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Topological stability for homeomorphisms with global attractor

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Abstract. We prove that every topologically stable homeomorphism with global attractor of $\mathbb{R}^n$ is topologically stable on its global attractor. The converse is not true. On the other hand, if a homeomorphism with global attractor of a locally compact metric space is expansive and has the shadowing property, then it is topologically stable. This extends the Walters stability theorem (Walters, *On the pseudo-orbit tracing property and its relationship to stability. The structure of attractors in dynamical systems*, 1978, pp. 231–244).

1 Introduction

Global attractors hold a pivotal position in the realm of differential equations due to their prevalence in various significant equations, as emphasized in [16]. They often play the role of stabilizing factors, demonstrating resilience even in the face of minor perturbations, as gauged through the Hausdorff distance [14, 16]. Nevertheless, this stability is not an inherent trait across the board. A case in point is the ordinary differential equation (ODE) depicted in Chapter 3 of [14], expressed as

$$\dot{x} = (1-x)x^2 \qquad (\forall x \in \mathbb{R}). \tag{1.1}$$

Within this ODE, the interval $[0,1]$ stands as a global attractor. However, when subjected to perturbation, as illustrated by the modified equation:

$$\dot{x} = (1-x)(x^2+\varepsilon) \qquad (\forall x \in \mathbb{R}), \tag{1.2}$$

the global attractor dwindles to the singleton set $\{1\}$.

Another compelling illustration lies in the homeomorphism $g : \mathbb{R} \to \mathbb{R}$, defined as

$$g(x) = (1+e^{-x^2})x \qquad (\forall x \in \mathbb{R}). \tag{1.3}$$

This homeomorphism features the global repelling fixed point at 0. Consequently, its inverse, $f = g^{-1} : \mathbb{R} \to \mathbb{R}$, boasts $\{0\}$ as a global attractor. However, a nuanced perspective emerges when considering $f(0) = 0$ and the limit as x tends to infinity:

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$\lim_{x\to\infty}(f(x) - x) = 0$. This revelation opens the door for tailored perturbations, where by selecting a sufficiently large positive value, denoted as M, a new fixed point emerges at $x = M$, with the interval $[0, M]$ assuming the role of the global attractor.

These intriguing phenomena are often characterized as "implosions" (as seen in the example (1.1) and its perturbation) and "explosions" (as witnessed in example (1.2) and its perturbation). Notably, explosions have been the subject of in-depth scrutiny with regard to both the nonwandering set and the chain recurrent set, as elucidated in [19] and [4], respectively. Recent research endeavors have delved into explosions concerning the generalized recurrent set, bearing relevance to the field of Lyapunov functions. These investigations and their broader implications are well-documented in [3] and the referenced literature therein.

These examples motivate the search of necessary and sufficient conditions for the stability of the global attractor under small perturbations. The contemporary paradigm of stability in dynamical systems is anchored in the realm of topology, a concept that has evolved to address the fundamental aspects of system behavior. This innovative perspective on stability was first introduced by Walters, as documented in his seminal work [23], with the intent of assuming the role traditionally held by structural stability in topological dynamics [1]. It is worth noting that a many dynamical systems, especially those inhabiting compact manifolds such as Morse-Smale and Anosov diffeomorphisms, or the broader class of Axiom A diffeomorphisms that adhere to the strong transversality condition, are inherently topologically stable [13]. Moreover, an interesting observation in the realm of expansive homeomorphisms within compact metric spaces is that they also exhibit topological stability, a phenomenon commonly referred to as the "Walters stability theorem" [22]. The pursuit of extending this remarkable theorem to the challenging noncompact domain has spurred endeavors in multiple directions. Researchers have embarked on ambitious endeavors to bridge the gap between compact and noncompact settings, with notable contributions documented in a body of literature that includes works such as [5, 11, 12, 18]. These exploratory undertakings strive to generalize the Walters stability theorem and unearth the underlying principles governing the stability of dynamical systems in an increasingly diverse and complex landscape.

