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Trigonometric convexity of the multidimensional indicator

Aleksandr Mkrtchyan and Armen Vagharshakyan

Abstract. The notion of indicator of an analytic function, that describes the function's growth along rays, was introduced by Phragmen and Lindelöf. Trigonometric convexity is a defining property of the indicator. For multivariate cases, an analogous property of trigonometric convexity was not known so far. We prove the property of trigonometric convexity for the indicator of multivariate analytic functions, introduced by Ivanov. The results that we obtain are sharp. Derivation of a multidimensional analogue of the inverse Fourier transform in a sector and obtaining estimates on its decay is an important step of our proof.

1 Introduction

1.1 Main results

We review the notion of indicator for analytic functions in one complex variable (see Section 1.2) and its multidimensional analogues for functions in several complex variables (see Section 1.3); in particular, we treat the notion of multidimensional indicator after Ivanov. We also consider a multidimensional analogue of the inverse Fourier transform in a sector and obtain estimates on its decay (see Section 2).

We remind that the indicator is a characteristic of a function, that describes a function's growth on directions (axes starting from the origin). In particular, the indicator and its property of trigonometric convexity plays an important role in certain methods of analytic continuation, and allows to avoid some extra hypotheses and cumbersome formulations in theorems. This claim is highlighted by Polya's theorem [22], stated later in this article, and the theorems of Lindelöf, Arakeyan [1], Carlson [5, 6], and others. All of these results are concerned with analytic continuation of power series in one complex variable by means of interpolating its coefficients by entire functions.

Meanwhile, as far as power series in several complex variables are concerned, many of the corresponding questions are still open. We remark that the latter questions are motivated by applications in mathematical physics, in particular by applications in

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thermodynamics [19, 27]. They are also related to multidimensional hypergeometric series [17, 24].

As explained in Remark 1.8, our main Theorem 1.1 establishes an analogue of trigonometric convexity for the multidimensional indicator after Ivanov.

We remark that derivation of a multidimensional analogue of the inverse Fourier transform in a sector and obtaining an estimate on its decay is an important step of our proof. For a general survey on decay of the Fourier transform, we refer the reader to the book of Iosevich and Liflyand [9]. We preface the formulation of Theorem 1.1 by the following definitions:

Definition 1.1 Denote by $\Delta_{\alpha_j} \subset \mathbb{C}$ the open sector determined by the angle $0 < \alpha_j < \pi/2$ as follows:

$$\Delta_{\alpha_j} = \{z_j \in \mathbb{C}\backslash\{0\} : \left|\arg\left(z_j\right)\right| < \alpha_j\}.$$

Definition 1.2 Recall that a function f is of finite exponential type $(h_1, \ldots, h_n)$ in $\Delta_{\alpha_1} \times \cdots \times \Delta_{\alpha_n}$ if for any $\varepsilon > 0$ there exists a constant $k_\varepsilon \geq 0$ such that

$$\begin{aligned} &\left|f\left(z_1, \ldots, z_n\right)\right| \leq k_\varepsilon e^{(h_1+\varepsilon)|z_1|+\cdots+(h_n+\varepsilon)|z_n|}, \\ &\text{for all } z_j \in \Delta_{\alpha_j},\ 1 \leq j \leq n. \end{aligned} \tag{1.1}$$

Note that, in this article, we tacitly assume that $h_1, \ldots, h_n \geq 0$.

Definition 1.3 Denote by $Exp\,(\alpha_1, \ldots, \alpha_n)$ the class of functions f that are analytic and are of finite exponential type in $\Delta_{\alpha_1} \times \cdots \times \Delta_{\alpha_n}$.

Definition 1.4 Following Ivanov [10], for $f \in Exp\,(\alpha_1, \ldots, \alpha_n)$, denote by $T_f\left(\vec{\theta}\right)$ the following:

$$\begin{aligned} T_f\left(\vec{\theta}\right) = \{\vec{v} \in \mathbb{R}^n : \ln\left|f\left(\vec{r}e^{i\vec{\theta}}\right)\right| \leq v_1 r_1 + \cdots + v_n r_n + C_{\vec{v},\vec{\theta}}, \\ \text{for some } C_{\vec{v},\vec{\theta}},\ \text{for all } \vec{r} \in \mathbb{R}^n_+\}; \end{aligned} \tag{1.2}$$

here, $\vec{r}e^{i\vec{\theta}}$ denotes the vector $\left(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}\right)$.

Theorem 1.1 *Let a function $f \in Exp\,(\alpha_1, \alpha_2)$ and the numbers $A_1^+,\ A_2^+,\ A_1^-,\ A_2^-$ satisfy*

$$\begin{aligned} (A_1^+, A_2^+) &\in \overline{T_f\,(\alpha_1, \alpha_2)}, \\ (A_1^-, A_2^-) &\in \overline{T_f\,(-\alpha_1, -\alpha_2)}, \\ (A_1^+, A_2^-) &\in \overline{T_f\,(\alpha_1, -\alpha_2)}, \\ (A_1^-, A_2^+) &\in \overline{T_f\,(-\alpha_1, \alpha_2)}. \end{aligned}$$

Then we have

$$(C_1, C_2) \in \overline{T_f\,(\theta_1, \theta_2)},$$

where the constants C_1, C_2 are determined by the following formulas:

$$C_1 \sin\left(2\alpha_1\right) = A_1^+ \sin\left(\theta_1 + \alpha_1\right) + A_1^- \sin\left(\alpha_1 - \theta_1\right), \tag{1.3}$$

$$C_2 \sin\left(2\alpha_2\right) = A_2^+ \sin\left(\theta_2 + \alpha_2\right) + A_2^- \sin\left(\alpha_2 - \theta_2\right). \tag{1.4}$$

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Theorem 1.1 can be paraphrased as follows:

Remark 1.2. Let a function $f \in Exp(\alpha_1, \alpha_2)$ and the numbers A_1^+, A_2^+, A_1^-, A_2^- satisfy

$$|f(z_1, z_2)| \le k_\varepsilon e^{(A_1^+ + \varepsilon)|z_1| + (A_2^+ + \varepsilon)|z_2|}, \quad \text{for } \arg(z_1) = \alpha_1,\ \arg(z_2) = \alpha_2,$$
$$|f(z_1, z_2)| \le k_\varepsilon e^{(A_1^+ + \varepsilon)|z_1| + (A_2^- + \varepsilon)|z_2|}, \quad \text{for } \arg(z_1) = \alpha_1,\ \arg(z_2) = -\alpha_2,$$
$$|f(z_1, z_2)| \le k_\varepsilon e^{(A_1^- + \varepsilon)|z_1| + (A_2^+ + \varepsilon)|z_2|}, \quad \text{for } \arg(z_1) = -\alpha_1,\ \arg(z_2) = \alpha_2,$$
$$|f(z_1, z_2)| \le k_\varepsilon e^{(A_1^- + \varepsilon)|z_1| + (A_2^- + \varepsilon)|z_2|}, \quad \text{for } \arg(z_1) = -\alpha_1,\ \arg(z_2) = -\alpha_2, \tag{1.5}$$

where by the value of the function f on the mentioned rays we mean f's non-tangential limit (see [25, Remark 2.1] for details). Then, for any $\varepsilon > 0$, there exists a constant $k'_\varepsilon > 0$ such that

$$|f(z_1, z_2)| \le k'_\varepsilon e^{(C_1 + \varepsilon)|z_1| + (C_2 + \varepsilon)|z_2|}, \quad \text{for } \arg(z_1) = \theta_1,\ \arg(z_2) = \theta_2, \tag{1.6}$$

where the constants C_1, C_2 are determined by the following formulas:

$$C_1 \sin(2\alpha_1) = A_1^+ \sin(\theta_1 + \alpha_1) + A_1^- \sin(\alpha_1 - \theta_1),$$
$$C_2 \sin(2\alpha_2) = A_2^+ \sin(\theta_2 + \alpha_2) + A_2^- \sin(\alpha_2 - \theta_2).$$

In Section 3, we derive Theorem 1.1 from Theorem 1.3 that serves as a two-dimensional analogue of the Fourier inversion formula (see [25, Theorem 1.2] and [7]) for functions of exponential type in a sector. We preface the formulation of Theorem 1.3 by the following definitions:

Definition 1.5 Let $f \in Exp(\alpha_1, \alpha_2)$, and let the numbers $A_1^+, A_2^+, A_1^-, A_2^-$ satisfy (1.5). Define the function m as f's two-dimensional concatenated Laplace transform. Namely, the domain of function m is the Cartesian product $\Omega_1 \times \Omega_2$, where

$$\Omega_1 = \Omega_1^+ \cup \Omega_1^-,$$
$$\Omega_2 = \Omega_2^+ \cup \Omega_2^-,$$

and in turn

$$\Omega_1^+ = \left\{\omega_1 : Re\left(\omega_1 e^{i\alpha_1}\right) < -A_1^+\right\},$$
$$\Omega_1^- = \left\{\omega_1 : Re\left(\omega_1 e^{-i\alpha_1}\right) < -A_1^-\right\},$$
$$\Omega_2^+ = \left\{\omega_2 : Re\left(\omega_2 e^{i\alpha_2}\right) < -A_2^+\right\},$$
$$\Omega_2^- = \left\{\omega_2 : Re\left(\omega_2 e^{-i\alpha_2}\right) < -A_2^-\right\} \tag{1.7}$$

(see Figure 1). The function m is defined on $\Omega_1 \times \Omega_2$ by the following four formulas:

$$m(\omega_1, \omega_2) = \frac{-1}{4\pi^2} \int_{e^{i\alpha_2}[0,+\infty)} \int_{e^{i\alpha_1}[0,+\infty)} f(\zeta_1, \zeta_2)\, e^{\omega_1\zeta_1 + \omega_2\zeta_2} d\zeta_1 d\zeta_2, \tag{1.8}$$
$$\text{for } \omega_1 \in \Omega_1^+,\ \omega_2 \in \Omega_2^+,$$
$$m(\omega_1, \omega_2) = \frac{-1}{4\pi^2} \int_{e^{i\alpha_2}[0,+\infty)} \int_{e^{-i\alpha_1}[0,+\infty)} f(\zeta_1, \zeta_2)\, e^{\omega_1\zeta_1 + \omega_2\zeta_2} d\zeta_1 d\zeta_2,$$
$$\text{for } \omega_1 \in \Omega_1^+,\ \omega_2 \in \Omega_2^-,$$

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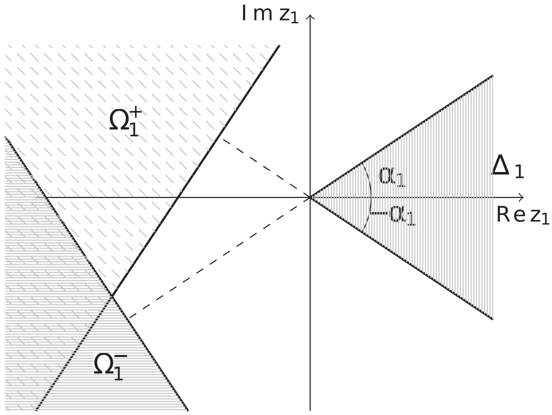

Figure 1: The set Ω.

$$m(\omega_1,\omega_2) = \frac{-1}{4\pi^2}\int_{e^{-i\alpha_2}[0,+\infty)}\int_{e^{i\alpha_1}[0,+\infty)} f(\zeta_1,\zeta_2)\, e^{\omega_1\zeta_1+\omega_2\zeta_2} d\zeta_1 d\zeta_2,$$
$$\text{for } \omega_1 \in \Omega_1^-,\ \omega_2 \in \Omega_2^+,$$
$$m(\omega_1,\omega_2) = \frac{-1}{4\pi^2}\int_{e^{-i\alpha_2}[0,+\infty)}\int_{e^{-i\alpha_1}[0,+\infty)} f(\zeta_1,\zeta_2)\, e^{\omega_1\zeta_1+\omega_2\zeta_2} d\zeta_1 d\zeta_2,$$
$$\text{for } \omega_1 \in \Omega_1^-,\ \omega_2 \in \Omega_2^-.$$

Definition 1.6 For a class of functions $Exp(\alpha_1,\alpha_2)$, denote by Γ_1 and Γ_2 the curves given by the following parametrizations:

$$\gamma_1 : \mathbb{R} \to \mathbb{C},$$
$$\gamma_1(t) = p_1 - ie^{i\alpha_1}|t|, \quad \text{for } t \in (-\infty, 0],$$
$$\gamma_1(t) = p_1 + ie^{-i\alpha_1}|t|, \quad \text{for } t \in (0, +\infty] \tag{1.9}$$

