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WEAK GEODESIC RAYS IN THE SPACE OF KÄHLER POTENTIALS AND THE CLASS $\mathcal{E}(X, \omega)$

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To Timea

Abstract Suppose that (X, ω) is a compact Kähler manifold. In the present work we propose a construction for weak geodesic rays in the space of Kähler potentials that is tied together with properties of the class $\mathcal{E}(X, \omega)$. As an application of our construction, we prove a characterization of $\mathcal{E}(X, \omega)$ in terms of envelopes.

Keywords: complex Monge–Ampère equations; geodesic rays; finite energy classes

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1. Introduction and main results

Given (X^n, ω), a connected compact Kähler manifold, the space of smooth Kähler potentials is the set

$$\mathcal{H} := \left\{ v \in C^\infty(X) : \omega + i\partial\bar{\partial}v > 0 \right\}.$$

This space has a Fréchet manifold structure as an open subset of $C^\infty(X)$. For $v \in \mathcal{H}$, one can identify $T_v\mathcal{H}$ with $C^\infty(X)$. As found by Mabuchi, one can define a Riemannian metric on $T_v\mathcal{H}$ [18]:

$$\langle \xi, \eta \rangle_v := \int_X \xi\eta(\omega + i\partial\bar{\partial}v)^n, \quad \xi, \eta \in T_v\mathcal{H}.$$

A smooth curve $(\alpha, \beta) \ni t \to \phi_t \in \mathcal{H}$ with $\alpha, \beta \in \mathbb{R} \cup \{-\infty, +\infty\}$ is a geodesic in this space if

$$\ddot{\phi}_t - \tfrac{1}{2}\langle \nabla\dot{\phi}_t, \nabla\dot{\phi}_t \rangle_{\phi_t} = 0, \quad t \in (\alpha, \beta).$$

As discovered independently by Semmes [23] and Donaldson [15], the above equation can be understood as a complex Monge–Ampère equation. With the notation

$$S_{\alpha\beta} = \{ s \in \mathbb{C} : \alpha < \text{Re } s < \beta \},$$

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let ω be the pullback of the Kähler form ω to the product $S_{\alpha\beta} \times X$. Let $u \in C^\infty(S_{\alpha\beta} \times X)$ be the complexification of ϕ, defined by $u(s, x) := \phi(\text{Re } s, x)$. Then ϕ is a geodesic if and only if the following equation is satisfied for u:

$$(\pi^*\omega + i\partial\overline{\partial}u)^{n+1} = 0, \tag{1}$$

where $\pi : S \times X \to X$ is the projection map to the second component. By analogy with the smooth setting, a curve $(\alpha, \beta) \ni t \to u_t \in \text{PSH}(X, \omega)$ is called a weak subgeodesic segment if its complexification $u : S_{\alpha\beta} \times X \to R$ is a locally bounded $\pi^*\omega$-plurishubharmonic (ω-psh) function. If additionally (1) is satisfied in the Bedford and Taylor sense [2], then $t \to u_t$ is called a weak geodesic segment.

When $\alpha = 0$ and $\beta = \infty$, we call $t \to u_t$ a weak geodesic ray. Given such a curve, we would like to understand the limit $u_\infty := \lim_{t\to\infty} u_t$ whenever it exists. This problem is partially motivated by Donaldson's program on the existence and uniqueness of constant scalar curvature Kähler metrics in a fixed Kähler class. According to this program, one should study the limit behavior of geodesics rays (as well as certain functionals along the rays) as $t \to \infty$. However, for a general weak geodesic ray, $\lim_{t\to\infty} u_t$ does not exist, and we need a normalization procedure that fixes this issue. This is the main motivation of our first result, which holds for arbitrary weak geodesic segments not just rays.

If the weak geodesic $t \to u_t$ is in C^1, Berndtsson [4, § 2.2] observed that the range of the tangent vectors $\dot{u}_t : X \to \mathbb{R}$ is the same for any $t \in (\alpha, \beta)$. If u is a smooth strong geodesic, more can be said, as it is well known that $\dot{u}_t = \dot{u}_0 \circ F_t$, for some family of symplectomorphisms $F_t : X \to X$ [22, Formula (27)]. Our first result is a generalization of these observations to arbitrary weak geodesic segments.