Within the scope of this paper, we explore the intricate domain of global attractor stability in the presence of slight perturbations. To begin, we establish that when a homeomorphism possesses a global attractor in the Euclidean space $\mathbb{R}^n$ and exhibits topological stability, the global attractor remains steadfast in the face of these perturbations. This resilience extends even further, as we observe that the homeomorphism, in this context, is not only globally stable but also retains its topological stability specifically on its global attractor. However, it is crucial to note that the converse of this assertion does not hold, as we shall elucidate. In a distinct line of inquiry, when a homeomorphism boasts a global attractor within the realm of a locally compact metric space and also exhibits expansiveness coupled with the shadowing property, it invariably qualifies as topologically stable. Our research endeavors culminate in a precise statement of these results, underlining the intricate interplay between global attractors, topological stability, expansiveness, and shadowing properties.

Consider a metric space X. Denote by id_X the identity map of X. Define the C^0-distance between maps $l, r : X \to X$ by

$$d_{C^0}(l, r) = \sup_{x \in X} d(l(x), r(x)).$$

This distance satisfies all requirements of a metric except that it may take infinite value. It is then referred to as an ∞-*metric* (p. 1014 in [6]).

The following is the classical definition of topologically stable homeomorphisms [22].

Definition 1.1 A homeomorphism of a metric space $f : X \to X$ is *topologically stable* if, for every $\varepsilon > 0$, there is $\delta > 0$ such that for any homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \le \delta$, there is a continuous $h : X \to X$ such that $d_{C^0}(h, \mathrm{id}_X) \le \varepsilon$ and $f \circ h = h \circ g$.

The second main definition is the following one. Following [16], given subsets of a metric space $A, B \subset X$ we define

$$\mathrm{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b).$$

Now, we remind the following the standard definition [16].

Definition 1.2 A *global attractor* of a continuous map $f : X \to X$ is a nonempty compact subset $\mathbb{A} \subset X$ which is *invariant* (i.e., $f(\mathbb{A}) = \mathbb{A}$) and *attracts bounded sets* namely

$$\mathrm{dist}(f^i(B), \mathbb{A}) \to 0 \quad \text{as} \quad i \to \infty, \quad \text{for all bounded } B \subset X.$$

It is natural to compare this definition with the concept of *Conley attractor*: a compact invariant set $\mathcal{A} \subset X$ exhibiting a closed neighborhood U such that

$$\mathcal{A} = \bigcap_{n \ge 0} f^n(U).$$

It is imperative to establish a clear distinction between global attractors and Conley attractors, as they occupy distinct positions in the realm of dynamical systems. It is evident that every global attractor automatically qualifies as a Conley attractor, given their inherent relationship. However, the inverse does not hold, primarily due to the semilocal nature of Conley attractors. See, for instance, the homeomorphisms in Figure 1 where the points p and q are Conley attractors, and the segment from p to q passing through the saddle point s is the global attractor. Notice that this homeomorphism is topologically stable.

In stark contrast, the global attractor, if it exists, is a unique entity that comprehensively encompasses all Conley attractors within its domain. This uniqueness is easily demonstrated through a straightforward argument: Should there be another candidate global attractor denoted as $\mathbb{A}'$, the distance between $\mathbb{A}'$ and the original global attractor $\mathbb{A}$ is shown to approach zero, as expressed by $\mathrm{dist}(f^i(\mathbb{A}'), \mathbb{A}) \to 0$. Consequently, this leads to the conclusion that $\mathrm{dist}(\mathbb{A}', \mathbb{A})$ is also zero, implying that

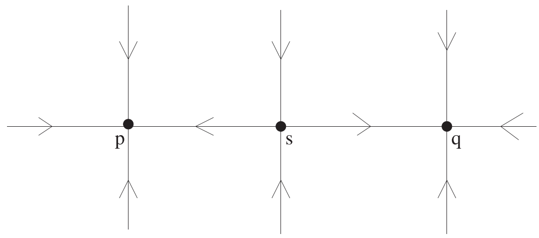

Figure 1: Comparing the global and Conley's attractors.

$\mathbb{A}'$ is a subset of $\mathbb{A}$. By virtue of symmetry, the roles of $\mathbb{A}$ and $\mathbb{A}'$ can be reversed, leading to the conclusion that $\mathbb{A}$ and $\mathbb{A}'$ are, in fact, one and the same: $\mathbb{A} = \mathbb{A}'$.