(see Figure 2), and the real number p_1 appearing in parametrization (1.9) is chosen in such a way that it satisfies inequality

$$p_1 \cos(\alpha_1) < -h_1 \tag{1.10}$$

and correspondingly the curve Γ_2 is parameterized by

$$\gamma_2 : \mathbb{R} \to \mathbb{C},$$
$$\gamma_2(t) = p_2 - ie^{i\alpha_2}|t|, \quad \text{for } t \in (-\infty, 0],$$
$$\gamma_2(t) = p_2 + ie^{-i\alpha_2}|t|, \quad \text{for } t \in (0, +\infty], \tag{1.11}$$

and the real number p_2 appearing in parametrization (1.9) is chosen in such a way that it satisfies inequality

$$p_2 \cos(\alpha_2) < -h_2. \tag{1.12}$$

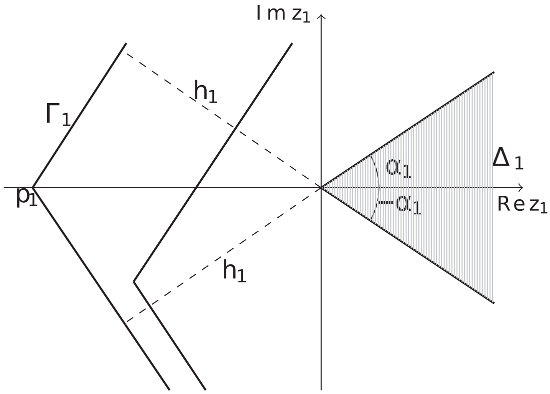

Figure 2: The curve Γ_1.

Theorem 1.3 *For a function $f \in Exp(\alpha_1, \alpha_2)$, the following Fourier inversion formula holds:*

$$f(z_1, z_2) = \int_{\Gamma_1} \int_{\Gamma_2} m(\omega_1, \omega_2) e^{-\omega_1 z_1 - \omega_2 z_2} d\omega_2 d\omega, \quad \text{for } z_1 \in \Delta_{\alpha_1},\ z_2 \in \Delta_{\alpha_2}. \tag{1.13}$$

Theorem 1.3 is proved in Section 2.

We complement Theorem 1.1 by the following remarks:

Remark 1.4. Theorem 1.1 is stated for functions of two complex variables. The corresponding theorem for functions of n complex variables also holds.

Remark 1.5. Theorem 1.1 is sharp, that is, there exists a function f for whom the assumptions of Theorem 1.1 are satisfied, and the inequality (1.6) is an equality.

Remark 1.5 is proved in Section 4.

Remark 1.6. Integral representation (1.13) in Theorem 1.3 could be compared to multivariate integrals of Mellin–Barnes type [8, 20, 26].

1.2 Indicators of functions in one complex variable

We say that an entire function $f : \mathbb{C} \to \mathbb{C}$ is of exponential type if

$$\limsup_{z \to \infty} \frac{\ln^+ |f(z)|}{|z|} < +\infty. \tag{1.14}$$

The notion of indicator of an entire function of exponential type was introduced by Phragmen and Lindeleph [3, 21] as follows:

$$h_f(\alpha) := \limsup_{r \to \infty} \frac{\ln \left| f\left(re^{i\alpha}\right)\right|}{r}, \quad \alpha \in \mathbb{R}. \tag{1.15}$$

The indicator $h_f(\alpha)$ describes the growth of function f along the ray $e^{i\alpha}[0, +\infty)$. It follows from the definition (1.15) that the indicator $h_f(\alpha)$ is a real-valued 2π-periodic function. It also follows that the indicator of the product of two functions does not exceed the sum of the indicators of the factors,

$$h_{fg}(\alpha) \leq h_f(\alpha) + h_g(\alpha),$$

and that the indicator of the sum of two functions does not exceed the larger of the two indicators,

$$h_{f+g}(\alpha) \leq \max\left(h_f(\alpha), h_g(\alpha)\right).$$

One of the main properties of the indicator $h_f(\alpha)$ is its *trigonometric convexity* [4, 15]: if $\alpha_1 < \alpha < \alpha_2$ and $\alpha_2 - \alpha_1 < \pi$, then the following inequality holds:

$$h_f(\alpha)\sin(\alpha_2 - \alpha_1) \leq h_f(\alpha_1)\sin(\alpha_2 - \alpha) + h_f(\alpha_2)\sin(\alpha - \alpha_1). \tag{1.16}$$

The following property follows from the trigonometric convexity (1.16) [15]: if the indicator is bounded on an open interval, then it is continuous. The latter claim does not hold for a closed interval.

The notion of indicator is known to be an important tool in some methods regarding to finding analytic continuation [1, 5, 6, 22]. In particular, the indicator appears in problems relating to analytic continuation of power series via interpolation of coefficients. It also plays a role in problems relating to localization of singularities of power series [2].

Let

$$f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$$

be the power-series representation of the entire function f. Consider its Borel transform defined by the Laurent series in the following way:

$$g(\omega) = \sum_{k=0}^{\infty} a_k \omega^{-k-1}. \tag{1.17}$$

The interrelation between the set of singularities of g and the indicator of f is described by Polya's theorem [14, 23].

Theorem 1.7 (Polya) *Let f be an entire function of exponential type. Denote by $K \subset \mathbb{C}$ the convex set whose support function*

$$k(\theta) = \sup_{\omega \in K} Re\left(\omega e^{-i\theta}\right)$$

is determined by f's indicator as follows:

$$k(-\theta) = h_f(\theta). \tag{1.18}$$

Then f can be restored by

$$f(z) = \frac{1}{2\pi i} \int_\Gamma g(\omega) e^{z\omega} d\omega, \tag{1.19}$$

where Γ is a closed contour containing the set K, and g (1.17) is the Borel transform of f. Additionally, K is the smallest convex set such that g is analytic in $\mathbb{C} \backslash K$.

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1.3 Indicators of functions in several complex variables

The works of Ronkin [23], Lelon [13], Levin [15], Ivanov [10], Kiselman [11], and others are dedicated to exploring multidimensional analogues of the indicator function. We say that $f(\vec{z}) = f(z_1, \ldots, z_n)$ is an n-valued entire function of exponential type if f is holomorphic in $\mathbb{C}^n$, i.e.,

$$f(\vec{z}) \in \mathcal{O}(\mathbb{C}^n),$$

and if there exist constants $k, \sigma_1, \ldots, \sigma_n$ such that

$$|f(\vec{z})| \le k e^{\sigma_1|z_1| + \cdots + \sigma_n|z_n|}, \quad \text{for } \vec{z} \in \mathbb{C}^n. \tag{1.20}$$

In the case of several complex variables, different characteristics of an analytic function's growth in directions have been introduced. For example, introduce the *radial indicator* of the function f as follows [13, 23]:

$$L_r(z, f) = \limsup_{t\to\infty} \frac{\ln|f(tz)|}{t}, \quad z \in \mathbb{C}^n$$

and correspondingly introduce regularization of the function $L(z, f)$ as follows:

$$L^*(z, f) = \limsup_{z'\to z} L(z, f). \tag{1.21}$$

We will refer to $L^*(z, f)$ as *regularized radial indicator* of the function *f*. Just as in the one-dimensional case, the function $L^*(z, f)$ is semi-continuous from above. Note that the regularized radial indicator is a plurisubharmonic function in $\mathbb{C}^n$. Consequently, more information about its properties may be found in works related to plurisubharmonic functions in potential theory [11] and their different generalizations [12, 16].

