Canad. Math. Bull. Vol. **48** (4), 2005 pp. 505–522

On the Generalized d'Alembert's and Wilson's Functional Equations on a Compact Group

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Abstract. Let G be a compact group.

Let σ be a continuous involution of G. In this paper, we are concerned by the following functional equation

$$\int_G f(xtyt^{-1})\,dt + \int_G f(xt\sigma(y)t^{-1})\,dt = 2g(x)h(y), \quad x, y \in G,$$

where $f, g, h\colon G \mapsto \mathbb{C}$, to be determined, are complex continuous functions on G such that f is central. This equation generalizes d'Alembert's and Wilson's functional equations. We show that the solutions are expressed by means of characters of irreducible, continuous and unitary representations of the group G.

1 Introduction, Notations and Preliminaries

1.1 Let G be a compact group endowed with a fixed normalized Haar measure denoted by dt. The unit element of the group G is denoted by e. We denote by $L_\infty(G)$, the Banach space of all complex measurable and essentially bounded functions on G. $\mathcal{C}(G)$ designates the Banach space of continuous complex valued functions on G. The Banach space of all complex integrable functions on G is denoted by $L_1(G)$. For each function ϕ on the group G, we define the new functions $\check{\phi}$, $\overline{\phi}$ and $\tilde{\phi}$ on G by $\check{\phi}(x) := \phi(x^{-1})$, $\overline{\phi}(x) := \overline{\phi(x)}$ and $\tilde{\phi}(x) := \overline{\phi(x^{-1})}$, for all $x \in G$. The algebra of all regular complex measures on G will be denoted by $M(G)$. We recall that the convolution of $M(G)$ is given by $\langle \mu \star \nu, \phi \rangle = \int_G \int_G \phi(ts)\, d\mu(t) d\mu(s)$ and its involution is defined by $\mu^* = \check{\overline{\mu}}$ where $\langle \overline{\mu}, \phi \rangle = \overline{\langle \mu, \phi \rangle}$ and $\langle \check{\mu}, \phi \rangle = \langle \mu, \check{\phi} \rangle$ for all $\phi \in \mathcal{C}(G)$. Let $f \in \mathcal{C}(G)$; f is called a central function (see [3]), if

$$f(yx) = f(xy), \quad x, y \in G. \tag{1.1.1}$$

We recall that a character of a representation $(\pi, \mathcal{H}_\pi)$ of G is a complex-valued function χ_π defined on G by

$$\chi_\pi(x) = \operatorname{tr}(\pi(x)), \quad x \in G, \tag{1.1.2}$$

where tr means trace.

Received by the editors August 25, 2003; revised April 19, 2004.
AMS subject classification: 39B32, 39B42, 22D10, 22D12, 22D15.
Keywords: Compact groups, Functional equations, Central functions, Lie groups, Invariant differential operators..

https://doi.org/10.4153/CMB-2005-047-3 Published online by Cambridge University Press

1.2 The aim of this paper is to solve the following integral equation

$$\int_G f(xtyt^{-1})\,dt + \int_G f(xt\sigma(y)t^{-1})\,dt = 2g(x)h(y), \quad x, y \in G, \tag{1.2.1}$$

where f, g, h to be determined, are in the space $\mathcal{C}(G)$ such that f is central. This equation is a generalization of the d'Alembert type functional equation

$$\int_G f(xtyt^{-1})\,dt + \int_G f(xt\sigma(y)t^{-1})\,dt = 2f(x)f(y), \quad x, y \in G. \tag{1.2.2}$$

It is also a generalization of the Wilson type functional equation

$$\int_G f(xtyt^{-1})\,dt + \int_G f(xt\sigma(y)t^{-1})\,dt = 2f(x)g(y), \quad x, y \in G. \tag{1.2.3}$$

We will show that the solutions are given by means of characters of irreducible, continuous and unitary representations of G.

1.3 First, we study the functional equation (1.2.2). In Theorems 3.1 and 3.2, we prove that if $f \in \mathcal{C}(G)$, then the map $h \mapsto \int_G h(x)f(x)\,dx$ is a character of the commutative subalgebra $C(P(L_1(G)))$ if and only if f is a solution of the functional equation (1.2.2), where $P(h) = \frac{1}{2}(h + h \circ \sigma)$ and $C(h)(x) = \int_G h(txt^{-1})\,dt$, for all $x \in G$.

In Theorem 3.3, we give a description of solutions of (1.2.2). The solutions are precisely expressed by the formula

$$\psi = \frac{1}{2}(\varphi + \varphi \circ \sigma),$$

where φ is a solution of the functional equation

$$\int_G \varphi(xtyt^{-1})\,dt = \varphi(x)\varphi(y), \quad x, y \in G. \tag{1.3.1}$$

For more information about this equation see [3, 6, 8, 9]. As a consequence we obtain in Corollary 3.4 that f is a solution of (1.2.2) if and only if there exists an irreducible, continuous and unitary representation $(\pi, \mathcal{H}_\pi)$ of G such that

$$f = \frac{1}{2d(\pi)}(\chi_\pi + \chi_\pi \circ \sigma),$$

where $d(\pi)$ is a dimension of $\mathcal{H}_\pi$. In Theorem 3.7, we consider the case where G is a compact connected Lie group. We prove that the solutions of (1.2.2) are the eigenfunctions of some differential operators associated with left invariant differential operators on G. Secondly, we discuss the functional equation (1.2.3). In Theorem 4.2, we show that the solutions, such that f is central, are of the form

$$f = \alpha\frac{\chi_\pi + \chi_\pi \circ \sigma}{2d(\pi)} + \beta\frac{\chi_\pi - \chi_\pi \circ \sigma}{2d(\pi)}, \quad g = \frac{\chi_\pi + \chi_\pi \circ \sigma}{2d(\pi)},$$

where α, β range over $\mathbb{C}$.