In the context of homeomorphisms with global attractors, it is essential to note that they encompass all homeomorphisms of compact metric spaces, denoted as $f : X \to X$, given that the entire space X inherently satisfies the prerequisites specified in the corresponding definition of a global attractor, namely, it is compact, exhibits invariance, and effectively attracts all bounded sets. However, it is crucial to underscore that many such homeomorphisms do not exhibit Conley attractors, with the identity transformation serving as a prime example.

The realm of homeomorphisms with global attractors extends to a diverse array of scenarios, including contractions on complete metric spaces, the time-1 map of the equation (1.1), the inverse of the homeomorphism (1.2), the Lorenz equation, solenoid diffeomorphisms within $\mathbb{R}^3$, the illustrative example presented in (1.3), and an array of other instances. In the case of contractions, the global attractor is given by the Banach contracting principle whereas the solenoid is the well-known example of a chaotic hyperbolic attractor suggested by Smale in his seminal paper [20]. For a more detailed exploration of the dynamics and characteristics of such homeomorphisms, one can refer to the extensive insights offered in [16] or consult the comprehensive analysis in [17].

Given a metric space X, we define the *Hausdorff distance* between subsets $A, B \subset X$ by

$$D(A,B) = \max\{\operatorname{dist}(A,B), \operatorname{dist}(B,A)\}.$$

The following alternate notion of stability is related to the *A-stability* (p. 25 in [8]) or Definition 1.1 (p. 4 in [10]). We denote $i_A : A \to X$ the inclusion map.

Definition 1.3 A homeomorphism with global attractor of a metric space $f : X \to X$ is *topologically stable on its global attractor* $\mathbb{A}$ if, for every $\varepsilon > 0$, there is $\delta > 0$ such that for any homeomorphism $g : X \to X$ with $d_{C^0}(f,g) \le \delta$, there are $\mathbb{A}_g \subset X$ compact and a continuous map $h_g : \mathbb{A}_g \to \mathbb{A}$ (often called *topological semiconjugacy*) such that:

- $\mathbb{A}_g$ is a global attractor of g;
- $D(\mathbb{A}, \mathbb{A}_g) < \varepsilon$;
- $d_{C^0}(h_g, i_{\mathbb{A}_g}) < \varepsilon$ and $f|_{\mathbb{A}} \circ h_g = h_g \circ g|_{\mathbb{A}_g}$.

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The Hausdorff distance $D(\mathbb{A}, \mathbb{A}_g)$ in the above definition is used to quantify the extent of implosion or explosion experienced by the global attractor when subjected to minor perturbations (this is not possible with $\text{dist}(\mathbb{A}, \mathbb{A}_g)$ only). This measure provides a crucial gauge for assessing the dynamic response of the global attractor in the face of external influences, enabling us to discern the degree of stability or vulnerability exhibited in the system.

On the other hand, not every homeomorphism with a global attractor is topologically stable on its global attractor. A simple counterexample is the time one map of (1.1).

With these definitions, we can state the following result.

Theorem 1.4 *Every topologically stable homeomorphism with global attractor of an Euclidean space $\mathbb{R}^n$ is topologically stable on its global attractor.*

We do not know whether, conversely, the topological stability on the global attractor implies the topological stability of the homeomorphism. At first glance, we could believe that a counterexample is the inverse $f : \mathbb{R} \to \mathbb{R}$ of the homeomorphism $g : \mathbb{R} \to \mathbb{R}$ described in (1.3). However, this f is not topologically stable on its global attractor $\{0\}$, because, arbitrarily small perturbation of f may have $[0, M]$ as a global attractor for some $M > 1$ and then $D(0, [0, M]) = M > 1$. We thank the anonymous referee who pointed this out to us.

An anonymous referee raised an inquiry regarding the potential extension of the theorem presented above from $\mathbb{R}^n$ to noncompact manifolds. It appears that the affirmative holds, and we intend to explore this matter in our upcoming investigations.