Theorem 1 (Theorem 3.4). *Given a weak geodesic* $(\alpha, \beta) \ni t \to u_t \in \text{PSH}(X, \omega)$*,* $(\alpha, \beta \in \mathbb{R} \cup \{-\infty, +\infty\})$*, for any* $a, b, c, d \in (\alpha, \beta)$ *one has the following.*

(i) $\inf_X \frac{u_a - u_b}{a - b} = \inf_X \frac{u_c - u_d}{c - d} =: m_u$.

(ii) $\sup_X \frac{u_a - u_b}{a - b} = \sup_X \frac{u_c - u_d}{c - d} =: M_u$.

Hence, $t \to u_t$ *is Lipschitz continuous in* t*, with Lipschitz constant* $\max\{|M_u|, |m_u|\}$.

With this result at hand, we turn to constructing weak geodesic rays. We say that a weak geodesic segment $t \to u_t$ is linear if $u_t = u_0 + M_u t = u_0 + m_u t$. If additionally $m_u = M_u = 0$, then $t \to u_t$ is constant. Clearly, u_t is not linear if and only if $M_u > m_u$. In this case, if $\alpha > -\infty$, then one can always obtain $M_u = 0$ and $m_u = -1$ by a translation and rescaling:

$$\tilde{u}_t = u_{\alpha + \frac{t-\alpha}{M_u - m_u}}(x) - \frac{M_u(t - \alpha)}{M_u - m_u}.$$

One is naturally led to the following notion, which will be especially useful in our investigation of weak geodesic rays.

Definition 1. We say that a weak geodesic segment $(\alpha, \beta) \ni t \to u_t \in \text{PSH}(X, \omega)$ is normalized if it is constant or $M_u = 0$ and $m_u = -1$.

Given a normalized weak geodesic ray $t \to u_t$, by Lipschitz continuity (Theorem 1) we have that $u_0 = \lim_{t\to 0} u_t \in \text{PSH}(X, \omega) \cap L^\infty(X)$. Since this last limit is uniform in X, it

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also follows that $\sup_X(u_{t_1} - u_{t_0}) = 0$, $t_1, t_0 \in [0, +\infty)$, $t_0 < t_1$, and hence

$$\begin{cases} u_{t_1} \leqslant u_{t_0}, & 0 \leqslant t_0 < t_1, \\ \inf_X u_0 \leqslant \sup_X u_t \leqslant \sup_X u_0, & t \in (0, +\infty). \end{cases}$$

It is well known that for any $C \in \mathbb{R}$ sets of the type

$$\{v \in \mathrm{PSH}(X, \omega) | -C \leqslant \sup v \leqslant C\}$$

are compact in $\mathrm{PSH}(X, \omega)$ equipped with the $L^1(X)$ topology. This implies that the decreasing pointwise limit $u_\infty = \lim_{t\to+\infty} u_t$ is ω-psh and it is different from $-\infty$.

For $\phi, \psi \in \mathrm{PSH}(X, \omega)$, $\psi \leqslant \phi$ with ϕ bounded and ψ possibly unbounded, our goal is to construct a normalized weak geodesic ray $t \to v_t$ such that $v_0 = \phi$ and $v_\infty = \psi$. As we shall see, this is not always possible, but, whenever it can be done, our construction below will provide such a ray (see Corollary 4.5).

We introduce the following set of normalized weak geodesic rays:

$$\mathcal{R}(\phi, \psi) = \Big\{v_t \text{ is a normalized weak ray with } v_0 = \lim_{t\to 0} v_t = \phi \text{ and } v_\infty = \lim_{t\to\infty} v_t \geqslant \psi\Big\},$$

where the limits are pointwise, but by Theorem 1 the first is perforce uniform. By $(0, l) \ni t \to u^l_t \in \mathrm{PSH}(X, \omega)$ we denote the unique weak geodesic segments joining ϕ with $\max\{\phi - l, \psi\}$.

Additionally, let c_ψ be the limit

$$c_\psi = \lim_{l\to+\infty} \frac{AM(\max\{-l, \psi\})}{l},$$

where $AM(\cdot)$ is the Aubin–Mabuchi energy of a bounded ω-psh function. As we shall see in § 2.3, the constant c_ψ is well defined and finite.

