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SETS WITH NO EMPTY CONVEX 7-GONS

BY
J. D. HORTON*

ABSTRACT. Erdös has defined $g(n)$ as the smallest integer such that any set of $g(n)$ points in the plane, no three collinear, contains the vertex set of a convex n-gon whose interior contains no point of this set. Arbitrarily large sets containing no empty convex 7-gon are constructed, showing that $g(n)$ does not exist for $n \geq 7$. Whether $g(6)$ exists is unknown.

Esther Klein raised the following combinatorial geometry problem [5]. For $n \geq 3$, let $f(n)$ be the smallest integer such that for any set of $f(n)$ points in the plane, no three collinear, contains the vertex set of a convex n-gon. Determine $f(n)$. It is easy to show that $f(3)=3$ and $f(4)=5$. That $f(5)=9$ was proved in [4]. Erdös and Szekeres [1], [2] determined that $2^{n-2}+1 \leq f(n) \leq \binom{2n-4}{n-2}+1$.

Erdös has raised a similar question. For $n \geq 3$, define $g(n)$ to be the smallest integer such that any set of $g(n)$ points in the plane, no three collinear, contains the vertex set of a convex n-gon whose interior contains no point of the set. We call a n-gon, with no points of the set in its interior, *empty*. Again, $g(3)=3$ and $g(4)=5$. Harborth [3] has proved that $g(5)=10$. However, it is not known whether $g(6)$ exists. The main result of this note is that $g(7)$, and hence $g(n)$ for all $n \geq 7$, does not exist.

We construct, for any k, a set of 2^k points with no empty convex 7-gon. Let $a_1a_2 \cdots a_k$ be the binary expansion of the integer i, $0 \leq i < 2^k$. Note that leading 0's are not omitted. Let $c = 2^k+1$, and define $d(i) = \sum a_j c^{i-1}$, summing from $j=1$ to $j=k$. Let p_i be the point $(i, d(i))$, and define S_k to be the set of points $\{p_i \mid i = 0, 1, \ldots, 2^k-1\}$. Observations:

(a) $\{p_i \mid i < 2^{k-1}\}$ = the left half of $S_k = L$.

(b) $\{p_i \mid i \geq 2^{k-1}\}$ = the right half of $S_k = R$, which is a translate of L.

(c) $\{p_i \mid i \text{ is even}\}$ = the bottom half of $S_k = B$.

(d) $\{p_i \mid i \text{ is odd}\}$ = the top half of $S_k = T$, which is a translate of B.

(e) L, R, B, and T are all scaled translates of each other. For example, halving the first coordinate while multiplying the second coordinate by c, takes B onto L.

(f) The 180° rotation of the plane about $((2^k-1)/2, \sum c^i/2)$ takes T onto B.

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(g) All points of T are above any line joining two points of B. The value of c was chosen large enough to make this true. Similarly, all points of B are below any line joining two points of T.

(h) If i and j both have the same last x digits in their binary expansions, and h has a different sequence of x rightmost digits, then whether p_h is above or below the line joining p_i and p_j is determined by the sequences of the last x digits.

Consider any empty convex n-gon A in S_k. We may assume A is contained entirely in neither T nor B. Otherwise if A is contained in B, apply the linear transformation that takes B onto L. A will be transformed into an empty convex n-gon in L. Similarly, if A is contained in T, apply the linear transformation that takes T onto L. Repeat this procedure until a transformed image of A meets both T and B.

Next, consider how many points of A can be in B. Assume p_i and p_j are in $A \cap B$. By (g) above, no point p_h of B with $i<h<j$, can be above the line segment joining p_i and p_j, since otherwise no point of T could be in A. As well, I claim that $d(h)<d(i)$ and $d(h)<d(j)$. Since p_h is below the line joining p_i and p_j, clearly one of these statements is true. Assume $d(h)<d(i)$, but $d(h)>d(j)$. Let x be the position of the right-most digit at which h and i differ in their binary expansions; let y be the position of the right-most digit at which h and j differ. In both cases, the number with the larger functional value must have a 1 in the position, and the other number a 0. If $x<y$ then p_j must be below the line joining p_i and p_h, by observation (h). But then p_h is above the line joining p_i and p_j, a contradiction. Hence we can assume that $y<x$. In this case, consider $l=j-2^{k-x}$. The right-most position in which the binary expansions of l and j differ is x, where l has a 1 and j has a 0. On the other hand, l and i must agree in the last $k-x$ positions. By observation (h), p_j is below the line joining p_i and p_l. But since $j-i>j-h\geq 2^{k-y}>2^{k-x}=j-l$, $i<l<j$. Then p_l must be both above and below the line joining p_i and p_j, a contradiction. Similarly, $d(j)<d(h)<d(i)$ leads to a contradiction. Therefore $d(h)<d(i)$ and $d(h)<d(j)$.

If $A \cap B$ contained four points $i<h<l<j$, then $d(h)<d(l)$ and $d(l)<d(h)$. Hence $A \cap B$ cannot contain more than three points. By observation (f) above, $A \cap T$ cannot contain more than three points either. Hence A has no more than 6 points.

Whether g(6) exists is still unknown, although the author believes that g(6) does exist.

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School of Computer Science
University of New Brunswick
Fredericton, New Brunswick, E3B 5A3