Finally, we show that the solutions of (1.2.1) can be listed as follows:

$$
\begin{aligned}
f(x) &= ab\frac{\chi_\pi(x) + \chi_\pi(\sigma(x))}{2d(\pi)} + ac\frac{\chi_\pi(x) - \chi_\pi(\sigma(x))}{2d(\pi)},\\
g(x) &= b\frac{\chi_\pi(x) + \chi_\pi(\sigma(x))}{2d(\pi)} + c\frac{\chi_\pi(x) - \chi_\pi(\sigma(x))}{2d(\pi)},\\
h(x) &= a\frac{\chi_\pi(x) + \chi_\pi(\sigma(x))}{2d(\pi)}.
\end{aligned}
$$

where χ_π is a character of an irreducible, continuous and unitary representation π of G, $d(\pi)$ is the dimension of π and a, b, c are complex numbers. This paper contains also some results concerning the equations (1.2.1), (1.2.2) and (1.2.3) and properties of their solutions.

2 General Properties

In this part, we are going to study the general properties. Let G be a compact group. For all $f \in \mathcal{C}(G)$, we put

$$(Cf)(x) = \int_G f(txt^{-1})\,dt, \quad x, y \in G,$$

and

$$\mathcal{S}(G) = \{f \in L_1(G) : f(xy) = f(yx), \quad x, y \in G\}.$$

$\mathcal{S}(G)$ is a commutative subalgebra (under the convolution) of the $*$-Banach algebra $L_1(G)$. For the notion of central function, see [3].

Proposition 2.1 *For all $f \in (G)$, we have the following properties:*

(i) $(Cf)(e) = f(e)$.

(ii) $(\check{Cf}) = C\check{f}$ *and* $(\tilde{Cf}) = C\tilde{f}$.

(iii) $C(Cf) = Cf$.

(iv) f *is central if and only if* $Cf = f$.

(v) *the map* $f \mapsto Cf$ *is an orthogonal projection on the commutative Banach algebra* $\mathcal{S}(G)$.

Proof By easy computations. ■

Proposition 2.2 *Any solution of the functional equation* (1.2.2) *has, for all* $x, y \in G$, *the properties*

$$f(e) = 1, \quad f \circ \sigma = f, \quad Cf = f \quad \text{and} \quad \int_G f(xtyt^{-1})\,dt = \int_G f(ytxt^{-1})\,dt.$$

Proof By easy computations. ■

https://doi.org/10.4153/CMB-2005-047-3 Published online by Cambridge University Press

The next proposition explains why we restrict ourselves to continuous solutions.

Proposition 2.3 *Let $f \in L_\infty(G)$ verifying the functional equation* (1.2.2), *then f is continuous.*

Proof If we replace x by xs in (1.2.2), after integration we obtain

$$\int_G \int_G f(xsyt)\,dsdt + \int_G \int_G f(xs\sigma(y)t)\,dsdt = 2\Big(\int_G f(xs)\,ds\Big) f(y), \quad x, y \in G.$$

Let $f \in L_\infty(G)$ be a solution of (1.2.2) and let $\mu = dt$, then for all $\phi \in L_1(G)$ and $y \in G$, we have

$$\begin{aligned}
&\mu \star \phi \star f \star \mu(y) + \mu \star \phi \star f \star \mu \circ \sigma(y)\\
&\quad = \int_G \phi \star f \star \mu(t^{-1}y)\,dt + \int_G \phi \star f \star \mu(t^{-1}\sigma(y))\,dt\\
&\quad = \int_G \int_G f \star \mu(x^{-1}ty)\phi(x)\,dtdx + \int_G \int_G f \star \mu(x^{-1}t\sigma(y))\phi(x)\,dtdx\\
&\quad = \int_G \int_G \int_G f(x^{-1}tys)\phi(x)\,dsdtdx + \int_G \int_G \int_G f(x^{-1}t\sigma(y)s)\phi(x)\,dsdtdx\\
&\quad = 2f(y) \int_G \int_G f(x^{-1}s)\phi(x)\,dsdx\\
&\quad = 2\langle \phi \star f, 1\rangle f(y).
\end{aligned}$$

Consequently f is continuous. ∎

For later use we note the following results:

Proposition 2.4 ([2]) *Let $f \in \mathcal{C}(G)$. Then we have*

(i) $\int_G \int_G f(ztxt^{-1}sys^{-1})\,dtds = \int_G \int_G f(ztyt^{-1}sxs^{-1})\,dtds,\ z, x, y \in G.$
(ii) *If f is central then f satisfies the condition:*

$$\int_G f(xtyt^{-1})\,dt = \int_G f(ytxt^{-1})\,dt, \quad x, y \in G.$$

Proposition 2.5 *Let $f, g \in \mathcal{C}(G)$ such that f is not identically* 0 *and (f, g) is a solution of* (1.2.3). *Then g is a solution of* (1.2.2). *Conversely if g is a solution of* (1.2.2), *then for any $a \in G$, $(L_a g, g)$ is a solution of* (1.2.3).

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Proof Let $a \in G$ such that $f(a) \neq 0$. Then

$$\begin{aligned}
2f(a)\Big(\int_G g(xtyt^{-1})\,dt + \int_G g(xt\sigma(y)t^{-1})\,dt\Big) \\
&= \int_G\int_G f(asxtyt^{-1}s^{-1})\,dsdt + \int_G\int_G f(as\sigma(y)t\sigma(x)t^{-1}s^{-1})\,dsdt \\
&\quad + \int_G\int_G f(asxt\sigma(y)t^{-1}s^{-1})\,dsdt + \int_G\int_G f(asyt\sigma(x)t^{-1}s^{-1})\,dsdt \\
&= 2\int_G f(asxs^{-1})\,dsg(y) + 2\int_G f(as\sigma(x)s^{-1})\,dsg(y) \\
&= 4f(a)g(x)g(y).
\end{aligned}$$

from which we deduce that g is a solution of (1.2.2). ■

Proposition 2.6 *Let g be a solution of* (1.2.2). *Let $a \in G$ and define the function*

$$f(x) = \int_G g(xtat^{-1})\,dt + \int_G g(xt\sigma(a)t^{-1})\,dt, \quad x \in G.$$

Then (f, g) is a solution of (1.2.3).