Another characteristic of an analytic functions growth was introduced by Ivanov; namely Ivanov [10] has introduced the set $T_f(\vec{\theta})$ as follows:

$$T_f(\vec{\theta}) = \{\vec{v} \in \mathbb{R}^n : \ln\left|f\left(\vec{r}e^{i\vec{\theta}}\right)\right| \le v_1 r_1 + \cdots + v_n r_n + C_{\vec{v},\vec{\theta}}, \\ \text{for some } C_{\vec{v},\vec{\theta}}, \text{ for all } \vec{r} \in \mathbb{R}^n_+\}; \tag{1.22}$$

here, $\vec{r}e^{i\vec{\theta}}$ is the vector $(r_1e^{i\theta_1}, \ldots, r_ne^{i\theta_n})$. The set $T_f(\vec{\theta})$ implicitly reflects the notion of an indicator of an entire function.

For example, for the closure of the set $T_\varphi(\theta_1, \theta_2)$ defined for the function $\varphi(\zeta_1, \zeta_2) = \cos(\zeta_1\zeta_2)^{\frac{1}{2}}$ of exponential type, we have the following one [18]:

$$\overline{T_\varphi(\theta_1, \theta_2)} = \left\{\vec{v} \in \mathbb{R}^2 : v_1v_2 = \frac{1}{4}\left|\sin\left(\frac{\theta_1 + \theta_2}{2}\right)\right|^2, \quad v_1 \ge 0, \quad v_2 \ge 0\right\}.$$

There exist many more analogues of indicator for functions in several complex variables. However, for none of them, a property resembling trigonometric convexity was obtained.

Remark 1.8. Following our main result (Theorem 1.1), we now formulate trigonometric convexity for multidimensional indicator after Ivanov.

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Let a function $f \in Exp\left(\alpha_1, \ldots, \alpha_n\right)$ and the numbers $A_1^+,\ A_1^-, \ldots, A_n^+,\ A_n^-$ satisfy

$$\left(A_1^{l_1}, \ldots, A_n^{l_n}\right) \in \overline{T_f\left(l_1\alpha_1, \ldots, l_n\alpha_n\right)},$$

where $l_j = \pm, \quad j = 1, \ldots, n$. Then

$$\left(C_1, \ldots, C_n\right) \in \overline{T_f\left(\theta_1, \ldots, \theta_n\right)},$$

where the constants $C_1, \ldots, C_n$ are determined from the following formulas:

$$C_j \sin\left(2\alpha_j\right) = A_j^+ \sin\left(\theta_j + \alpha_j\right) + A_j^- \sin\left(\alpha_j - \theta_j\right), \quad j = 1, \ldots, n.$$

2 Two-dimensional sectorial Fourier inversion formula

2.1 Two-dimensional concatenated Fourier transform

Due to (1.7) for any pair of complex numbers $\left(\omega_1, \omega_2\right) \subset \Omega_1^+ \times \Omega_2^+$, we can pick $\varepsilon > 0$ so that the following inequalities are satisfied:

$$\begin{aligned} Re\left(\omega_1 e^{i\alpha_1}\right) + \varepsilon &< -A_1^+, \\ Re\left(\omega_2 e^{i\alpha_2}\right) + \varepsilon &< -A_2^+. \end{aligned} \tag{2.1}$$

Then we have the following estimate on the function m defined by the first formula in (1.8):

$$\begin{aligned} &\left|m\left(\omega_1, \omega_2\right)\right| \overset{(1.8)}{\le} \\ &\le \frac{1}{4\pi^2} \int_{e^{i\alpha_1}[0,+\infty)} \int_{e^{i\alpha_2}[0,+\infty)} \left|f\left(\zeta_1, \zeta_2\right)\right| \left|e^{\omega_1\zeta_1 + \omega_2\zeta_2}\right| \left|d\zeta_2\right| \left|d\zeta_1\right| \le \\ &\overset{(1.5)}{\le} \frac{k_\varepsilon}{4\pi^2} \left[\int_{e^{i\alpha_1}[0,+\infty)} e^{\left(A_1^+ + \varepsilon + Re\left(\omega_1 e^{i\alpha_1}\right)\right)\left|\zeta_1\right|} \left|d\zeta_1\right|\right] \cdot \\ &\cdot \left[\int_{e^{i\alpha_2}[0,+\infty)} e^{\left(A_2^+ + \varepsilon + Re\left(\omega_2 e^{i\alpha_2}\right)\right)\left|\zeta_2\right|} \left|d\zeta_2\right|\right] = \\ &\overset{(2.1)}{=} \frac{k_\varepsilon}{4\pi^2} \cdot \frac{-1}{A_1^+ + \varepsilon + Re\left(\omega e^{i\alpha_1}\right)} \cdot \frac{-1}{A_2^+ + \varepsilon + Re\left(\omega_2 e^{i\alpha_2}\right)} < +\infty. \end{aligned} \tag{2.2}$$

Due to the estimate (2.2), the first of four double integrals in (1.8) is absolutely convergent. And that double integral determines a function that is analytic in two complex variables on the set $\Omega_1^+ \times \Omega_2^+$. The same claims are true for the other three double integrals in the definition (1.8).

We now address the formal ambiguity in the definition (1.8) of function m, arising from the fact that some of the four domains that appear in (1.8) might intersect. Due to Lemma 4.1 in [25] (a lemma that uses the exponential estimate (1.1) on the function f and that is based on application of the Phragmen–Lindelöf maximum principle), the first and the second double integrals in (1.8) are equal on the intersection of their corresponding domains: $\Omega_1^+ \times \Omega_2^+$ and $\Omega_1^+ \times \Omega_2^-$. Due to the estimate (2.2), Fubini's theorem applies to each of the four double integrals in (1.8). By changing the order of integration in the first and third double integrals and referring Lemma 4.1 in [25], we

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see that the first and the third double integrals are equal on the intersection of their corresponding domains.