Theorem 2 (Theorem 4.1). *For any $\phi, \psi \in \mathrm{PSH}(\omega)$ with ϕ bounded and $\psi \leqslant \phi$, the weak geodesic segments u^l form an increasing family. The upper semicontinuous regularization of their limit $v(\phi, \psi) = \mathrm{usc}(\lim_{l\to\infty} u^l)$ is a weak geodesic ray for which the following hold.*

(i) $v(\phi, \psi)_t = \mathrm{usc}(\lim_{l\to\infty} u^l_t)$ *for any* $t \in (0, +\infty)$.

(ii) $v(\phi, \psi) \in \mathcal{R}(\phi, \psi)$*; more precisely,* $v(\phi, \psi) = \inf_{v\in\mathcal{R}(\phi,\psi)} v$*. In particular,* $t \to v(\phi, \psi)_t$ *is constant if and only if* $\mathcal{R}(\phi, \psi)$ *contains only the constant ray* ϕ.

(iii) $AM(v(\phi, \psi)_t) = AM(\phi) + c_\psi t$*; in particular,* $t \to v(\phi, \psi)_t$ *is constant if and only if* $\psi \in \mathcal{E}(X, \omega)$.

In a nutshell, the above theorem says that the ray $v(\phi, \psi)$ is the lower envelope of the elements of $\mathcal{R}(\phi, \psi)$, and it is constant if and only if $\mathcal{R}(\phi, \psi)$ contains only the constant ray $t \to \phi$, which in turn is equivalent to $\psi \in \mathcal{E}(X, \omega)$.

For the definition of the class $\mathcal{E}(X, \omega) \subset \mathrm{PSH}(X, \omega)$, we refer the reader to § 2.3. This class was introduced in [17], and it was used to solve global Monge–Ampère equations with very rough data. As an intermediate result, we prove in § 2.3 that $\psi \in \mathcal{E}(X, \omega)$ if and only if $c_\psi = 0$.

Ever since the importance of geodesic rays was pointed out in Donaldson's program [15], there have been many papers on methods how to construct weak geodesic rays. We mention [1, 11, 12, 19, 20], to indicate only a few articles in a very fast expanding literature. At first sight, our construction below is perhaps most reminiscent of [11] (we construct our ray out of segments as well), but our conclusions and the questions investigated seem to be entirely different.

The elements $v \in \mathcal{E}(X, \omega)$ are usually unbounded but have very mild singularities. In particular, by [17, Corollary 1.8], at any $x \in X$ the Lelong number of v is zero. However, as noted in [17], this property does not characterize $\mathcal{E}(X, \omega)$. Our next theorem tries to fill this void, that is, to characterize elements of $\mathcal{E}(X, \omega)$ in terms of the mildness of their singularities.

Given upper semi-continuous functions b_0, b_1, one can define the envelopes

$$P(b_0) = \sup\{\psi \leqslant b_0 : \psi \in \mathrm{PSH}(X, \omega)\},$$
$$P(b_0, b_1) = P(\min\{b_0, b_1\}).$$

As the upper semicontinuous regularization $\mathrm{usc}(P(b_0))$ is a competitor and $P(b_0) \leqslant \mathrm{usc}(P(b_0))$, it follows that $P(b_0) \in \mathrm{PSH}(X, \omega)$, and the same is true for $P(b_0, b_1)$.

For $\psi, \psi' \in \mathrm{PSH}(X, \omega)$, we say that ψ and ψ' have the same singularity type if there exists $C > 0$ s.t.

$$\psi' - C < \psi < \psi' + C.$$

This induces an equivalence relation on $\mathrm{PSH}(X, \omega)$, and we denote each class by $[\psi]$, given a representative $\psi \in \mathrm{PSH}(X, \omega)$.

Suppose now that $\phi, \psi \in \mathrm{PSH}(X, \omega)$ with $\phi \in L^\infty(X)$. In [20], the envelope of ϕ with respect to the singularity type of ψ was considered in the following manner:

$$P_{[\psi]}(\phi) = \mathrm{usc}\Big(\lim_{C \to +\infty} P(\psi + C, \phi)\Big).$$

Assuming that ψ has analytic singularities, one can show that ψ has the same singularity type as $P_{[\psi]}(\phi)$. After making this observation, in [20, Remark 4.6], [21, Remark 3.9] the authors ask whether this holds for general ψ. This is not the case, as our next result says that, given a continuous potential ϕ, the singularities of the envelope $P_{[\psi]}(\phi)$ disappear once $\psi \in \mathcal{E}(X, \omega)$.