Proof By easy computations. ■

3 On the Functional Equation

$$\int_G f(xtyt^{-1})\,dt + \int_G f(xt\sigma(y)t^{-1})\,dt = 2f(x)f(y)$$

The following results explain some relations existing between solutions of the functional equation (1.2.2) and continuous characters of the commutative algebra $C(P(L_1(G)))$.

Theorem 3.1 *Let $f \in \mathcal{C}(G)$ be a solution of* (1.2.2). *Then the map $h \mapsto \langle h, f\rangle := \int_G h(x)f(x)\,dx$ is a continuous character of the commutative Banach algebra*

$$C(P(L_1(G))).$$

Proof Let $f \in \mathcal{C}(G)$ be a solution of (1.2.2). Let $h, g \in L_1(G)$, then we have

$$\begin{aligned}
\langle C(Ph) \star C(Pg), f\rangle &= \langle C(\frac{h + h\circ\sigma}{2}) \star C(\frac{g + g\circ\sigma}{2}), f\rangle \\
&= \frac{1}{4}\int_G\int_G\Big[(g(x)h(y) + g(x)(h\circ\sigma)(y) + (g\circ\sigma)(x)h(y) \\
&\quad + (g\circ\sigma)(x)(h\circ\sigma)(y))\int_G f(xtyt^{-1})\,dt\Big]\,dxdy.
\end{aligned}$$

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Since f is central and $f \circ \sigma = f$, we get

$$\begin{aligned}\langle C(Ph) \star C(Pg), f\rangle &= \frac{1}{2}\int_G\int_G g(x)h(y)[f(xtyt^{-1})\,dt + f(xt\sigma(y)t^{-1})\,dt]dxdy\\ &= \int_G f(x)g(x)\,dx\int_G f(y)g(y)\,dy\\ &= \int_G f(x)\frac{g(x)+g(\sigma(x))}{2}\,dx\int_G f(y)\frac{h(y)+h(\sigma(y))}{2}\,dy\\ &= \langle C(Ph), f\rangle\langle C(Pg), f\rangle.\end{aligned}$$

■

Theorem 3.2 *Let $\chi\colon C(P(L_1(G))) \mapsto \mathbb{C}^\star$ be a continuous character of $C(P(L_1(G)))$. Then there exists $f \in \mathcal{C}(G)$ solution of the functional equation* (1.2.2) *such that $\chi(h) = \langle h, f\rangle$, for all $h \in C(P(L_1(G)))$.*

Proof Let χ be a non-zero continuous character of the Banach algebra $C(P(L_1(G)))$, since the map $L_1(G) \to \mathbb{C}^\star\colon g \mapsto \chi(C(Pg))$ is continuous and linear, then there exists $f \in \mathcal{L}_\infty(G)$ such that $\chi(C(Pg)) = \langle g, f\rangle$. Since $\langle g, f\rangle = \chi(C(Pg)) = \chi(C(P(Cg))) = \langle Cg, f\rangle = \langle g, Cf\rangle$. It follows that f is central. On the other hand we have $\chi(C(Pg)) = \chi(C(\frac{P(Cg)+P(Cg)\circ\sigma}{2})) = \langle P(C(g)), f\rangle = \langle g, P(Cf)\rangle = \langle g, Pf\rangle$. Then we get $f = Pf$, *i.e.*, $f \circ \sigma = f$. Now for all $g, h \in L_1(G)$, we have

$$\begin{aligned}\langle C(Ph) \star C(Pg), f\rangle &= \frac{1}{2}\int_G\int_G g(x)h(y)[f(xtyt^{-1})\,dt + f(xt\sigma(y)t^{-1})\,dt]\,dxdy\\ &= \langle C(Ph), f\rangle\langle C(Pg), f\rangle\\ &= \int_G f(x)h(x)\,dx\int_G f(y)g(y)\,dy,\end{aligned}$$

hence it follows that $\int_G f(xtyt^{-1})\,dt + \int_G f(xt\sigma(y)t^{-1})\,dt = 2f(x)f(y)$, for all $x, y \in G$, which concludes the proof of the theorem. ■

Now we are going to determine all non-zero complex-valued continuous solutions of the functional equation (1.2.2). We adapt the proof used in [4].

Theorem 3.3

The only continuous non-zero solutions of the functional equation (1.2.2) *are the functions of the form*

$$f(x) = \frac{\varphi(x) + \varphi(\sigma(x))}{2}, \quad x \in G$$

where φ is a solution of (1.3.1).

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Lemma 3.3.1 *Let $f \in \mathcal{C}(G) \setminus \{0\}$ be a solution of* (1.2.2)*. For a fixed $\alpha \in \mathbb{C}$ and $a \in G$, we define*

$$\varphi(x) = f(x) + \alpha\Big(\int_G f(xtat^{-1})\,dt - \int_G f(xt\sigma(a)t^{-1})\,dt\Big), \quad x \in G,$$

then $f = (\varphi + \varphi \circ \sigma)/2$.