As for the first and the fourth double integrals, the intersection of their corresponding domains lies within the intersection of any two of the four domains. Hence in there, the first integral equals the second integral, and in turn, the second integral equals the fourth integral.

Consequently, any two of the four definitions of the function m in (1.8) are equivalent on the intersection of their corresponding domains. And, the function m defined by formulas (1.8) is analytic on $\Omega_1 \times \Omega_2$ in each of its variables.

Remark 2.1. Thanks to the estimate (2.2), the function m is bounded on any subset of $\Omega_1^+ \times \Omega_2^+$ that is bounded away from its boundary $\partial\left(\Omega_1^+ \times \Omega_2^+\right)$. Similarly, one can prove that the function m is bounded on any subset of $\Omega_1 \times \Omega_2$ that is bounded away from the boundary $\partial\left(\Omega_1 \times \Omega_2\right)$.

2.2 Verification of the sectorial Fourier inversion formula

Due to the condition (1.1), for any $z_2 \in \Delta_{\alpha_2}$, the Fourier inversion formula (see [25, Theorem 1.2]) applies to the function $z_1 \to f\left(z_1, z_2\right)$: that is, we have

$$f(z_1, z_2) = \int_{\Gamma_1} g\left(\omega_1, z_2\right) e^{-\omega_1 z_1} d\omega_1, \quad \text{for } z_2 \in \Delta_{\alpha_2}, \tag{2.3}$$

where the curve Γ_1 is defined by (1.9), and the function $\omega_1 \to g\left(\omega_1, z_2\right)$ is well defined by the following formulas:

$$\begin{aligned} g\left(\omega_1, z_2\right) &= \frac{1}{2\pi i} \int_{e^{-i\alpha_1}[0,+\infty)} f\left(\zeta_1, z_2\right) e^{\omega_1\zeta_1} d\zeta_1, \quad \text{for } Re\left(\omega_1 e^{-i\alpha_1}\right) < -h_1, \\ g\left(\omega_1, z_2\right) &= \frac{1}{2\pi i} \int_{e^{i\alpha_1}[0,+\infty)} f\left(\zeta_1, z_2\right) e^{\omega_1\zeta_1} d\zeta_1, \quad \text{for } Re\left(\omega_1 e^{i\alpha_1}\right) < -h_1. \end{aligned} \tag{2.4}$$

We now derive the estimate (2.6) for function $g\left(\omega_1, z_2\right)$, that is uniform in its first variable ω_1 for $\omega_1 \in \Gamma_1$ and that depends exponentially on its second variable z_2.

Due to inequality (1.10) imposed on the real number p_1, we can pick $\varepsilon > 0$ such that

$$h_1 + \varepsilon + p_1 \cos\left(\alpha_1\right) < 0. \tag{2.5}$$

Let $\omega_1 \in \Gamma_1$. Then, due to parameterization (1.9), we have $\omega_1 = \gamma_1(t)$, $t \in \mathbb{R}$. Specifically, assume that $t \geq 0$, so that the second formula of the two formulas (2.4) holds for $g\left(\gamma_1(t), z_2\right)$. We estimate

$$\begin{aligned} \left|g\left(\gamma_1(t), z_2\right)\right| &\overset{(2.4)}{\leq} \frac{1}{2\pi} \int_{e^{i\alpha_1}[0,+\infty)} \left|f\left(\zeta_1, z_2\right)\right| \cdot \left|e^{\gamma_1(t)\zeta_1}\right| \left|d\zeta_1\right| \leq \\ &\overset{(1.1)}{\leq} \frac{1}{2\pi} \int_{e^{i\alpha_1}[0,+\infty)} k_\varepsilon e^{(h_1+\varepsilon)|\zeta_1| + (h_2+\varepsilon)|z_2|} \cdot e^{Re(\gamma_1(t)\zeta_1)} \left|d\zeta_1\right| = \\ &= \frac{k_\varepsilon e^{(h_2+\varepsilon)|z_2|}}{2\pi} \int_{e^{i\alpha_1}[0,+\infty)} e^{\left(h_1+\varepsilon+Re\left(\gamma_1(t)e^{i\alpha_1}\right)\right)|\zeta_1|} \left|d\zeta_1\right| = \end{aligned} \tag{2.6}$$

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$$\overset{(1.9)}{=} \frac{k_\varepsilon e^{(h_2+\varepsilon)|z_2|}}{2\pi} \int_{e^{i\alpha_1}[0,+\infty)} e^{(h_1+\varepsilon+p_1\cos(\alpha_1))|\zeta_1|} |d\zeta_1| =$$
$$\overset{(2.5)}{=} \left[\frac{k_\varepsilon}{2\pi} \cdot \frac{-1}{h_1+\varepsilon+p_1\cos(\alpha_1)} \right] \cdot e^{(h_2+\varepsilon)|z_2|}.$$

And, we would get the same estimate on $g(\gamma_1(t), z_2)$ if we assumed that $t < 0$. Due to the estimate (2.6), for any $-\infty < t < +\infty$, the Fourier inversion formula (see [25, Theorem 1.2]) applies to the function $z_2 \to g(\gamma_1(t), z_2)$, and we have

$$g(\gamma_1(t), z_2) = \int_{\Gamma_2} m(\gamma_1(t), \omega_2) e^{-\omega_2 z_2} d\omega_2, \quad \text{for } z_2 \in \Delta_{\alpha_2}, \tag{2.7}$$

where the curve Γ_2 is parameterized by (1.11), and for any $\infty < t < +\infty$, the function $\omega_2 \to m(\gamma_1(t), \omega_2)$ is well defined by the formulas (1.8).

Remark 2.2. Note that due to the uniform bound (2.6), the curve Γ_2, involved in formula (2.7), is defined by formula (1.11) in such a way that it does not depend on the choice of $\omega_1 = \gamma_1(t)$.

By combining the Fourier inversion formulas (2.3) and (2.7), we obtain representation (1.13) of function f as two consecutive integrals. Due to (2.2), we estimate Fubini's theorem applies to the consecutive integrals in the representation (1.13) of function f, and we can consider those consecutive integrals as a double integral.

