Bull. Aust. Math. Soc. **108** (2023), 459–463
doi:10.1017/S0004972723000059

REVERSED HARDY–LITTLEWOOD–PÓLYA INEQUALITIES WITH FINITE TERMS

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(Received 2 November 2022; accepted 23 December 2022; first published online 3 February 2023)

Abstract

We prove a reversed Hardy–Littlewood–Pólya inequality with finite terms. We also give the limit of the best constant.

2020 *Mathematics subject classification*: primary 26D15; secondary 40B05.

Keywords and phrases: reversed Hardy–Littlewood–Pólya inequality, best constant, estimates of the upper/lower bound.

1. Introduction

Let $a_i, b_i \geq 0$ $(i = 1, 2, \ldots)$. The Hardy–Littlewood–Pólya inequality [3, Theorem 381, page 288] states that

$$\sum_{i,j=1,\, i\neq j}^{\infty} \frac{a_i b_j}{|i-j|^{\lambda}} \leq K_{p,q}\Big(\sum_{i=1}^{\infty} a_i^p\Big)^{1/p}\Big(\sum_{i=1}^{\infty} b_i^q\Big)^{1/q}, \tag{1.1}$$

where $p, q > 1$, $1/p + 1/q > 1$, $\lambda = 2 - (1/p + 1/q)$. In 2015, Huang, Li and Yin used the Hardy–Littlewood–Sobolev inequality [7] to generalise (1.1) to the case of higher dimensions. In addition, they also proved that the best constant can be approximated by the corresponding functional with finite terms [4].

In 2015, Dou and Zhu in [2] established a reversed Hardy–Littlewood–Sobolev inequality:

$$\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)\,dx\,dy}{|x-y|^{\lambda}}\Big| \geq C\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)} \quad \text{for all } f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n),$$

where $n \geq 1$ and $p, q \in (n/(n-\lambda), 1)$ satisfy $1/p + 1/q + \lambda/n = 2$. In addition, they proved the best constant is attained. From this inequality, a reversed discrete inequality

This research was supported by NSFC (No. 11871278) of China.

in higher dimensions was also deduced in [5]:

$$\sum_{i,j\in\mathbb{Z}^n} \frac{|f_i||g_j|}{|i-j|^\lambda} + \sum_{j\in\mathbb{Z}^n} |f_j||g_j| \ge C\|f\|_{l^p}\|g\|_{l^q} \quad \text{for all } (f,g)\in l^p(\mathbb{Z}^n)\times l^q(\mathbb{Z}^n),$$

where $f=(f_i)_{i\in\mathbb{Z}^n}$, $g=(g_j)_{j\in\mathbb{Z}^n}$, $\lambda<0$, $n/(n-\lambda)<p,q<1$ and $1/p+1/q+\lambda/n\le 2$. When $n=1$ and replacing $\mathbb{Z}^n$ by $\mathbb{N}$, we denote the best constant by

$$L_{p,q,\lambda} := \inf\Big\{\sum_{i,j=1}^{\infty}\frac{|f_i||g_j|}{|i-j|^\lambda} + \sum_{j=1}^{\infty}|f_j||g_j| : \|f\|_{l^p} = \|g\|_{l^q} = 1\Big\}. \tag{1.2}$$

In 2011, Li and Villavert [6] proved the Hardy–Littlewood–Pólya inequality with finite terms:

$$\sum_{i,j=1,\,i\ne j}^{N}\frac{a_ib_j}{|i-j|} \le K_N\Big(\sum_{i=1}^N a_i^2\Big)^{1/2}\Big(\sum_{i=1}^N b_i^2\Big)^{1/2}, \tag{1.3}$$

where the constant K_N satisfies

$$2\ln N - 2 \le K_N \le 2(\ln N - \ln 2) + 2.$$

Comparing (1.3) with (1.1) for $p=q=2$ shows that (1.3) is an inequality in the critical case. In contrast to the estimate of K_N above, the best constant for the upper-critical inequality is bounded with respect to N [4, Lemma 2.2]. The bounds for the best constant are helpful in giving a better understanding of the Coulomb energy in the Thomas–Fermi model describing electron gas and N-body systems [8]. The results in higher dimensions can be found in [1].