Proof Using that $f = f \circ \sigma$, $f(xy) = f(yx)$, $\int_G f(xtyt^{-1})\,dt = \int_G f(ytxt^{-1})\,dt$, $x, y \in G$, we get

$$\begin{aligned}\varphi(\sigma(x)) &= f(\sigma(x)) + \alpha\Big(\int_G f(\sigma(x)tat^{-1})\,dt - \int_G f(\sigma(x)t\sigma(a)t^{-1})\,dt\Big)\\ &= f(x) + \alpha\Big(\int_G f(xt\sigma(a)t^{-1})\,dt - \int_G f(xtat^{-1})\,dt\Big).\end{aligned}$$

Adding this to $\varphi(x)$ we find that $\varphi(x) + \varphi(\sigma(x)) = 2f(x)$. ■

We will next examine whether φ is a solution of (1.3.1).

Lemma 3.3.2 *Let $f \in \mathcal{C}(G)$ be a solution of* (1.2.2)*. For any $x, y \in G$, we define*

$$\eta(x, y) = \varphi(x)\varphi(y) - \int_G \varphi(xtyt^{-1})\,dt,$$

then we have the following identities

$$\begin{aligned}\eta(x, y) =& \Big[\int_G f(xt\sigma(y)t^{-1})\,dt - \int_G f(xtyt^{-1})\,dt\Big]\\ &\times\Big[\alpha^2\Big(\int_G f(at\sigma(a)t^{-1})\,dt - \int_G f(atat^{-1})\,dt\Big) + \frac{1}{2}\Big]\end{aligned}$$

Proof Let $x, y \in G$, then we have

$$\begin{aligned}\eta(x, y) &= \varphi(x)\varphi(y) - \int_G \varphi(xtyt^{-1})\,dt\\ &= \frac{1}{2}\Big[\int_G f(xt\sigma(y)t^{-1})\,dt - \int_G f(xtyt^{-1})\,dt\Big]\\ &\quad - \frac{\alpha}{2}\int_G\int_G f(xtyt^{-1}sas^{-1})\,dtds\\ &\quad + \frac{\alpha}{2}\int_G\int_G f(xtyt^{-1}s\sigma(a)s^{-1})dtds + \frac{\alpha}{2}\int_G\int_G f(xtat^{-1}s\sigma(y)s^{-1})\,dtds\\ &\quad - \frac{\alpha}{2}\int_G\int_G f(xtat^{-1}s\sigma(y)s^{-1})\,dtds\end{aligned}$$

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$$+\frac{\alpha}{2}\Big[\int_G\int_G f(ytxt^{-1}sas^{-1})\,dsdt+\int_G\int_G f(yt\sigma(a)t^{-1}s\sigma(x)s^{-1})\,dtds\Big]$$
$$+\alpha^2\int_G f(xtat^{-1})\,dt\int_G f(ytat^{-1})\,dt$$
$$-\alpha^2\int_G f(xtat^{-1})dt\int_G f(yt\sigma(a)t^{-1})\,dt$$
$$-\frac{\alpha}{2}\Big[\int_G\int_G f(ytxt^{-1}s\sigma(a)s^{-1})\,dtds+\int_G\int_G f(ytat^{-1}s\sigma(x)s^{-1})\,dtds\Big]$$
$$-\alpha^2\int_G f(xt\sigma(a)t^{-1})\,dt\int_G f(ytat^{-1})\,dt$$
$$+\alpha^2\int_G f(xt\sigma(a)t^{-1})dt\int_G f(yt\tau(a)t^{-1})dt.$$

Since

$$\int_G\int_G f(ytxt^{-1}szs^{-1})\,dtds=\int_G\int_G f(xtyt^{-1}szs^{-1})\,dtds,$$
$$\int_G f(ytxt^{-1})\,dt=\int_G f(xtyt^{-1})\,dt,$$

$f\circ\sigma=f$, it follows that

$$\begin{aligned}
&\varphi(x)\varphi(y)-\int_G\varphi(xtyt^{-1})\,dt\\
&\quad=\frac{1}{2}\Big[\int_G f(xt\sigma(y)t^{-1})dt-\int_G f(xtyt^{-1})dt\Big]\\
&\qquad+\frac{\alpha^2}{2}\Big[\int_G\int_G\int_G f(ytxt^{-1}lsas^{-1}al^{-1})\,dtdsdl\\
&\qquad\qquad+\int_G\int_G\int_G f(ytxt^{-1}ls\sigma(a)s^{-1}\sigma(a)l^{-1})\,dtdsdl\Big]\\
&\qquad+\frac{\alpha^2}{2}\Big[\int_G\int_G\int_G f(\sigma(y)txt^{-1}lsas^{-1}\sigma(a)l^{-1})\,dtdsdl\\
&\qquad\qquad+\int_G\int_G\int_G f(\sigma(y)txt^{-1}lsas^{-1}\sigma(a))\,dtdsdl\Big]\\
&\qquad-\frac{\alpha^2}{2}\Big[\int_G\int_G\int_G f(ytxt^{-1}lsas^{-1}\sigma(a))\,dtdsdl\\
&\qquad\qquad+\int_G\int_G\int_G f(ytxt^{-1}lsas^{-1}\sigma(a)l^{-1})\,dtdsdl\Big]
\end{aligned}$$

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$$-\frac{\alpha^2}{2}\Big[\int_G\int_G\int_G f(\sigma(y)txt^{-1}ls\sigma(a)s^{-1}\sigma(a)l^{-1})\,dtdsdl$$
$$+\int_G\int_G\int_G f(\sigma(y)txt^{-1}lsas^{-1}al^{-1})\,dtdsdl\Big]$$
$$=\Big[\int_G f(xt\sigma(y)t^{-1})dt-\int_G f(xtyt^{-1})\,dt\Big]$$
$$\times\Big[\alpha^2\Big(\int_G f(at\sigma(a)t^{-1})\,dt-\int_G f(atat^{-1})\,dt\Big)+\frac{1}{2}\Big].$$

Proof of Theorem 3.3

Case 1: If there exists $a\in G$ such that

$$\int_G f(atat^{-1})\,dt-\int_G f(at\sigma(a)t^{-1})\,dt\neq 0,$$

then we may choose $\alpha\in\mathbb{C}$ such that

$$\alpha^2\Big[\int_G f(at\sigma(a)t^{-1})\,dt-\int_G f(atat^{-1})\,dt\Big]+\frac{1}{2}=0.$$

That is to say $\varphi(x)\varphi(y)=\int_G\varphi(xtyt^{-1})\,dt$.

