LINEAR FUNCTIONALS ON HOMOGENEOUS POLYNOMIALS

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The space H_m of homogeneous polynomials in n real variables $x_1, x_2, \ldots, x_n$ of degree m may be considered as an inner product space with inner product $(p, q) = \int_{|x|=1} p(x)q(x)\, ds(x)$, $p, q \in H_m$; where ds is the rotation-invariant measure on $S^{n-1} = \{x \in R^n : |x| = 1\}$, $\int_{S^{n-1}} ds = 1$. The problem solved in this paper is the following: given a linear functional ϕ on H_m, find $P\phi \in H_m$ so that $\phi(p) = (p, P\phi)$ for all $p \in H_m$.

Let $K_i = \{p \in H_i : \Delta p = 0\}$, $i = 0, 1, 2, \ldots$ ($\Delta \equiv$ Laplacian). Any $p \in H_m$ has a unique expansion (the Laplace series)

$p(x) = \Sigma_{i=0}^{[m/2]} |x|^{2i} \tilde{p}_{m-2i}(x)$, $\tilde{p}_r \in K_r$, ($[t]$ = largest integer $\leq t$).
It is known that $q \in K_i$, $r \in K_j$, $i \neq j$ implies $(q, r) = 0$. Let α be a multi-index $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, α_i are non-negative integers, $|\alpha| = \Sigma_{i=1}^n \alpha_i$, $\alpha! = \alpha_1!\alpha_2! \ldots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, $\binom{|\alpha|}{\alpha} = (|\alpha|!)/\alpha!$ the multinomial coefficient. $p \in H_m$ can be written as $p(x) = \Sigma_{|\alpha|=m} A_\alpha x^\alpha$; the A_α's are real coefficients. A basis for the dual space of H_m is given by the set $\{\phi_\alpha : |\alpha| = m\}$, where

$$\phi_\alpha(\Sigma_{|\beta|=m} A_\beta x^\beta) = A_\alpha / \binom{m}{\alpha}$$

(the inclusion of the multinomial coefficient is motivated by its occurrence in Taylor's formula for several variables).

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THEOREM. Let α be a multi-index with $|\alpha| = m$; let $q_\alpha(x) = x^\alpha$.

Then $P\phi_\alpha(x) = \Sigma_{i=0}^{[m/2]} \frac{2^m i!\ \Gamma(m-i+n/2)}{m!\Gamma(n/2)} (q_\alpha)^\sim_{m-2i}(x)|x|^{2i}$, that is

$\phi_\alpha(p) = (p, P\phi_\alpha)$ for all $p \in H_m$.

The method of proof depends on the theory of spherical convolution, from which the following facts are needed (see [1]).

i) $P_n^\lambda(t)$ is the ultraspherical polynomial of degree n, index λ, normalized by $P_n^\lambda(1) = 1$, given by

$$(1-2rt + r^2)^{-\lambda} = \sum_{n=0}^{\infty} \frac{\Gamma(2\lambda+n)}{n!\,\Gamma(2\lambda)} P_n^\lambda(t) r^n .$$

ii) If g is measurable on $[-1, 1]$ and

$$||g||_1 = a_\lambda \int_{-1}^{1} |g(t)| \ (1-t^2)^{\lambda-1/2} dt < \infty,$$

where

$$a_\lambda \int_{-1}^{1} (1-t^2)^{\lambda-1/2} dt = 1,$$

then g has the ultraspherical expansion

$$g \sim \sum_{n=0}^{\infty} \left(\frac{n}{\lambda} + 1\right) \frac{\Gamma(2\lambda+n)}{n!\,\Gamma(2\lambda)} \hat{g}_n P_n^\lambda$$

where

$$\hat{g}_n = a_\lambda \int_{-1}^{1} g(t) P_n^\lambda(t) \ (1-t^2)^{\lambda-1/2} dt.$$

Note $|\hat{g}_n| \le ||g||_1$.

(iii) $g(t) = t^k$, $k = 0, 1, 2, \ldots$ has the expansion

$$t^k = \frac{\Gamma(\lambda+1)k!}{2^k} \sum_{i=0}^{[k/2]} \frac{(k-2i+\lambda)\ \Gamma(2\lambda+k-2i)}{\lambda\ \Gamma(k-i+\lambda+1)i!\ \ (k-2i)!\,\Gamma(2\lambda)} P_{k-2i}^\lambda(t)$$

(this formula is derived in Hua's book [2, p. 141]).

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iv) If $F \in L^1(S^{n-1}, ds)$ then F has a Laplace series $F \sim \sum_{m=0}^{\infty} \tilde{F}_m$, $\tilde{F}_m \in K_m$.

v) If F and g are as in iv) and ii) respectively, $\lambda = n/2 - 1$, $|x| = 1$, then

$$g*F(x) = \int_{S^{n-1}} F(y)g(x\cdot y)ds(y)$$

exists and is finite almost everywhere $[ds]$, $g_* F \in L^1(S^{n-1})$, and the Laplace series of $g_* F \sim \sum_{m=0}^{\infty} \hat{g}_m \tilde{F}_m$. (Note: $x\cdot y$ is the inner product).

Proof of Theorem. The given inner product is positive definite, since $(p,p) = 0$, implies $p(x) = 0$ all x on S^{n-1}, so $p \equiv 0$; hence given ϕ_α, $P\phi_\alpha$ exists in H_m (finite dimensional version of Riesz theorem). Let $g(t) = t^m$, then for $|x| = 1$,

$$g*P\phi_\alpha(x) = \int_{S^{n-1}} P\phi_\alpha(y)\,(x\cdot y)^m\, ds(y)$$

$$= \int_{S^{n-1}} P\phi_\alpha(y) \sum_{|\beta|=m} \binom{m}{\beta} x^\beta y^\beta\, ds(y)$$

(the multinomial theorem)

$$= x^\alpha = q_\alpha(x) \text{ by definition of } P\phi_\alpha .$$

By iii) and v)

$$\sum_{i=0}^{[m/2]} (q_\alpha)^\sim_{m-2i}(x) = \sum_{i=0}^{[m/2]} \frac{m!\,\Gamma(n/2)}{2^m i!\,\Gamma(m-i+n/2)} (P\phi_\alpha)^\sim_{m-2i}(x) \text{ for } |x| = 1.$$

K_i and K_j are orthogonal for $i \neq j$ so

$$(P\phi_\alpha)^\sim_{m-2i}(x) = \frac{2^m i!\,\Gamma(m-i+n/2)}{m!\,\Gamma(n/2)} (q_\alpha)^\sim_{m-2i}(x), \qquad i = 0, 1, \ldots, [m/2].$$

COROLLARY: Let $h_m(t) =$

$$\sum_{i=0}^{[m/2]} \left(\frac{2m-4i}{n-2} + 1\right) \frac{(n+m-2i-3)!}{(m-2i)!\,(n-3)!} \frac{2^m i!\,\Gamma(m-i+n/2)}{m!\,\Gamma(n/2)} P^{n/2-1}_{m-2i}(t)$$

then $P\phi_\alpha(x) = h_m * q_\alpha(x)$ for $|x| = 1$.

REFERENCES

1. C.F. Dunkl, Operators and harmonic analysis on the sphere. Trans. A.M.S. 125 (1966) 250-263.

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