ it follows that $(x_n, y_n)(C \propto C) \to (a, 1)(C \propto C)$ as $n \to \infty$. Since B normalizes C the projection $A \propto B \to A$ induces a continuous map $(A \propto B)/(C \propto C) \to A/C$ sending $(x, y)(C \propto C)$ to xC. By applying this map to the above convergent sequence we obtain that $x_nC \to aC$ in A/C. From $x_n y_n \in b_nC$ and $x_n \in A$, $y_n \in B$ it follows that $x_n \in A \cap B$. This shows that aC belongs to the closure of $(A \cap B)/C$ in A/C. Hence, we have proved (2) and in the same time we have completed the proof of the theorem.

Some applications

Proposition 1. *Let G be a real Lie group, H an analytic subgroup of G, N the radical of H and S a maximal semi-simple analytic subgroup of H. Then $N \cap S$ is contained in the center Z of S. If the index of $N \cap S$ in Z is finite, then $\bar{H} = \bar{N}S$ and $\bar{N}$ is the radical R of $\bar{H}$.*

We use $^-$ to denote closure of subsets in G.

Proof. $N \cap S$ is a discrete normal subgroup of the analytic group S and hence $N \cap S \subset Z$. From now on we shall assume that the index of $N \cap S$ in Z is finite.

By Hochschild, Theorem 2.1, p. 190 $\bar{N}$ is normal in $\bar{H}$ and consequently we have $R \supset N$. It follows from the same theorem that S is also a maximal semi-simple analytic subgroup of $\bar{H}$. Hence we have $\bar{H} = RS = RH$ and we can apply Theorem 1 to the triple $(\bar{H}, R, H)$.

We need to compute the closure of $R \cap H$. Since $N \cap S \subset R \cap S \subset Z$ and the index of $N \cap S$ in Z is finite it follows that $\bar{N}(R \cap S)$ is closed in G. From $R \cap H = R \cap NS = N(R \cap S)$ it follows now that $\overline{R \cap H} = \bar{N}(R \cap S)$.

By applying Theorem 1 we get $\bar{H} = \overline{R \cap H} \cdot H = \bar{N}(R \cap S)H = \bar{N}S$. We have $\bar{N} \subset R$ so that $L(\bar{N}) \subset L(R)$. Since also $L(\bar{H}) = L(\bar{N}) + L(S)$ and $L(R) \cap L(S) = 0$, it follows that $L(\bar{N}) = L(R)$ and so $\bar{N} = R$.

Proposition 2. *Let G be an analytic subgroup of $GL(V)$ where V is a finite-dimensional real vector space. Then the commutator subgroup G' of G is closed in $GL(V)$.*

Proof. Let R be the radical and S a maximal semi-simple analytic subgroup of G. By Hochschild, Proposition 4.1, p. 221 the center of S is finite. Since

$$L(G') = [L(G), L(G)] = [L(G), L(R)] + L(S)$$

we have $G' = (G, R)S$, where (G, R) is the subgroup of G generated by all commutators $(x, y) = xyx^{-1}y^{-1}$ with $x \in G$ and $y \in R$. By Hochschild, Theorem 3.2, p. 128 the group (G, R) is unipotent on V and hence it is closed in $GL(V)$. By Proposition 1 we conclude that G' is closed in $GL(V)$.

We shall say that a real Lie group G is *linear* if it has a faithful continuous finite-dimensional representation.

PROPOSITION 3. *Let G be a linear real Lie group and H a semi-simple analytic subgroup of G. Then H is closed in G.*

Proof. We can consider G as a subgroup of $GL(V)$ for a suitable finite-dimensional real vector space V. Thus H is an analytic subgroup of $GL(V)$. Since H is also semi-simple we have $H' = H$. By Proposition 2 H is closed in $GL(V)$ and consequently closed in G.