Case 2: Suppose that $\int_G f(xtxt^{-1})\,dt=\int_G\psi(xt\sigma(x)t^{-1})\,dt$, for all $x\in G$. Noting that in this case

$$\int_G f(xtxt^{-1})\,dt=\int_G f(xt\sigma(x)t^{-1})\,dt=f(x)^2,\quad\forall x\in G.$$

Let $X=\int_G f(xtyt^{-1})\,dt$, $Y=\int_G f(xt\sigma(y)t^{-1})\,dt$. Then we have $X+Y=2f(x)f(y)$ and by computation we show that $XY=f(x)^2f(y)^2$. Making use of this we obtain that $X=f(x)f(y)=\int_G f(xtyt^{-1})\,dt$. Conversely, for all φ satisfying the functional equation (1.3.1) it is easy to see that $f=\frac{1}{2}(\varphi+\varphi\circ\sigma)$ is a solution of (1.2.2). ■

Corollary 3.4 *Let $f\in\mathcal{C}(G)\setminus\{0\}$. Then f is a solution of* (1.2.2) *if and only if there exists an irreducible, continuous and unitary representation $(\pi,\mathcal{H}_\pi)$ of G such that*

$$f=\frac{1}{2d(\pi)}(\chi_\pi+\chi_\pi\circ\sigma),$$

where $d(\pi)$ is a dimension of $\mathcal{H}_\pi$.

Proof By [3, 5, 6], we have that φ is a solution of (1.3.1) if and only if there exists $(\pi,\mathcal{H}_\pi)$ an irreducible, continuous and unitary representation of G such that $\varphi=\frac{\chi_\pi}{d(\pi)}$, where $d(\pi)$ denotes the dimension of the space $\mathcal{H}_\pi$. ■

Next, we suppose that G is a connected compact Lie group, and we shall characterize the solutions of (1.2.2) in terms of eigenfunctions of some differential operators.

For each fixed $a \in G$, we define the left (resp. the right) translation operators as follows $(L_a f)(x) = f(a^{-1}x)$ (resp. $(R_a f)(x) = f(xa)$) and we will say that the operator T is left (resp. right) invariant if $(L_a T)f = T(L_a f)$ (resp. $(R_a T)f = T(R_a f)$). Let $\mathbb{D}(G)$ denote the algebra of left invariant differential operators on G and $Z(G)$ denote the center of $\mathbb{D}(G)$.

For any differential operator D on G, we define the differential operator $\tilde{D}$ by

$$(\tilde{D}f)(x) := \frac{1}{2} D\{C(L_{x^{-1}} f) + C(L_{x^{-1}} f) \circ \sigma\}(e),$$

where $f \in \mathcal{C}^\infty(G)$ and $x \in G$.

Proposition 3.5 *Let D be a differential operator on G, then $\tilde{D}$ satisfies the following properties:*

(i) $\tilde{\tilde{D}} = \tilde{D}$.
(ii) $\tilde{D} \in Z(G)$.
(iii) *If $D \in Z(G)$, then $\tilde{D} = \frac{1}{2}\{D + D^\sigma\}$, where $D^\sigma = D(f \circ \sigma) \circ \sigma$.*
(iv) $(\tilde{D}f)(e) = \frac{1}{2} D\{Cf + Cf \circ \sigma\}(e)$. *In particular if $Cf = f$ and $f \circ \sigma = f$, then we have $(\tilde{D}f)(e) = (Df)(e)$.*
(v) *If f is a solution of* (1.2.2)*, then $(\tilde{D}f) = (Df)(e)f = \lambda(D)f$.*

Proof By easy computations we have (i) and (iv).

(ii) Let $f \in \mathcal{C}^\infty(G)$ and let $a \in G$, for all $x \in G$, we have

$$\begin{aligned} L_a(\tilde{D}f)(x) &= (\tilde{D}f)(a^{-1}x) \\ &= \frac{1}{2} D\{C(L_{x^{-1}a} f) + C(L_{x^{-1}a} f) \circ \sigma\}(e) \\ &= \frac{1}{2} D\{C(L_{x^{-1}}(L_a f)) + C(L_{x^{-1}}(L_a f)) \circ \sigma\}(e) \\ &= \tilde{D}(L_a f)(x) \end{aligned}$$

and

$$\begin{aligned} R_a(\tilde{D}f)(x) &= (\tilde{D}f)(xa) \\ &= \frac{1}{2} D\{C(L_{(xa)^{-1}} f) + C(L_{(xa)^{-1}} f) \circ \sigma\}(e) \\ &= \frac{1}{2} D\{C(L_{x^{-1}}(R_a f)) + C(L_{x^{-1}}(R_a f)) \circ \sigma\}(e) \\ &= \tilde{D}(R_a f)(x). \end{aligned}$$

Then we obtain that $\tilde{D} \in Z(G)$.