Next, we give sufficient conditions for the topological stability of a given homeomorphism with global attractor. Recall that a homeomorphism $f : X \to X$ is *expansive* if there is $e > 0$ such that if $x, y \in X$ and $d(f^n(x), f^n(y)) \le e$ for every $n \in \mathbb{Z}$, then $x = y$. This concept is due to Utz [21]. Given $\delta > 0$, a bi-infinite sequence $(x_i)_{i\in\mathbb{Z}}$ is called *δ-pseudo orbit* if $d(f(x_i), x_{i+1}) \le \delta$ for all $i \in \mathbb{Z}$. We say that the sequence *can beδ-shadowed* if there is $x \in X$ such that $d(f^i(x), x_i) \le \delta$ for every $i \in \mathbb{Z}$. A homeomorphism $f : X \to X$ has the *shadowing property* if, for every $\varepsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit can be ε-shadowed.

The previously mentioned Walters stability theorem (Theorem 4, p. 236 in [22]) asserts that every expansive homeomorphism with the shadowing property of a compact metric space is topologically stable. We would like to extend this theorem to homeomorphisms with global attractors on metric spaces. To do that we need an extra hypothesis on the space. Recall that a metric space is *locally compact* if every point has a compact neighborhood.

Theorem 1.5 *Let $f : X \to X$ be a homeomorphism with global attractor of a locally compact metric space. If f is expansive and has the shadowing property, then f is topologically stable.*

This result can be applied to Examples 2.3 and 2.5. The paper is organized as follows: In Section 2, we give some preparatory lemmas. In Section 3, we use these lemmas to prove the theorems.

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2 Preliminary lemmas

To prove Theorem 1.4, we will use the following lemma. It was extracted from Theorem 1.8 (p. 39 in [9]; and seemed well known). Denote by $B[x,r]$, the closed r-ball centered at x.

Lemma 2.1 *For every $n \in \mathbb{N}$ and every compact $\mathbb{A} \subset \mathbb{R}^n$, there is $\rho > 0$ such that if $h : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $d_{C^0}(h, \mathrm{id}_{\mathbb{R}^n}) < \rho$, then $\mathbb{A} \subset h(\mathbb{R}^n)$.*

Proof We can assume that $\mathbb{A} = B[0,\gamma]$ for some $\gamma > 0$. Denote $B = B[0, 2\gamma]$ and ∂B the boundary of B. Choose $\rho > 0$ small enough such that for every $z \in B[0,\gamma]$, every $x \in \partial B$ and $y \in \mathbb{R}^n$ with $\|x - y\| < \rho$ it is true that the line traced from z to y intersects ∂B at some point u with $\|u - x\| < 4\gamma$. Suppose by contradiction that there is $h : \mathbb{R}^n \to \mathbb{R}^n$ continuous such that

$$d_{C^0}(h, \mathrm{id}_{\mathbb{R}^n}) < \rho \quad \text{and} \quad B[0,\gamma] \notin h(\mathbb{R}^n).$$

Choose $z \in B[0,\gamma] \setminus h(\mathbb{R}^n)$. Define $H : B \to \partial B$ by $H(x) = u$, where u is as above with $y = h(x)$. Then, H is continuous and because $\|h(x) - x\| < 4\gamma$ $(\forall x \in \partial B)$ we also have $H(x) \neq -x$ for every $x \in \partial B$. From this, we can construct a homotopy from $H|_{\partial B} : \partial B \to \partial B$ to $\mathrm{id}_{\partial B}$ through the minimal circle arc in ∂B from $H(x)$ to x. This would imply that ∂B is a retract of B which is impossible by the Brouwer fixed point theorem. ∎

To prove Theorem 1.5, we need two lemmas. The first is a generalization of Walters's stability theorem to certain metric spaces including the compact ones. To motivate the definition, we recall that a metric space X is *proper* when every closed ball of X is compact (see Gromov [7]). Every compact metric space is proper but not conversely. Noncompact examples are the closed subset of the Euclidean space $\mathbb{R}^n$, image of those sets under bi-Lipschitz maps, complete Riemannian manifolds, complete Finsler manifolds, complete sub-Riemannian or sub-Finsler manifolds or finite products of these spaces.

Actually, we will consider a little more general metric spaces defined as follows. A metric space X is *uniformly locally compact* if there is $0 < \beta < \infty$ such that every closed β-ball of X is compact. These spaces were first considered by A. Weil in his study of uniform spaces [24].