In this paper, we always assume $a_i, b_i \ge 0$ $(i=1,2,\dots,N)$. We will prove the following reversed Hardy–Littlewood–Pólya inequality with finite terms.

THEOREM 1.1. *Let $\lambda<0$ and $p,q\in(0,1)$ satisfy $1/p+1/q\le 2-\lambda$. Then we can find a constant $L>0$ which only depends on p,q,λ,N such that*

$$\sum_{i,j=1,\,i\ne j}^{N}\frac{a_ib_j}{|i-j|^\lambda} + \sum_{i=1}^N a_ib_i \ge L\Big(\sum_{i=1}^N a_i^p\Big)^{1/p}\Big(\sum_{i=1}^N b_i^q\Big)^{1/q}. \tag{1.4}$$

Denote the best constant in (1.4) by

$$L_{p,q,\lambda,N} := \min\Big\{\sum_{i,j=1,\,i\ne j}^{N}\frac{a_ib_j}{|i-j|^\lambda} + \sum_{i=1}^N a_ib_i : \sum_{i=1}^N a_i^p = \sum_{i=1}^N b_i^q = 1\Big\}. \tag{1.5}$$

THEOREM 1.2. *Let $\lambda<0$ and $p,q\in((1-\lambda)^{-1},1)$ satisfy $1/p+1/q\le 2-\lambda$. Then $L_{p,q,\lambda,N}\to L_{p,q,\lambda}$ when $N\to\infty$.*

https://doi.org/10.1017/S0004972723000059 Published online by Cambridge University Press

2. Proof of Theorems 1.1 and 1.2

PROOF OF THEOREM 1.1. Write $a = (a_1, a_2, \ldots, a_N)$ and $b = (b_1, b_2, \ldots, b_N)$. Set

$$J(a,b) = \sum_{i,j=1,\, i\neq j}^{N} \frac{a_i b_j}{|i-j|^{\lambda}} + \sum_{i=1}^{N} a_i b_i - L_{p,q,\lambda,N}\Big(\sum_{i=1}^{N} a_i^p\Big)^{1/p}\Big(\sum_{i=1}^{N} b_i^q\Big)^{1/q}.$$

Clearly, $J(a,b) \geq 0$ for all $a, b \in \mathbb{R}_+^N := \{x = (x_1, x_2, \ldots, x_N) : x_i \geq 0, i = 1, 2, \ldots, N\}$. However,

$$\mathbb{S}(N) := \Big\{(a,b) : \sum_{i=1}^{N} a_i^p = \sum_{i=1}^{N} b_i^q = 1\Big\}$$

is compact in $\mathbb{R}_+^N \times \mathbb{R}_+^N$ and hence the minimisation problem (1.5) has solutions in $\mathbb{S}(N)$. Thus, we can find $(a(N), b(N)) \in \mathbb{S}(N)$ such that $J(a(N), b(N)) = 0$. We call $(a(N), b(N))$ the minimiser of J. Therefore, both the partial derivatives of J are equal to zero at $(a(N), b(N))$. Namely,

$$\Big[\frac{d}{dt}J(a(N)+ta, b(N))\Big]_{t=0} = \Big[\frac{d}{dt}J(a(N), b(N)+tb)\Big]_{t=0} = 0$$

for any $(a,b) \in \mathbb{R}_+^N \times \mathbb{R}_+^N$. From this result, by simple calculation, we see that

$$\begin{cases} L_{p,q,\lambda,N}\ a(N)_i^{p-1} = \displaystyle\sum_{j=1}^{N} \frac{b(N)_j}{|i-j|^{\lambda}} + b(N)_i \\ L_{p,q,\lambda,N}\ b(N)_i^{q-1} = \displaystyle\sum_{j=1}^{N} \frac{a(N)_j}{|i-j|^{\lambda}} + a(N)_i. \end{cases} \tag{2.1}$$

Noting $(a(N), b(N)) \neq (0,0)$ (because $(a(N), b(N)) \in \mathbb{S}(N)$), from (2.1) we see that

$$\min\{a(N)_i, b(N)_i\} > 0 \quad \text{for } 1 \leq i \leq N.$$

Therefore, $L_{p,q,\lambda,N} > 0$.