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(iii) Let $D \in Z(G)$; for all $x, y \in G$, we have

$$C(L_{x^{-1}}f)(y) = \int_G (L_{x^{-1}}f)(tyt^{-1})\,dt,$$

and

$$D(C(L_{x^{-1}}f))(y) = \int_G (L_{x^{-1}}Df)(tyt^{-1})\,dt.$$

Then we get

$$D(C(L_{x^{-1}}f))(e) = (Df)(x)$$

and

$$D(C(L_{x^{-1}}f)\circ\sigma)(e) = (D(f\circ\sigma)\circ\sigma)(x),$$

and then

$$(\tilde{D}f) = \frac{1}{2}\{Df + D(f\circ\sigma)\circ\sigma\}.$$

(v) Let $f \in \mathcal{C}^\infty(G)$ be a solution of (1.2.2), then

$$\begin{aligned} C(L_{x^{-1}}f)(y) + C(L_{x^{-1}}f)(\sigma(y)) &= \int_G f(xtyt^{-1})\,dt + \int_G f(xt\sigma(y)t^{-1})\,dt \\ &= 2f(x)f(y). \end{aligned}$$

For $y = e$, we get

$$(\tilde{D}f) = f(Df)(e) = \lambda(D)f. \qquad \blacksquare$$

Proposition 3.6 *Let $f \in \mathcal{C}^\infty(G)$ be a non-zero solution of* (1.2.2), *then f is analytic.*

Proof Let L be the Laplace–Beltrami operator on G, we have $L \in Z(G)$ and $\tilde{L} = \frac{1}{2}\{L + L^\sigma\}$. In addition this operator is elliptic, and f is an eigenfunction of $\tilde{L}$, we deduce that f is analytic. ■

Theorem 3.7 *Let G be a compact connected Lie group and let $f \in \mathcal{C}^\infty(G)$. Then the following statements are equivalent:*

(1) *f is a solution of* (1.2.2).
(2) (i) *$f(e) = 1$, $Cf = f$ and $f\circ\sigma = f$,*
 (ii) *f is analytic,*
 (iii) *f is a eigenfunction of the operators $\tilde{D}$, for all $D \in \mathbb{D}(G)$.*

Proof (1) ⇒ (2) follows directly from Propositions 3.5 and 3.6. Conversely, suppose that (2) holds, with $\tilde{D}f = \lambda(D)f$, for all $D \in \mathbb{D}(G)$, where $\lambda(D) = (Df)(e)$. For a fixed $x \in G$, we define the function

$$F(y) = \frac{1}{2}\Big\{\int_G f(xtyt^{-1})\,dt + \int_G f(xt\sigma(y)t^{-1})\,dt\Big\}, \quad y \in G.$$

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Since f is central and $f \circ \sigma = f$, then we get

$$F(y) = \frac{1}{2}\Big\{ \int_G L_{(t^{-1}xt)^{-1}} f)(y)\, dt + \int_G (R_{t\sigma(x)t^{-1}} f)(y)\, dt \Big\}.$$

Consequently, for all $D \in \mathbb{D}(G)$, we have

$$(\tilde{D}F)(y) = \frac{1}{2}\Big\{ \int_G \tilde{D}(L_{(t^{-1}xt)^{-1}} f)(y)\, dt + \int_G \tilde{D}(R_{t\sigma(x)t^{-1}} f)(y)\, dt \Big\}.$$

Since $\tilde{D} \in Z(G)$, then we obtain

$$(\tilde{D}F)(y) = Df(e)F(y).$$

In particular for $y = e$, we have

$$(\tilde{D}F)(e) = Df(e)F(e).$$

Hence, by Proposition 3.5(iv), it follows that

$$(DF)(e) = D(f)(e)F(e),$$

i.e.,

$$D(F - F(e)f)(e) = 0,$$

for all $D \in \mathbb{D}(G)$. Since $F - F(e)f$ is an analytic function on the connected Lie group G, then by [5, Ch. II], we obtain

$$F - F(e)f \equiv 0$$

on G. We conclude that

$$\int_G f(xtyt^{-1})\, dt + \int_G f(xt\sigma(y)t^{-1})\, dt = 2f(x)f(y), \quad x, y \in G. \qquad \blacksquare$$

Corollary 3.8 *Let G be a compact connected Lie group and let $f \in \mathcal{C}^\infty(G)$. Then the following statements are equivalent:*

(1) *f is a solution of* (1.2.2).

(2) (i) *$f(e) = 1$, $Cf = f$ and $f \circ \sigma = f$,*

(ii) *f is analytic,*

(iii) *$\frac{1}{2}(Df + Df \circ \sigma) = \lambda(D)f$, for all $D \in Z(G)$.*

Proof By using Proposition 3.5, we have for all $D \in \mathbb{D}(G)$, $\tilde{\tilde{D}} = \tilde{D}$, $\tilde{D} \in Z(G)$ and $\tilde{D} = \frac{1}{2}(Df + Df \circ \sigma)$, for all $D \in Z(G)$. $\blacksquare$

4 On the Functional Equation

$$\int_G f(xtyt^{-1})\,dt + \int_G f(xt\sigma(y)t^{-1})\,dt = 2f(x)g(y)$$

In this section, we study the functional equation (1.2.3) and we determine the solutions of this equation in the case where f is central. We shall need the following proposition during the proof of the theorem.

Proposition 4.1 *Let $f, g \in \mathcal{C}(G)\setminus\{0\}$ constitute a solution of the functional equation*

$$\int_G f(xtyt^{-1})\,dt = f(x)g(y) + g(x)f(y), \quad x, y \in G. \tag{4.0.1}$$

Then there exists a constant $b \in \mathbb{C}$ such that

$$\int_G g(xtyt^{-1})\,dt = g(x)g(y) + b^2 f(x)f(y), \quad x, y \in G,$$

and f, g have one of the following forms:

(1) *there exists a function φ solution of* (1.3.1) *and a constant c such that*

$$f = c\varphi, \quad g = \frac{\varphi}{2}.$$

(2) *there exist two functions φ_1, φ_2 solutions of* (1.3.1) *and a constant b such that*

$$f = \frac{b(\varphi_1 - \varphi_2)}{2}, \quad g = \frac{\varphi_1 + \varphi_2}{2}.$$