Lemma 2.2 *Every expansive homeomorphism with the shadowing property of a uniformly locally compact metric space is topologically stable.*

Proof Let $f : X \to X$ be an expansive homeomorphism with the shadowing property of a uniformly locally compact metric space X. Let β be the positive number given by the uniform local compactness of X. Let e be an expansivity constant of f. Fix $\varepsilon > 0$, $0 < \varepsilon' < \frac{1}{8}\min\{\varepsilon, e, \beta\}$ and take $\delta > 0$ from the shadowing property of f for this ε'.

Let $g : X \to X$ be a homeomorphism with $d_{C^0}(f, g) < \delta$. Given $x \in X$, we have that

$$d(f(g^i(x)), g^{i+1}(x)) = d(f(g^i(x)), g(g^i(x))) \le d_{C^0}(f, g) < \delta, \qquad \forall i \in \mathbb{Z}.$$

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Then, $\{g^i(x)\}_{i\in\mathbb{Z}}$ is a δ-pseudo orbit and so, by the shadowing property, there is $y \in X$ such that

$$d(f^i(y), g^i(x)) \le \varepsilon', \qquad \forall i \in \mathbb{Z}.$$

We have that there is only one y satisfying the above inequalities (for the given x). Indeed, if there were another $y' \in X$ satisfying

$$d(f^i(y'), g^i(x)) \le \varepsilon' \qquad \forall i \in \mathbb{Z},$$

we would have

$$d(f^i(y), f^i(y')) \le 2\varepsilon' < e, \qquad \forall i \in \mathbb{Z}.$$

Since e is an expansivity constant, $y = y'$ proving the assertion.

Then, by making $y = h(x)$, we obtain a map $h : X \to X$ satisfying

$$d(f^i(h(x)), g^i(x)) \le \varepsilon', \qquad \forall i \in \mathbb{Z}. \tag{2.1}$$

By replacing $i = 0$ above, we obtain $d(h(x), x) \le \varepsilon' < \varepsilon$ for all $x \in X$, hence

$$d_{C^0}(h, \mathrm{id}_X) \le \varepsilon' \quad \text{and so} \quad d_{C^0}(h, \mathrm{id}_X) < \varepsilon. \tag{2.2}$$

Also, by replacing i by $i + 1$ and x by $g(x)$ in (2.1), we obtain, respectively, that

$$d(f^i(f(h(x))), g^{i+1}(x)) \le \varepsilon' \quad \text{and} \quad d(f^i(h(g(x))), g^{i+1}(x)) \le \varepsilon',$$

$\forall i \in \mathbb{Z}$. Then, the triangle inequality implies

$$d(f^i(f(h(x))), f^i(h(g(x)))) \le 2\varepsilon' < e, \qquad \forall i \in \mathbb{Z}.$$

Since e is an expansivity constant, we conclude that $f(h(x)) = h(g(x))$ for all $x \in X$ proving

$$f \circ h = h \circ g.$$

It remains to prove that h is continuous. The argument is similar to one used in [2]. By contradiction, suppose that it is not. Then, there is a convergent sequence $x_n \to x$ such that $h(x_n) \not\to h(x)$ as $n \to \infty$. Then, up to passing to a subsequence if necessary, we can assume that there is $\Delta > 0$ such that

$$d(h(x_n), h(x)) \ge \Delta, \qquad \forall n \in \mathbb{N}. \tag{2.3}$$

Next, since $x_n \to x$, up to discarding some finite set of this sequence, we can assume

$$d(x_n, x) \le \frac{\beta}{4}, \qquad \forall n \in \mathbb{N}.$$

Since

$$\begin{aligned} d(h(x_n), h(x)) &\le d(h(x_n), x_n) + d(x_n, x) + d(h(x), x) \\ &\overset{(2.2)}{\le} 2\varepsilon' + \sup_{n\in\mathbb{N}} d(x_n, x) \\ &< \frac{\beta}{4} + \frac{\beta}{4} = \frac{\beta}{2}, \qquad \forall n \in \mathbb{N}, \end{aligned}$$

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we conclude that $h(x_n)$ is contained in the closed β-ball centered at $h(x)$ for every $n \in \mathbb{N}$. From this, we have that $\{h(x_n)\}_{n\in\mathbb{N}}$ has a convergent subsequence. For simplicity, we assume that the sequence itself is convergent, namely $h(x_n) \to z$ for some $z \in X$.