3 Estimates for two-dimensional indicator after Ivanov

3.1 A calculation relating to Figure 3

We now justify that the length $|[0, c]|$ of the interval $[0, c]$ constructed in Figure 3 indeed equals C_1. Indeed, we observe three triangles determined by their vertices $0, q_1, a;\ 0, q_1, b$, and $0, q_1, c$, to see that

$$\begin{aligned} Re\left(q_1 e^{i\alpha_1}\right) &= -|[0, a]|, \\ Re\left(q_1 e^{-i\alpha_1}\right) &= -|[0, b]|, \\ Re\left(q_1 e^{i\theta_1}\right) &= -|[0, c]|. \end{aligned} \tag{3.1}$$

Denote $x = Re(q_1), y = Im(q_1)$. Then we can paraphrase (3.1) as

$$\begin{aligned} x\cos(\alpha_1) - y\sin(\alpha_1) &= -A_1^+, \\ x\cos(\alpha_1) + y\sin(\alpha_1) &= -A_1^-, \\ x\cos(\theta_1) - y\sin(\theta_1) &= -|[0, c]|, \end{aligned}$$

so that

$$|[0, c]| = \frac{A_1^+}{2}\left[\frac{\sin(\theta_1)}{\sin(\alpha_1)} + \frac{\cos(\theta_1)}{\cos(\alpha_1)}\right] + \frac{A_1^-}{2}\left[\frac{\cos(\theta_1)}{\cos(\alpha_1)} - \frac{\sin(\theta_1)}{\sin(\alpha_1)}\right].$$

Consequently, due to (1.3), we have $|[0, c]| = C_1$.

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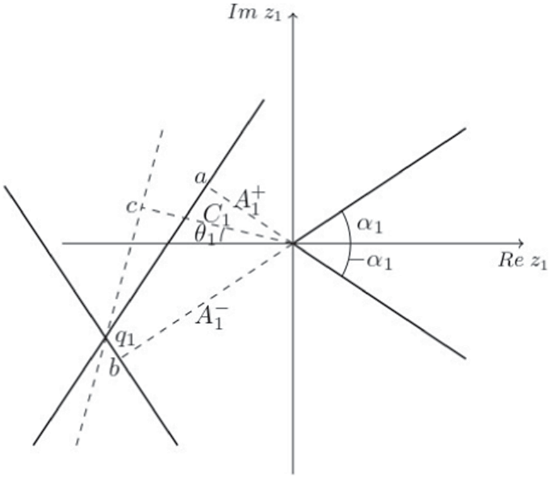

Figure 3: Construction of C_1.

3.2 An auxiliary estimate

We now estimate the integral

$$\int_{\Lambda_1} e^{-Re(\omega_1 z_1)} \, |d\omega_1|, \quad \text{for } \arg(z_1) = \theta_1, \tag{3.2}$$

where the curve Λ_1 is defined by Figure 4. Due to the choice (1.3) of the constant C_1 and parametrization (1.9) of the curve Γ_1, the curve Λ_1 is a union of three subcurves: the finite segment $\overline{\Lambda_1 \backslash \Gamma_1}$ and two infinite segments Λ_1^+ and Λ_1^- correspondingly parameterized by

$$\begin{aligned} &\gamma_1(t), \quad t \geq t_+ > 0, \\ &\gamma_1(t), \quad t \leq t_- < 0, \end{aligned} \tag{3.3}$$

where t_+ and t_- are determined by $s_1^- = \gamma_1(t_-)$, $s_1^+ = \gamma_1(t_+)$.

Step 1.

Note that due to the choice (1.3) of the constant C_1,

$$\begin{aligned} &Re\,(\omega_1 z_1) = Re\left(\omega_1 e^{i\theta_1}\right)|z_1| = -\left(C_1 + \delta\right)|z_1|, \\ &\text{for } \omega_1 \in \overline{\Lambda_1 \backslash \Gamma_1}, \ \arg(z_1) = \theta_1. \end{aligned} \tag{3.4}$$

And, consequently,

$$\int_{\Lambda_1 \backslash \Gamma_1} e^{-Re(\omega_1 z_1)} \, |d\omega| = e^{(C_1+\delta)|z_1|} \, |\Lambda_1 \backslash \Gamma_1|. \tag{3.5}$$

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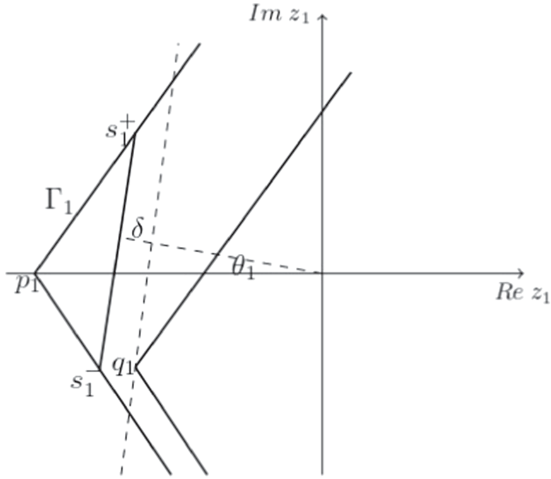

Figure 4: Construction of Λ.

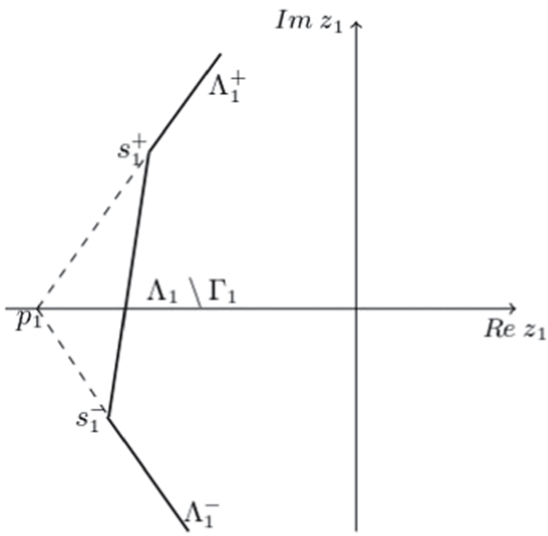

Figure 5: Construction of Λ_1.

Step 2.