Proof Let $f, g \in \mathcal{C}(G)\setminus\{0\}$ be a solution of (4.0.1). If there exists a constant $\lambda \in \mathbb{C}$ such that $g = \lambda f$, then the functional equation (4.0.1) is reduced to

$$\int_G f(xtyt^{-1})\,dt = 2\lambda f(x)f(y),$$

which implies that $2\lambda f = \varphi$ is a solution of (1.3.1) and we have

$$f = \frac{\varphi}{2\lambda}, \quad g = \frac{\varphi}{2}.$$

If f, g are linearly independent, then by using equation (4.0.1) we obtain for all $x, y, z \in G$

$$f(x)\int_G g(ytzt^{-1})\,dt + g(x)\int_G f(ytzt^{-1})\,dt = \int_G f(xtyt^{-1})\,dt g(z) + f(z)\int_G g(xtyt^{-1})\,dt.$$

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Then we get

$$(**)\quad f(x)\Big(\int_G g(ytzt^{-1})\,dt - g(y)g(z)\Big) = f(z)\Big(\int_G g(xtyt^{-1})\,dt - g(x)g(y)\Big).$$

Since $f \neq 0$, let $z_0 \in G$ such that $f(z_0) \neq 0$, then

$$\int_G g(xtyt^{-1})\,dt - g(x)g(y) = f(x)\psi(y),$$

where

$$\psi(y) = \frac{\int_G g(ytz_0t^{-1})\,dt - g(y)g(z_0)}{f(z_0)}.$$

By using $(**)$ we obtain

$$f(z)f(x)\psi(y) = f(x)f(y)\psi(z),$$

from which we see that ψ is a constant multiple of f, so

$$\psi(y) = cf(y) = b^2 f(y), \quad b \in \mathbb{C},$$

and the functions $\varphi_1 = g + bf$, $\varphi_2 = g - bf$ are solutions of (1.3.1) ■

Theorem 4.2 *Let $f, g \in \mathcal{C}(G) \setminus \{0\}$ such that f is central. If (f, g) is a solution of* (1.2.3)*, then there exist $(\pi, \mathcal{H}_\pi)$ an irreducible, continuous and unitary representation of G and $\alpha, \beta \in \mathbb{C}$ such that*

$$g = \frac{\chi_\pi + \chi_\pi \circ \sigma}{2d(\pi)}, \quad f = \alpha\frac{\chi_\pi + \chi_\pi \circ \sigma}{2d(\pi)} + \beta\frac{\chi_\pi - \chi_\pi \circ \sigma}{2d(\pi)}.$$

Proof Let (f, g) be a solution of (1.2.3); then by Proposition 2.5 we get that g satisfies (1.2.2). We deduce, by using Corollary 3.4, that $g = \frac{\chi_\pi + \chi_\pi \circ \sigma}{2d(\pi)}$. By decomposing f into its even and odd parts we write

$$f(x) = \frac{f(x) + f(\sigma(x))}{2} + \frac{f(x) - f(\sigma(x))}{2} = f_1(x) + f_2(x).$$

We see that $f_1(\sigma(x)) = f(x)$ and $f_2(\sigma(x)) = -f(x), x \in G$. Since f is central, f_1 is central and $\int_G f(xtyt^{-1})\,dt = \int_G f(ytxt^{-1})\,dt$. Then we have

$$(4.0.2)\qquad \int_G f_1(xtyt^{-1})\,dt + \int_G f_1(xt\sigma(y)t^{-1})\,dt = 2f_1(x)g(y), \quad x, y \in G.$$

Since f_1 is central and $f_1 \circ \sigma = f_1$, we find that $f_1 = f_1(e)g = \alpha g$. On the other hand f_2 is a solution of the functional equation

$$(4.0.3)\qquad \int_G f_2(xtyt^{-1})\,dt + \int_G f_2(xt\sigma(y)t^{-1})\,dt = 2f_2(x)g(y), \quad x, y \in G.$$

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So

$$\int_G f_2(ytxt^{-1})\,dt + \int_G f_2(yt\sigma(x)t^{-1})\,dt = 2f_2(y)g(x), \quad x, y \in G, \tag{4.0.4}$$

and adding the equations (4.0.3) and (4.0.4), and in view of $f_2(\sigma(x)) = -f_2(x)$ and $\int_G f_2(xtyt^{-1})\,dt = \int_G f_2(ytxt^{-1})\,dt$, we have

$$\int_G f_2(xtyt^{-1})\,dt = f_2(x)g(y) + g(x)f_2(y), \quad x, y \in G.$$

By using Proposition 4.1(2), there exists $(\pi, \mathcal{H}_\pi)$ an irreducible, continuous and unitary representation of G and $\alpha, \beta \in \mathbb{C}$ such that

$$f = \alpha \frac{\chi_\pi + \chi_\pi \circ \sigma}{2d(\pi)} + \beta \frac{\chi_\pi - \chi_\pi \circ \sigma}{2d(\pi)}. \qquad \blacksquare$$

5 On the Functional Equation

$$\int_G f(xtyt^{-1})dt + \int_G f(xt\sigma(y)t^{-1})dt = 2g(x)h(y)$$

In this section, we study the properties of the functional equation (1.2.1) and we determine the solutions of this equation in the case where f is central.

Theorem 5.1 *Let $(f, g, h) \in (\mathcal{C}(G) \setminus \{0\})^3$ be a solution of the functional equation* (1.2.1). *Then*

(i) *h is a central function and $h \circ \sigma = h$.*
(ii) *If f is central, then g is central.*
(iii) *There exists a function ϕ solution of the functional equation* (1.2.2) *such that (g, ϕ) and $(\check{h}, \check{\phi})$ are solutions of* (1.2.3).
(iv) *If G is a connected Lie group, then g and $\check{h}$ are eigenfunctions of the operators $\tilde{D}$ for all $D \in \mathbb{D}(G)$. Precisely we have*

$$\tilde{D}g = (D\phi)(e)g, \quad \tilde{D}\check{h} = (D\check{\phi})(e)\check{h}, \quad D \in \mathbb{D}(G).$$

Proof By easy computations we have (i) and (ii).