Now, (2.1) implies

$$d(f^i(h(x_n)), g^i(x_n)) \leq \varepsilon', \qquad \forall i \in \mathbb{Z},\ n \in \mathbb{N}.$$

Then, since f and g are continuous, by fixing i and letting $n \to \infty$ above, we obtain

$$d(f^i(z), g^i(x)) \leq \varepsilon', \qquad \forall i \in \mathbb{Z}.$$

By (2.1) once more, we have

$$d(f^i(h(x)), g^i(x)) \leq \varepsilon', \qquad \forall i \in \mathbb{Z},$$

hence

$$d(f^i(z), f^i(h(x))) \leq 2\varepsilon' < e, \qquad \forall i \in \mathbb{Z}.$$

Since e is an expansivity constant, we obtain $h(x) = z$. However, by letting $n \to \infty$ in (2.3), we get $d(z, h(x)) \geq \Delta > 0$ that's absurd. This contradiction proves that h is continuous and finishes the proof. ∎

This lemma can be applied to the following examples.

Example 2.3 Any expansive homeomorphism with the shadowing property of a compact metric space (more precisely, under the conditions of Walters stability theorem).

Example 2.4 Any hyperbolic linear homeomorphism of $\mathbb{R}^n$ (or any other finite dimensional Banach space).

In particular, all such operators are topologically stable (an outline of the proof of this fact was done by Robbin [15]).

Example 2.5 The family of homeomorphisms $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + x^{3n}$, $\forall x \in \mathbb{R}$ and $n \in \mathbb{N}$.

The examples in this family are expansive with the shadowing property but not uniformly expansive. Then, we deduce their topological stability not from [22] (or [12]) but from Lemma 2.2.

Now, recall that a continuous map of a metric space $f : X \to X$ is *dissipative* if it possesses a compact absorbing set C; that is, for any bounded $B \subset X$ there is $n_0(B) \in \mathbb{N} \cup \{0\}$ such that

$$f^n(B) \subset C, \qquad \forall n \geq n_0(B).$$

This is the discrete version of the corresponding definition for semigroups (Definition 10.2, p. 264 in [16]).

Lemma 2.6 *Let X be a metric space. If there is a dissipative homeomorphism $f : X \to X$, then X is proper and so uniformly locally compact.*

Proof Let C be the compact absorbing set. Since any closed ball B is bounded one has $f^n(B) \subset C$ for some n. Since f is a homeomorphism, $f^n(B)$ is closed and so compact since it is contained in C. Since f is a homeomorphism, $B = f^{-n}(f^n(B))$ is also compact proving the result. ■

The corollary below follows from Lemmas 2.2 and 2.6.

Corollary 2.7 *Every dissipative expansive homeomorphism with the shadowing property is topologically stable.*

3 Proof of the theorems

First, we prove Theorem 1.4.

Proof Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a topologically stable homeomorphism with global attractor. Let $\rho > 0$ be given by Lemma 2.1 for $\mathbb{A}$. Fix $0 < \varepsilon < \rho$, and let δ be given by the topological stability of f for this ε. Take a homeomorphism $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $d_{C^0}(f, g) < \delta$. Then, the topological stability of f provides $h : \mathbb{R}^n \to \mathbb{R}^n$ continuous such that $d_{C^0}(h, \mathrm{id}_{\mathbb{R}^n}) < \varepsilon$ and $f \circ h = h \circ g$. By Lemma 2.1, we have that $\mathbb{A} \subset h(\mathbb{R}^n)$. In particular, one has $\mathbb{A}_g = h^{-1}(\mathbb{A}) \neq \emptyset$. Let us prove that this set satisfies the required properties.