We rewrite

$$\int_{\Lambda_1^+} e^{-Re(\omega_1 z_1)} |d\omega_1| \overset{(3.3),(1.9)}{=} \int_{t_+}^{+\infty} e^{-Re(\gamma_1(t)z_1)} dt. \tag{3.6}$$

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Note that

$$|\theta_1|<\alpha_1<\pi/2.$$

Consequently,

$$Re\left(ie^{-i\alpha_1}z_1\right)>0, \quad \text{for } \arg(z_1)=\theta_1,$$

so that the following integral is convergent and equals

$$\int_{t_+}^{+\infty} e^{-Re(\gamma_1(t)z_1)}dt = \frac{e^{-Re(\gamma_1(t_+)z_1)}}{Re\left(ie^{-i\alpha_1}z_1\right)}, \quad \text{for } \arg(z_1)=\theta_1. \tag{3.7}$$

Note that $\gamma_1(t_+)\in\overline{\Lambda_1\backslash\Gamma_1}$. Consequently, due to (3.4), we have

$$Re\left(\gamma_1(t_+)z_1\right) = -\left(C_1+\delta\right)|z_1|, \quad \text{for } \arg(z_1)=\theta_1. \tag{3.8}$$

By combining (3.6)–(3.8), we get

$$\int_{\Lambda_1^+} e^{-Re(\omega_1 z_1)}|d\omega_1| \le \frac{e^{(C_1+\delta)|z_1|}}{Re\left(ie^{-i\alpha_1}z_1\right)}, \quad \text{for } \arg(z_1)=\theta_1. \tag{3.9}$$

Similarly,

$$\int_{\Lambda_1^-} e^{-Re(\omega_1 z_1)}|d\omega_1| \le \frac{e^{(C_1+\delta)|z_1|}}{Re\left(-ie^{i\alpha_1}z_1\right)}, \quad \text{for } \arg(z_1)=\theta_1. \tag{3.10}$$

Step 3.

By combining the estimates (3.5),(3.9), and (3.10), we get

$$\int_{\Lambda_1} e^{-Re(\omega_1 z_1)}|d\omega_1| \le k_{\alpha_1,\theta_1}\frac{e^{(C_1+\delta)|z_1|}}{|z_1|}, \quad \text{for } \arg(z_1)=\theta_1, \tag{3.11}$$

where the number k_{α_1,θ_1}, while depending on α and θ, does not depend on $|z|$. We remark that the estimate (3.11) that we have obtained is similar to the one in Lemma 3.1 in [25].

3.3 Proof of trigonometric convexity

As the function m is analytic on the set $\Omega_1\times\Omega_2$, by consecutive applications of the Cauchy integral theorem in variables z_1 and z_2, we can rewrite (1.13) as

$$f(z_1,z_2) = \int_{\Lambda_1}\int_{\Lambda_2} m(\omega_1,\omega_2)\, e^{-\omega_1 z_1-\omega_2 z_2}d\omega_2 d\omega_1, \tag{3.12}$$

where the curve Λ_1 is constructed by Figure 4. Due to the construction of the curve Λ_1, it is bounded away from the boundary $\partial\Omega_1$. Similarly, the curve Λ_2 is bounded away from the boundary $\partial\Omega_2$. Consequently, due to Remark 2.1, the function m is uniformly bounded (by some constant $k_\delta(m)$) on the Cartesian product $\Lambda_1\times\Lambda_2$. We estimate

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$$|f(z_1,z_2)| \overset{(1.13)}{\leq} \int_{\Lambda_1}\int_{\Lambda_2} |m(\omega_1,\omega_2)|\, e^{-Re(\omega_1 z_1)-Re(\omega_2 z_2)}\,|d\omega_2|\,|d\omega_1| \leq$$
$$\overset{(2.3)}{\leq} k_\delta(m)\left[\int_{\Lambda_1} e^{-Re(\omega_1 z_1)}\,|d\omega_1|\right]\cdot\left[\int_{\Lambda_2} e^{-Re(\omega_2 z_2)}\,|d\omega_2|\right] \leq$$
$$\overset{(3.11)}{\leq} k_\delta(m)k_{\alpha_1,\theta_1}k_{\alpha_2,\theta_2}\frac{e^{(C_1+\delta)|z_1|}}{|z_1|}\frac{e^{(C_2+\delta)|z_2|}}{|z_2|},$$
$$\text{for } \arg(z_1)=\theta_1,\ \arg(z_2)=\theta_2. \tag{3.13}$$

Additionally, due to the estimate (1.5), the function f is bounded in: the intersection of $\Omega_1 \times \Omega_2$, and a vicinity of 0. By combining this fact with the estimate (3.13), we obtain the following estimate on the function f:

$$|f(z_1,z_2)| \leq k_{\delta,\theta_1,\theta_2,\alpha_1,\alpha_2} e^{(C_1+\delta)|z_1|+(C_2+\delta)|z_2|},$$
$$\text{for } \arg(z_1)=\theta_1,\ \arg(z_2)=\theta_2,$$

where the constant $k_{\delta,\theta_1,\theta_2,\alpha_1,\alpha_2}$, while depending on $\delta, \theta_1, \theta_2, \alpha_1, \alpha_2$, does not depend on $|z_1|$ or $|z_2|$.

4 Proof of sharpness

Consider the entire function $f(z_1,z_2) = e^{z_1+z_2}$. The function f is of exponential type,

$$|f(z_1,z_2)| \leq e^{|z_1|\cos\theta_1+|z_2|\cos\theta_2}, \quad \text{for } z_1,z_2 \in \mathbb{C}.$$

For our choice of the function f, the set $T_f(\theta_1,\theta_2)$ defined by (1.2) equals

$$T_f(\theta_1,\theta_2) = \{(\nu_1,\nu_2) : \nu_1 \geq \cos\theta_1, \nu_2 \geq \cos\theta_2\} \tag{4.1}$$

(see Figure 6). In particular, for $\alpha_1 = \alpha_2 = \pm\frac{\pi}{4}$, we have

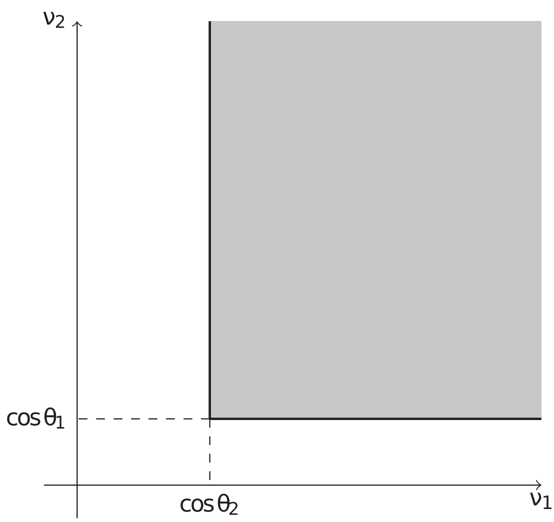

Figure 6: $T_f(\theta_1,\theta_2)$.

