SOME POSSIBLE AND SOME IMPOSSIBLE TRIPARTITIONS OF THE PLANE

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For each positive integer n it is possible to partition the Euclidean plane into n (disjoint) congruent connected sets [1], but if $n > 2$, it is impossible to partition the plane into n congruent continuumwise connected sets such that some one of the sets can be translated onto another one [2]. This paper is concerned with the possibility of partitioning the plane into three congruent sets without any topological restrictions whatever.

Definition. Case (r_1, r_2) is a partitioning of the plane into three sets A_1, A_2, A_3 such that a rotation through $2\pi r_j$ carries A_j onto A_{j+1} if $j \in \{1, 2\}$. A rotation through 0 is interpreted as a translation.

Remark 1. If case (r_1, r_2) is possible, and is realised by the ordered triple (A_1, A_2, A_3) of sets, then the following cases are also possible (as are all cases obtainable by repeated application of these rules):

(a) (r_1+m, r_2+n) (where m and n are integers), realised by (A_1, A_2, A_3);

(b) $(r_1+r_2, -r_2)$, realised by the permutation (A_1, A_3, A_2);

(c) $(-r_2, -r_1)$, realised by the permutation (A_3, A_2, A_1);

(d) (r_2, r_1), realised by $(\bar{A}_3, \bar{A}_2, \bar{A}_1)$, where the bar denotes reflection with respect to some fixed line; and

(e) $(-r_1, -r_2)$, realised by $(\bar{A}_1, \bar{A}_2, \bar{A}_3)$.

THEOREM 1. Case $(0, 0)$ is possible.

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Proof. If $j \in \{1, 2, 3\}$, let A_j be $\{(x, y) : [x] \equiv j \pmod 3\}$. (The brackets denote the greatest-integer function.) In (1) an example is given in which the three sets are connected but not arcwise connected.

THEOREM 2. Case (0, 1/2) is possible.

Proof. If $j \in \{1, 2, 3\}$, let A_j be the set of all (x, y) in the plane such that either x is an integer congruent to $j - 1$ modulo 3, or x is a non-integer such that $[x] \equiv j \pmod 3$. Then a reflection through the point $(3/2, 0)$ carries A_2 onto A_3.

This example also illustrates case (0, 0), and is in fact a special case of the following theorem.

THEOREM 3. If case (r, 1/2) is possible, then so is case (r, 0).

Proof. Let a coordinate system be chosen so that (with complex number notation for the plane)

$$z \in A_2 \Longleftrightarrow -z \in A_3 . \tag{1}$$

Then

$$z \in A_1 \Longleftrightarrow -z \in A_1 , \tag{2}$$

since both other possibilities contradict (1). Thus A_1 is symmetric about the origin. Hence A_3 is symmetric about some point α. But then A_2 is taken onto A_3 by the translation (through 2α) composed of a reflection through the origin followed by a reflection through α. Thus the same partitioning serves for case (r, 0).

THEOREM 4. Case (1/4, 0) is impossible.

Proof. The steps in the proof are indicated schematically in Figure 1. A coordinate system is chosen so that

$$z \in A_1 \Longleftrightarrow iz \in A_2 \quad \text{and} \tag{3}$$

$$z \in A_2 \Longleftrightarrow z + 1 \in A_3 . \tag{4}$$

The horizontal and vertical dashed lines are the coordinate axes. Points in the lattice of Gaussian integers which are needed in the proof of impossibility are indicated by dots or symbols j_m, the latter