Since h is continuous, $\mathbb{A}_g$ is closed. Also for all $y, y' \in \mathbb{A}_g$, one has $h(y) = x$ and $h(y') = x'$ for some $x, x' \in \mathbb{A}$. So,

$$d(y, y') \le d(y, h(y)) + d(x, x') + d(h(y'), y') \le 2\varepsilon + \mathrm{diam}(\mathbb{A}),$$

where $\mathrm{diam}(C) = \sup\{d(c, c') : c, c' \in C\}$ is the diameter of C, and $d(p, q) = \|p - q\|$ is the Euclidean distance. Since $\mathbb{A}$ is compact, $\mathrm{diam}(\mathbb{A}) < \infty$ hence $\mathrm{diam}(\mathbb{A}_g) < \infty$ and so $\mathbb{A}_g$ is bounded. Therefore, $\mathbb{A}_g$ is compact.

Also if $y \in \mathbb{A}_g$ (and then $h(y) = x \in \mathbb{A}$ for some $x \in \mathbb{A}$), one has $h(g(y)) = f(h(y)) = f(x) \in \mathbb{A}$ so $g(y) \in \mathbb{A}_g$, thus $g(\mathbb{A}_g) \subset \mathbb{A}_g$. Likewise, $h(g^{-1}(y)) = f^{-1}(h(y)) = f^{-1}(x) \in \mathbb{A}$, thus $y \in g(\mathbb{A}_g)$ proving $g(\mathbb{A}_g) = \mathbb{A}_g$ that is $\mathbb{A}_g$ is an invariant set of g.

Finally, we prove that $\mathbb{A}_g$ attracts bounded sets of $\mathbb{R}^n$ under g. It suffices to prove that it attracts compact sets $B \subset \mathbb{R}^n$. Suppose by contradiction that this is not true. Then, there is a compact $B \subset \mathbb{R}^n$ such that $\mathrm{dist}(g^i(B), h^{-1}(\mathbb{A})) \not\to 0$ as $i \to \infty$. Up to passing to a subsequence if necessary, we can assume that there are a sequence $b_i \in B$ and $\Delta > 0$ such that

$$\mathrm{dist}(g^i(b_i), h^{-1}(\mathbb{A})) \ge \Delta, \qquad \forall i \in \mathbb{N}. \tag{3.1}$$

However, $h(B)$ is compact (hence bounded) so

$$\mathrm{dist}(f^i(h(B)), \mathbb{A}) \to 0 \quad \text{as} \quad i \to \infty.$$

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Since $h(b_i) \in h(B)$, the above limit implies $\mathrm{dist}(f^i(h(b_i)), \mathbb{A}) \to 0$ as $i \to \infty$. In particular, $f^i(h(b_i))$ is bounded and so we can assume that $f^i(h(b_i)) \to w$ is convergent. Clearly, $w \in \mathbb{A}$. Since

$$\|g^i(b_i))\| \leq d(g^i(b_i), h(g^i(b_i))) + \|f^i(h(b_i))\|$$

for all i we have that $g^i(b_i)$ is also bounded. Therefore, we can assume that $g^i(b_i) \to z$ is convergent too. But then $h(g^i(b_i)) \to h(z)$ so $h(z) = w \in \mathbb{A}$, hence $z \in h^{-1}(\mathbb{A})$. Nevertheless, by letting $i \to \infty$ in (3.1), we get $\mathrm{dist}(z, h^{-1}(\mathbb{A})) \geq \Delta$ that's is absurd. Therefore, $\mathbb{A}_g$ attracts bounded subsets of $\mathbb{R}^n$ under g. Since $d_{C^0}(h, \mathrm{id}_{\mathbb{R}^n}) < \varepsilon$ and $\mathbb{A} \subset h(\mathbb{R}^n)$, we get $D(\mathbb{A}, \mathbb{A}_g) < \varepsilon$. Finally, by taking $h_g = h|_{\mathbb{A}_g}$, we get $h_g : \mathbb{A}_g \to \mathbb{A}$ such that $d_{C^0}(h_g, i_{\mathbb{A}_g}) \leq d_{C^0}(h, \mathrm{id}_{\mathbb{R}^n}) < \varepsilon$ and $f|_{\mathbb{A}} \circ h_g = h_g \circ g|_{\mathbb{A}_g}$ completing the proof. ∎

Finally, we prove Theorem 1.5.