Next, we prove that $L_{p,q,\lambda,N}$ has a positive lower bound which is independent of N. Multiplying $(2.1)_1$ by $a(N)_i$ and summing from 1 to N gives

$$L_{p,q,\lambda,N}\sum_{i=1}^{N} a(N)_i^p = \sum_{i,j=1}^{N} \frac{a(N)_i b(N)_j}{|i-j|^{\lambda}} + \sum_{i=1}^{N} a(N)_i b(N)_i. \tag{2.2}$$

Write

$$\begin{cases} \bar{a}(N) = (a(N)_1, a(N)_2, \ldots, a(N)_N, 0, \ldots), \\ \bar{b}(N) = (b(N)_1, b(N)_2, \ldots, b(N)_N, 0, \ldots). \end{cases}$$

Since $(a(N), b(N)) \in \mathbb{S}(N)$,

$$(\bar{a}(N), \bar{a}(N)) \in \mathbb{S} := \{(a,b) : \|a\|_{l^p} = \|b\|_{l^q} = 1\}.$$

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From (2.2) and (1.2), it follows that

$$L_{p,q,\lambda,N} = L_{p,q,\lambda,N} \sum_{i=1}^{N} a(N)_i^p = \sum_{i,j=1}^{\infty} \frac{\bar{a}(N)_i \bar{b}(N)_j}{|i-j|^\lambda} + \sum_{i=1}^{\infty} \bar{a}(N)_i \bar{b}(N)_i \geq L_{p,q,\lambda}. \tag{2.3}$$

Therefore, $L_{p,q,\lambda,N} > 0$ with the lower bound (2.3). This proves (1.4). □

REMARK 2.1. We claim that

$$\lim_{N\to\infty} \min_{1\leq i\leq N} \{a(N)_i, b(N)_i\} = 0.$$

Without loss of generality, we can assume $a(N)_1 = \min_{1\leq i\leq N}\{a(N)_i, b(N)_i\}$. From (2.1),

$$\begin{aligned} L_{p,q,\lambda,N} &= a(N)_1^{1-p}\Big(\sum_{j=2}^{N} \frac{b(N)_j}{(j-1)^\lambda} + b(N)_i\Big) \\ &\geq a(N)_1^{2-p}\Big(\sum_{j=2}^{N} \frac{1}{(j-1)^\lambda} + 1\Big) = a(N)_1^{2-p}\Big(\sum_{j=1}^{N-1} j^{-\lambda} + 1\Big) \\ &\geq a(N)_1^{2-p}\Big(\int_1^N (r-1)^{-\lambda}\,dr + 1\Big) = a(N)_1^{2-p}\Big(\frac{(N-1)^{1-\lambda}}{1-\lambda} + 1\Big). \end{aligned} \tag{2.4}$$

However, since $\mathbb{S}(N) \subset \mathbb{S}(N+1)$, it follows that $L_{p,q,\lambda,N}$ is nonincreasing with respect to N. Therefore, $L_{p,q,\lambda,N} \leq L_{p,q,\lambda,1}$. Taking $a_1 = b_1 = 1$, we see that $(a_1, b_1) \in \mathbb{S}(1)$ and hence $L_{p,q,\lambda,1} \leq a_1 b_1 = 1$. This gives the upper bound

$$L_{p,q,\lambda,N} \leq 1. \tag{2.5}$$

Combining (2.5) with (2.4) yields

$$a(N)_1^{2-p} = O(N^{\lambda-1}) \quad (N \to \infty).$$

This implies our claim.