Let $a, b \in G$ such that $g(a) \neq 0$ and $h(b) \neq 0$. Then for all $x, y \in G$ we have

$$\begin{aligned}
2h(b)\Big(\int_G g(atxt^{-1})\,dt + \int_G g(at\sigma(x)t^{-1})\,dt\Big)
&= \int_G 2h(b)g(atxt^{-1}\,dt + \int_G 2h(b)g(at\sigma(x)t^{-1})\,dt\\
&= \int_G\int_G f(atxt^{-1}sbs^{-1})\,dsdt + \int_G\int_G f(atxt^{-1}s\sigma(b)s^{-1})\,dsdt\\
&\quad + \int_G\int_G f(at\sigma(x)t^{-1}sbs^{-1})\,dsdt + \int_G\int_G f(at\sigma(x)t^{-1}s\sigma(b)s^{-1})\,dsdt
\end{aligned}$$

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$$= 2g(a)\int_G h(xtbt^{-1})\,dt + 2g(a)\int_G h(xt\sigma(b)t^{-1})\,dt.$$

Let

$$\begin{aligned}\phi(x) &= \frac{1}{2g(a)}\Big(\int_G g(atxt^{-1})\,dt + \int_G g(at\sigma(x)t^{-1})\,dt\Big)\\ &= \frac{1}{2h(b)}\Big(\int_G h(xtbt^{-1})\,dt + \int_G h(xt\sigma(b)t^{-1})\,dt\Big).\end{aligned}$$

Then we get

$$\begin{aligned}&2g(a)\Big(\int_G h(xtyt^{-1})\,dt + \int_G h(xt\sigma(y)t^{-1})\,dt\Big)\\ &\quad= \int_G\int_G f(asxtyt^{-1}s^{-1})\,dtds + \int_G\int_G f(as\sigma(x)t\sigma(y)t^{-1}s^{-1})\,dtds\\ &\qquad+ \int_G\int_G f(asxt\sigma(y)t^{-1}s^{-1})\,dtds + \int_G\int_G f(as\sigma(x)tyt^{-1}s^{-1})\,dtds\\ &\quad= 2h(y)\Big(\int_G g(asxs^{-1})\,ds + \int_G g(as\sigma(x)s^{-1})\,ds\Big),\end{aligned}$$

i.e.,

$$\int_G h(xtyt^{-1})\,dt + \int_G h(xt\sigma(y)t^{-1})\,dt = 2h(y)\phi(x),$$

and

$$\begin{aligned}&2h(b)\Big(\int_G g(xtyt^{-1})dt + \int_G g(xt\sigma(y)t^{-1})dt\Big)\\ &\quad= \int_G\int_G f(xtyt^{-1}sbs^{-1})\,dtds + \int_G\int_G f(xtyt^{-1}s\sigma(b)s^{-1})\,dtds\\ &\quad= \int_G\int_G f(xt\sigma(y)t^{-1}sbs^{-1})\,dtds + \int_G\int_G f(xt\sigma(y)t^{-1}s\sigma(b)s^{-1})\,dtds\\ &\quad= 2g(x)\Big(\int_G h(ysbs^{-1})\,ds + \int_G h(ys\sigma(b)s^{-1})\,ds\Big),\end{aligned}$$

i.e.,

$$\int_G g(xtyt^{-1})\,dt + \int_G g(xt\sigma(y)t^{-1})\,dt = 2g(x)\phi(y).$$

(iv) follows by using Theorem 3.7. ■

In the next theorem, we assume that $g = f$ in (1.2.1). As immediate consequences, we obtain the following theorem:

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Theorem 5.2 *Let* $(f,h) \in (\mathcal{C}(G) \setminus \{0\})^2$ *be a solution of the functional equation* (1.2.3)*, then*

(i) *h is a central function and $h \circ \sigma = h$.*
(ii) *h is a solution of* (1.2.2).
(iii) *If G is a connected Lie group, then $\tilde{D}f = (Dh)(e)f$, for all $D \in \mathbb{D}(G)$.*

Applying Theorem 5.1, we get the following theorem:

Theorem 5.3 *Let $f, g, h \in \mathcal{C}(G) \setminus \{0\}$ such that f is central, verifying the functional equation* (1.2.1)*. Then these functions are given by*

$$f(x) = ab\frac{\varphi(x) + \varphi(\sigma(x))}{2} + ac\frac{\varphi(x) - \varphi(\sigma(x))}{2},$$
$$g(x) = b\frac{\varphi(x) + \varphi(\sigma(x))}{2} + c\frac{\varphi(x) - \varphi(\sigma(x))}{2},$$
$$h(x) = a\frac{\varphi(x) + \varphi(\sigma(x))}{2},$$

where a, b, c are arbitrary complex numbers and φ is a solution of (1.3.1).

Corollary 5.4 *Let $f, g, h \in \mathcal{C}(G) \setminus \{0\}$ such that f is central. Then (f, g, h) is a solution of* (1.2.1) *if and only if there exists $(\pi, \mathcal{H}_\pi)$ an irreducible, continuous and unitary representation of G such that*

$$f(x) = ab\frac{\chi_\pi(x) + \chi_\pi(\sigma(x))}{2d(\pi)} + ac\frac{\chi_\pi(x) - \chi_\pi(\sigma(x))}{2d(\pi)},$$
$$g(x) = b\frac{\chi_\pi(x) + \chi_\pi(\sigma(x))}{2d(\pi)} + c\frac{\chi_\pi(x) - \chi_\pi(\sigma(x))}{2d(\pi)},$$
$$h(x) = a\frac{\chi_\pi(x) + \chi_\pi(\sigma(x))}{2d(\pi)},$$

where a, b, c are arbitrary complex numbers and $d(\pi)$ denotes the dimension of the representation π.

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