Proof Let $f : X \to X$ be a homeomorphism with global attractor of a locally compact metric space. Suppose that f is expansive and has the shadowing property. Since X is locally compact and f has a global attractor, f is dissipative. Then, f is topologically stable by Corollary 2.7. ∎

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References

[1] A. Andromov and L. Pontrjagin, *Structurally stable systems*. Dokl. Akad. Nauk URSS **14**(1937), 247–251.
[2] A. Artigue, *Lipschitz perturbations of expansive systems*. Discrete Contin. Dyn. Syst. 35(2015), 1829–1841.
[3] O. Bernardi, A. Florio, and J. Wiseman, *The generalized recurrent set, explosions and Lyapunov functions*. J. Dynam. Differential Equations **32**(2020), no. 4, 1797–1817.
[4] L. Block and J. E. Franke, *The chain recurrent set, attractors, and explosions*. Ergodic Theory Dynam. Systems **5**(1985), no. 3, 321–327.
[5] G. Cousillas, J. Groisman, and J. Xavier, *Topologically Anosov plane homeomorphisms*. Topol. Methods Nonlinear Anal. **54**(2019), 371–382.
[6] J. Dydak and C. S. Hoffland, *An alternative definition of coarse structures*. Topology Appl. **155**(2008), 1013–1021.
[7] M. Gromov, *Groups of polynomial growth and expanding maps*. Inst. Hautes Études Sci. Publ. Math. **53**(1981), 53–73.
[8] J. Hale, *Asymptotic behavior of dissipative systems*, Mathematical Surveys and Monographs, 25, American Mathematical Society, Providence, RI, 1988.
[9] M. Hirsch, *Differential topology*, Graduate Text in Mathematics, 33, Springer, New York, 1976.
[10] J. Lee and N. Nguyen, *Topological stability of Chafee-Infante equations under Lipschitz perturbations of the domain and equation*. J. Math. Anal. Appl. **517**(2023), Article No. 126628, 28 pp.
[11] K. Lee, N.-T. Nguyen, and Y. Yang, *Topological stability and spectral decomposition for homeomorphisms on noncompact spaces*. Discrete Contin. Dyn. Syst. **38**(2018), 2487–2503.
[12] J. Lee and A. Rojas, *A topological shadowing theorem*. Proc. Indian Acad. Sci. Math. Sci. **132**(2022), no. 1, Article No. 32, 9 pp.
[13] Z. Nitecki, *On semi-stability for diffeomorphisms*. Invent. Math. **14**(1971), 83–122.

[14] G. Raugel, *Global attractors in partial differential equations*. In: B. Fiedler (ed.), Handbook of dynamical systems II, Freie Universität Berlin, Berlin, 2002, pp. 887–982.
[15] J. W. Robbin, *Review of the article "on semi-stability for diffeomorphisms" (in Invent. Math. 14 (1971), 83–122, by Zbigniew Nitecki)*. Mathematical Reviews/MathSciNet MR0293671.
[16] J. C. Robinson, *Infinite-dimensional dynamical systems. An introduction to dissipative parabolic PDEs and the theory of global attractors*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.
[17] J. C. Robinson and J. J. Sánchez-Gabites, *On finite-dimensional global attractors of homeomorphisms*. Bull. Lond. Math. Soc. **48**(2016), 483–498.
[18] M. Sears, *Expansiveness of locally compact spaces*. Math. Systems Theory **7**(1974), 377–382.
[19] M. Shub and S. Smale, *Beyond hyperbolicity*. Ann. of Math. (2) **96**(1972), no. 3, 576–591.
[20] S. Smale, *Differentiable dynamical systems*. Bull. Amer. Math. Soc. **73**(1967), 747–817.
[21] W. R. Utz, *Unstable homeomorphisms*. Proc. Amer. Math. Soc. **1**(1950), 769–774.
[22] P. Walters, *On the pseudo-orbit tracing property and its relationship to stability. The structure of attractors in dynamical systems*. In Proc. Conf., North Dakota State Univ., Fargo, N.D., 1977, Lecture Notes in Mathematics, 668, Springer, Berlin, 1978, pp. 231–244.
[23] P. Walters, *Anosov diffeomorphisms are topologically stable*. Topology **9**(1970), 71–78.
[24] A. Weil, *Sur les espaces a structure uniforme et Sur la topologie generale*, Hermann, Paris, 1938.

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