PROOF OF THEOREM 1.2. By (1.2), we can find a minimising sequence $(a^{(m)}, b^{(m)}) \in \mathbb{S}$ such that

$$\sum_{i,j=1,\, i\neq j}^{\infty} \frac{a_i^{(m)} b_j^{(m)}}{|i-j|^\lambda} + \sum_{i=1}^{\infty} a_i^{(m)} b_i^{(m)} \leq L_{p,q,\lambda} + \frac{1}{m}.$$

The convergence of this series implies

$$\sum_{i,j=1,\, i\neq j}^{\infty} \frac{a_i^{(m),N_m} b_j^{(m),N_m}}{|i-j|^\lambda} + \sum_{i=1}^{\infty} a_i^{(m),N_m} b_i^{(m),N_m} \leq L_{p,q,\lambda} + \frac{2}{m} \tag{2.6}$$

when $N_m > m$ is sufficiently large. Here,

$$\begin{cases} a_i^{(m),N_m} = a_i^{(m)} & \text{when } i \leq N_m, \\ a_i^{(m),N_m} = 0 & \text{when } i > N_m, \end{cases}$$

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and $b_i^{(m),N_m}$ is defined by the same truncation. Since $(a^{(m)}, b^{(m)}) \in \mathbb{S}$,

$$\|a^{(m),N_m}\|_{l^p}^p \geq 1 - \frac{1}{m}, \quad \|b^{(m),N_m}\|_{l^q}^q \geq 1 - \frac{1}{m}, \tag{2.7}$$

when $N_m > m$ is sufficiently large. Therefore, noting that

$$\Big(\frac{a^{(m),N_m}}{\|a^{(m),N_m}\|_{l^p}}, \frac{b^{(m),N_m}}{\|b^{(m),N_m}\|_{l^q}}\Big) \in \mathbb{S}(N_m),$$

from (1.5), (2.6) and (2.7), we deduce

$$L_{p,q,\lambda,N_m} \leq \Big(L_{p,q,\lambda} + \frac{2}{m}\Big)\Big(1 - \frac{1}{m}\Big)^{-(1/p+1/q)}$$

for large N_m. Letting $m \to \infty$ and combining with (2.3) completes the proof. □

Acknowledgement

The authors thank the anonymous referee very much for the useful suggestions.

References

[1] Z. Cheng and C. Li, 'An extended discrete Hardy–Littlewood–Sobolev inequality', *Discrete Contin. Dyn. Syst.* **34** (2014), 1951–1959.

[2] J. Dou and M. Zhu, 'Reversed Hardy–Littlewood–Sobolev inequality', *Int. Math. Res. Not. IMRN* **2015**(19) (2015), 9696–9726.

[3] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd edn (Cambridge University Press, Cambridge, 1952).

[4] G. Huang, C. Li and X. Yin, 'Existence of the maximizing pair for the discrete Hardy–Littlewood–Sobolev inequality', *Discrete Contin. Dyn. Syst.* **35** (2015), 935–942.

[5] Y. Lei, Y. Li and T. Tang, 'Critical conditions and asymptotics for discrete systems of the Hardy–Littlewood–Sobolev type', *Tohoku Math. J.*, to appear.

[6] C. Li and J. Villavert, 'An extension of the Hardy–Littlewood–Pólya inequality', *Acta Math. Sci.* **31**(6) (2011), 2285–2288.

[7] E. Lieb, 'Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities', *Ann. of Math. (2)* **118** (1983), 349–374.

[8] E. Lieb, 'Coherent states as a tool for obtaining rigorous bounds', in: *Coherent States, Past, Present and Future: Proceedings of the International Symposium, Oak Ridge, 1993* (eds. D. H. Feng, J. R. Klauder and M. R. Strayer) (World Scientific, Singapore, 1994), 267–278.

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https://doi.org/10.1017/S0004972723000059 Published online by Cambridge University Press