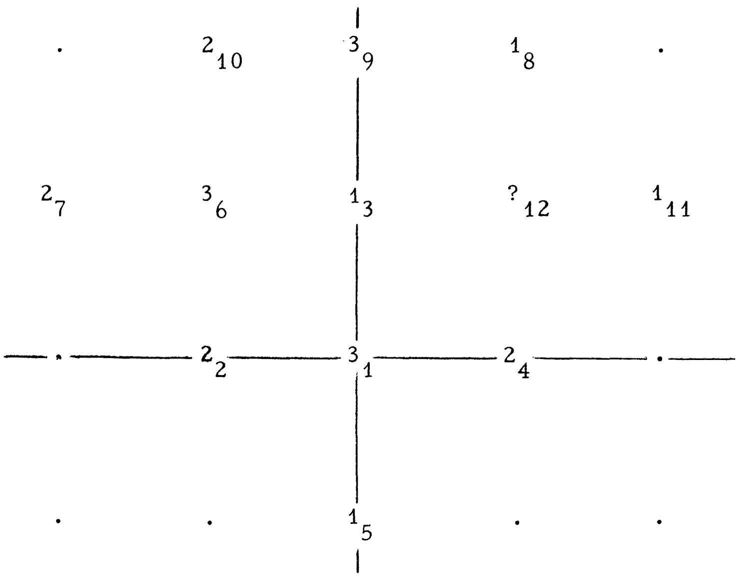

Figure 1. Case (1/4, 0)

denoting the m^{th} point whose assignment to a set can be determined from (3) and (4), where that set is A_j if $j \varepsilon \{1, 2, 3,\}$, and is no set if j is ?. Thus, the origin is marked 3_1, since it must belong to A_3 (a centre of rotation from A_1 to A_2 can't belong to either A_1 or A_2), and is the first point assignable to a set. The assignment of points numbered 1, 4, 6, 9, and 12 is by exclusion of all other possibilities, because of conflicts with previously assigned points. The presence of the question mark shows that the proposed partitioning is impossible.

Remark 2. Case (1/4, 1/2) is impossible, by Theorems 3 and 4. It is easy to see, using Remark 1, that all cases involving angles which are integral multiples of $\pi/2$ can be obtained from those treated in Theorems 1 through 4.

THEOREM 5. Case (1/3, r) is impossible if r is rational.

Proof. Let c be exp $(2\pi i/3)$ and let a be exp $(2\pi ir)$. Then $c^3 = 1$ and $c^2 + c + 1 = 0$, and for some complex number b we have (after choice of a coordinate system)

(5) $$z \varepsilon A_1 \longleftrightarrow cz \varepsilon A_2 \quad \text{and}$$

(6) $$z \varepsilon A_2 \longleftrightarrow az + b \varepsilon A_3 ,$$

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or, equivalently,

(7) $$z \in A_3 \iff (z - b)/a \in A_2 .$$

An indirect proof using (5) shows that

(8) $$z \in A_2 \rightarrow cz \in A_3$$

(one way implication only!). Use of (8) and (7) yields

(9) $$z \in A_2 \rightarrow (cz - b)/a \in A_2 .$$

From (5) it follows that

(10) $$0 \in A_3 .$$

Hence, by (7),

(11) $$-b/a \in A_2 .$$

Repeated application of (9), beginning with (11), yields

(12) $$\sum_{j=0}^{n-1} (c/a)^j(-b/a) \in A_2$$

for every positive integer n. Now, by hypothesis, a and c are roots of unity. Hence also c/a is a root of unity. If $a \neq c$, then for any positive integer n such that $(c/a)^n = 1$, the sum in (12) is 0, which contradicts (10). However, if $a = c$ (i.e., in the case (1/3, 1/3)), then let

(13) $$d \text{ be } bc/(c - 1) .$$

It follows from (13), (5), and (6), after a little computation, that

$$d \in A_1 \iff cd \in A_2 \iff d \in A_3 ,$$

which is impossible unless $d \in A_2$. Then (6) and (9) yield

$$cd + b \in A_3 \text{ and } (cd - b)/c \in A_2 ,$$

respectively. But this is impossible, since

$$cd + b = b(c^2 + c - 1)/(c - 1) = (cd - b)/c .$$

THEOREM 6. *Case* (1/6, 0) *is impossible*.

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Proof. Let c be $\exp(\pi i/3)$. A coordinate system is chosen so that

$$(14) \qquad z \in A_1 \longleftrightarrow cz \in A_2 \quad \text{and}$$

$$(15) \qquad z \in A_2 \longleftrightarrow z + 1 \in A_3 .$$

Let d be the centroid $(c+1)/3$ of the triangle with vertices 0, 1, and c. The proof of impossibility, like that of Theorem 3, is shown schematically, in Figures 2 and 3. The points indicated are those of the regular hexagonal tesselation of the plane in which the centre of one hexagon is at the origin and one of its vertices is at d. If d is assumed to belong to A_1, then Figure 2 is used; points numbered 4, 7, 9, 12, 13, 15, and 18 are assigned by indirect argument. If d is assumed to belong to A_2, then Figure 3 is used; points numbered 3, 6, and 8 are assigned by indirect argument. If d is assumed to belong to A_3, then set