PROPOSITION 4. *Let G be a real analytic group, S a semi-simple analytic subgroup of G, Z the center of G and Z_1 the identity component of Z. Then the following are equivalent:*

(i) *S is closed in G;*
(ii) *$S \cap Z$ is closed in G;*
(iii) *$S \cap Z_1$ is closed in G.*

Proof. Since Z is the kernel of the adjoint representation of G the group G/Z is linear. Since SZ/Z is a semi-simple analytic subgroup of G/Z, it is closed in G/Z by Proposition 3. Thus SZ is closed in G and we can apply Theorem 1 to the triple (G, Z, S). That Theorem gives at once that (i) $\Leftrightarrow$ (ii).

It is trivial that (ii) $\Rightarrow$ (iii). Now let us assume that (iii) holds. Since Z/Z_1 is discrete it is clear that $(S \cap Z)Z_1$ is closed in Z and in G. Let (x_n) be a sequence in $S \cap Z$ which converges to a point $a \in G$. In fact, we must have $a \in (S \cap Z)Z_1$. There exists $s \in S \cap Z$ such that $a \in sZ_1$. Since sZ_1 is open in Z we can assume that $x_n \in sZ_1$ for all n. Thus $x_n \in S \cap sZ_1 = sS \cap sZ_1 = s(S \cap Z_1)$. It follows from (iii) that $s(S \cap Z_1)$ is closed in G. Therefore $a \in s(S \cap Z_1) \subset S \cap Z$. Hence $S \cap Z$ is also closed in G and we have proved that (iii) $\Rightarrow$ (ii).

All the vector spaces that occur in the next proposition and its proof will be real and finite-dimensional and all the representations will be continuous.

PROPOSITION 5. *Let G be a linear real analytic group. Then there exists a faithful representation $\rho: G \to GL(W)$ such that $\rho(G)$ is closed in $GL(W)$.*

Proof. Since G is linear we can assume that G is an analytic subgroup of $A = GL(U)$ for some vector space U. By Proposition 2 G' is closed in A and also in G. Hence G/G' is an abelian analytic group. Therefore there exists a representation $\sigma: G \to B = GL(V)$ such that $\operatorname{Ker} \sigma = G'$ and $\sigma(G)$ is closed in B. Let $W = U \oplus V$ and identify the direct product $A \times B$ with its canonical image in $GL(W)$. If $\rho(a) = (a, \sigma(a)) \in A \times B$ then ρ is a faithful representation of G. Since $H = \rho(G)$ is contained in $A \times B$ and $A \times B$ is closed in $GL(W)$ it remains to show that H is closed in $A \times B$. Since $\sigma(G)$ is closed in B and $AH = A \times \sigma(G)$ it follows that AH is closed in $A \times B$. Hence, we can apply Theorem 1 to the triple $(A \times B, A, H)$. Since $A \cap H = G'$, it is closed in A and Theorem 1 gives that H is closed in $A \times B$.

The proof is finished.

PROPOSITION 6. *Let G be a real analytic group, R its radical, S a maximal semi-simple analytic subgroup of G and Z the center of G. Then $R \cap S \cap Z$ has finite index in $R \cap S$.*

Proof. There is a canonical isomorphism of $(R \cap S)/(R \cap S \cap Z)$ with $((R \cap S)Z)/Z$ as abstract groups. The latter group is contained in the center of SZ/Z. Using the adjoint representation of G we conclude that G/Z is a linear group and so is SZ/Z. By Hochschild, Theorem 4.1, p. 221 the center of SZ/Z is finite. Hence the two groups mentioned in the beginning of this proof are also finite.

The proposition is proved.

REFERENCES

1. M. Gotô, *Faithful representations of Lie groups I*, Math. Japanicae **1** (1948), 107–119.
2. M. Gotô, *Faithful representations of Lie groups II*, Nagoya Math. J. **1** (1950), 91–107.
3. G. Hochschild, *The structure of Lie groups*, San Francisco 1965.
4. G. Hochschild, *Complexification of real analytic groups*, Trans. Amer. Math. Soc. **125** (1966), 406–413.

DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO,
CANADA