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$$T_f\left(\pm\frac{\pi}{4},\pm\frac{\pi}{4}\right)=\left\{(v_1,v_2)\colon v_1\geq\frac{\sqrt{2}}{2},v_2\geq\frac{\sqrt{2}}{2}\right\}.$$

Thus,

$$A_1^+=A_2^+=A_1^-=A_2^-=\frac{\sqrt{2}}{2}.$$

Take $\theta_1=\theta_2=0$. By (1.3), we evaluate

$$C_1=C_2=\frac{\sqrt{2}}{4}\frac{2}{\sqrt{2}}+\frac{\sqrt{2}}{4}\frac{2}{\sqrt{2}}=1.$$

Thus, according to Theorem 1.1, the following inequality holds:

$$f(z_1,z_2)\leq k_\varepsilon e^{(1+\varepsilon)|z_1|+(1+\varepsilon)|z_2|},\quad \text{for } \arg z_1=\arg z_2=0.$$

On the other hand, by (4.1),

$$T_f(0,0)=\{(v_1,v_2):v_1\geq 1,v_2\geq 1\}.$$

That is, in this case, Theorem 1.1 is sharp.

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References

[1] N. Arakelian, *On efficient analytic continuation of power series*. Math. Sb. **52**(1985), no. 1, 21–39. https://doi.org/10.1070/SM1985v052n01ABEH002875
[2] N. Arakelian, W. Luh, and J. Muller, *On the localization of singularities of lacunar power series*. Complex Var. Elliptic Equ. **52**(2007), no. 7, 561–573. https://doi.org/10.1080/17476930701246396
[3] L. Bieberbach, *Analytische fortsetzun*, Springer, Berlin, 1955. https://doi.org/10.1007/978-3-662-01270-3
[4] R. Boas, *Entire functions*, Academic Press, New York, 1954.
[5] F. Carlson, *Sur une classe de series de Taylor*. Diss. Upsala, 1914.
[6] F. Carlson, *Uber ganzwertige Funktionen*. Math. Z. **11**(1921), 1–23.
[7] M. Dzhrbashyan and A. Avetisyan, *Integral representation of a certain class of functions analytic in an angular domain*. Sib. Math. J. **1**(1960), no. 3, 383–426.
[8] S. Friot and D. Greynat, *On convergent series representations of Mellin–Barnes integrals*. J. Math. Phys. **53**(2012), 023508. https://doi.org/10.1063/1.3679686
[9] A. Iosevich and E. Liflyand, *Decay of the Fourier transform. Analytic and geometric aspects*, Birkhäuser/Springer, Basel, 2014. https://doi.org/10.1007/978-3-0348-0625-1
[10] V. Ivanov, *A characterization of the growth of an entire function of two variables and its application to the summation of double power series*. Mat. Sb. (N.S.) **47** (**89**)(1959), no. 1, 3–16.
[11] C. Kiselman, *On entire functions of exponential type and indicators of analytic functional*. Acta Math. **117**(1967), 1–35. https://doi.org/10.1007/BF02395038
[12] A. Kondratyuk, *The Fourier series method for entire and meromorphic functions of completely regular growth*. Sb. Math. **35**(1979), no. 1, 63–84. https://doi.org/10.1070/SM1984v048n02ABEH002677
[13] P. Lelong and L. Gruman, *Entire functions of several complex variables*, Springer, Berlin, 1986. https://doi.org/10.1007/978-3-642-70344-7
[14] A. Leont'ev, *Entire functions*, Exponential Series, Nauka, Moscow, 1983.

[15] B. Levin, *Distribution of zeros of entire functions*, American Mathematical Society, Providence, RI, 1964. https://doi.org/10.1090/mmono/005
[16] K. Malyutin and N. Sadik, *The indicator of a delta-subharmonic function in a half-plane*. Ufa Math. J. 3(2011), no. 4, 84–91.
[17] A. Mkrtchyan, *Continuation of multiple power series in a sectorial domain*. Math. Nachr. **292**(2019), no. 11, 2441–2451. https://doi.org/10.1002/mana.201900042
[18] A. Mkrtchyan, *Continuability of multiple power series into sectorial domain by means of interpolation of coefficients*. Ufa Math. J. **14**(2022), no. 2, 1–8. https://doi.org/10.13108/2022-14-2-108
[19] M. Passare, D. Pochekutov, and A. Tsikh, *Amoebas of complex hypersurfaces in statistical thermodynamics*. Math. Phys. Anal. Geom. **16**(2013), no. 3, 89–108. https://doi.org/10.1007/s11040-012-9122-x
[20] M. Passare, A. Tsikh, and A. Yger, *Residue currents of the Bochner–Martinelli type*. Publ. Mat. **44**(2000), no. 1, 85–117.
[21] F. Phragmen and E. Lindelöf, *Sur une extension d'un principe classique de l'analyseet sur quelques proprietes des fonctions monogenes dans le voisinage d'un point singulier*. Acta Math. **31**(1908), 381–406.
[22] G. Pólya, *Untersuchungen über Lücken und Singularitäten von Potenzreihen*. Math. Z. **29**(1929), 549–640. https://doi.org/10.1007/BF01180553
[23] L. Ronkin, *Introduction to the theory of entire functions of several variables*, Translations of Mathematical Monographs, 44, American Mathematical Society, Providence, RI, 1974. https://doi.org/10.1090/mmono/044
[24] T. Sadykov and A. Tsikh, *Hypergeometric and algebraic functions in several variables*, Nauka, Moscow, 2014. https://doi.org/10.1090/mmono/044
[25] A. Vagharshakyan, *Sectorial Paley–Wiener theorem*. Math. Methods Appl. Sci. **46**(2023), 19173–19183. https://doi.org/10.1002/mma.9618
[26] A. Vidras and A. Yger, *On some generalizations of Jacobi's residue formula*. Ann. Sci. Éc. Norm. Supér. (4) **34**(2001), no. 1, 131–157. https://doi.org/10.1016/s0012-9593(00)01056-9
[27] V. Zorich, *Mathematical analysis of problems in the natural science*, Springer, Berlin–Heidelberg, 2011. https://doi.org/10.1007/978-3-642-14813-2_1

Siberian Federal University, Krasnoyarsk, Russia, Institute of Mathematics NAS, Yerevan, Armenia and Saint Petersburg University, Saint Petersburg, Russia
e-mail: AMkrtchyan@sfu-kras.ru

Institute of Mathematics NAS, Yerevan, Armenia and Yerevan State University, Yerevan, Armenia
e-mail: avaghars@kent.edu