$$B_1 = \{z : c - z \in A_1\} ,$$

$$B_2 = \{z : c - z \in A_3\} , \quad \text{and}$$

$$B_3 = \{z : c - z \in A_2\} .$$

(Note the permuted subscripts!) Then easy calculations show that B_1, B_2, and B_3 form a partitioning of the plane satisfying (14) and (15) (with "B" replacing "A" throughout), and that $d \in B_1$. Figure 2 shows that such sets B_1, B_2, and B_3 cannot exist; thus A_1, A_2, and A_3 cannot exist.

Remark 3. All cases in which the rotations are integral multiples of $\pi/3$ can be obtained from those treated in Theorems 1, 2, 3, 5, and 6, by use of Remark 1. For example, if case (1/6, 1/6), were possible, then so would be case (1/3, -1/6), which contradicts Theorem 5.

Remark 4. All constructions used in this paper made essential use of regular tesselations of the plane. What can be said about the possibility of other cases (r_1, r_2), for example if both r_1 and r_2 are rational? What can be said about partitions into four or more sets?

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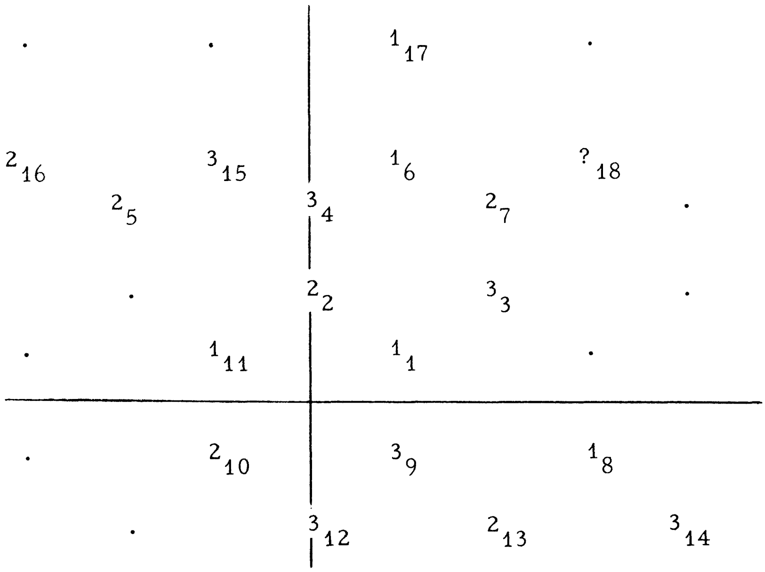

Figure 2. Case (1/6, 0) if $d \in A_1$.

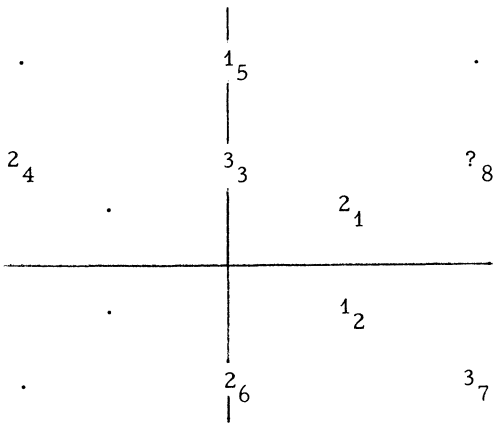

Figure 3. Case (1/6, 0) if $d \in A_2$.

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Added in proof: N.K. Krier has pointed out that Figure 1 can be simplified by omitting the points with subscripts 5, 6, 7, and 8, labelling the point (2, 0) as 3_5, and decreasing subscripts higher than 8 by 3.

REFERENCES

1. Problem E1515, The Amer. Math. Monthly, 69 (1962) 312. Solution, ibid. 70 (1963) 95-96.

2. Leroy F. Meyers, Partition of the plane into finitely many isometric continuumwise connected sets. Bull. Acad. Polon. Sci., sér. sci. math., astr. et phys., 13 (1965) 533-535.

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