Theorem 3 (Theorem 5.3). *Suppose that $\psi \in \mathrm{PSH}(X, \omega)$ and $\phi \in \mathrm{PSH}(X, \omega) \cap C(X)$. Then $\psi \in \mathcal{E}(X, \omega)$ if and only if*

$$P_{[\psi]}(\phi) = \phi.$$

Interestingly, one direction in the proof of this theorem relies on the findings of Theorem 2, although in the statement of this result weak geodesics are not mentioned at all. What seems even more intriguing, the above result can be used to construct very general geodesic segments joining points of $\mathcal{E}(X, \omega)$ (see [13]).

Suppose that ϕ, ψ are as specified in the above theorem, and that ψ has a non-zero Lelong number at $x \in X$. This means that in a neighborhood of x we have

$$\psi(y) < c\log\|y-x\| + d$$

for some $c, d > 0$. Having alternative definitions of the pluricomplex Green function in mind, one observes that this estimate implies that $P_{[\psi]}(\phi)$ has a logarithmic singularity at x. Hence, the condition $P_{[\psi]}(\phi) = \phi$ guarantees that all Lelong numbers of ψ are zero, justifying our earlier claim that the above theorem is a characterization of the elements of $\mathcal{E}(X,\omega)$ in terms of the mildness of their singularities.

Lastly, we compare our construction of weak geodesic rays to the one in [20]. For details on the terminology we refer to § 2.4. Courtesy of an argument provided by Ross and Witt-Nyström, the condition of 'small unbounded locus' can be removed from the definition of a test curve (Theorem 2.9). As a consequence of this and Theorem 1, we remark that all weak geodesic rays can be constructed using analytic test configurations (Corollary 5.2). Finally, we conclude that the geodesic rays we constructed in Theorem 2 can be recovered using very specific test curves.

Theorem 4 (Theorem 6.1). *Suppose that $\phi, \psi \in \mathrm{PSH}(\omega)$ with ϕ bounded and $\psi \leqslant \phi$. Then the weak geodesic ray $t \to v(\phi,\psi)_t$ is the same as the ray obtained from a special test curve $\tau \to \gamma^*_\tau$ using the method of [20].*

2. Preliminaries

2.1. Berndtsson's construction

In this section, we recall Berndtsson's construction [5, § 2.1] of a weak solution to the Dirichlet problem associated to the geodesic equation in the space of Kähler potentials:

$$\begin{gathered} u \in \mathrm{PSH}(S_{0,1}\times X,\omega)\cap L^\infty(S_{0,1}\times X)\\ (\pi^*\omega + i\partial\overline{\partial}u)^{n+1} = 0\\ u(t+ir,x) = u(t,x) \quad \forall x\in X, t\in(0,1), r\in\mathbb{R} \\ \lim_{t\to 0,1} u_t(x) = u_{0,1}(x), \forall x \in X, \end{gathered} \tag{2}$$

where $u_0, u_1 \in \mathrm{PSH}(X,\omega)\cap L^\infty(X)$ and the limits are uniform in X. We introduce the following set of weak subgeodesics:

$$\mathcal{S} = \Big\{v \text{ is a weak subgeodesic with } \lim_{t\to 0,1} v_t \leqslant u_{0,1}\Big\},$$

where the boundary limits are assumed to be only pointwise in X. Our candidate solution is defined by taking the upper envelope of this family:

$$u = \sup_{v\in S} v.$$

Since u_0, u_1 are bounded, for $A > 0$ big enough we have that $v_t = \max\{u_0 - At, u_1 - A(1-t)\} \in \mathcal{S}$. Since all elements of S are convex in t, we have the following estimate:

$$v_t \leqslant u_t \leqslant u_0 + t(u_1 - u_0). \tag{3}$$

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This estimate is also true for the upper semicontinuous regularization u^* of u as well; hence $u^* \in \mathcal{S}$, implying that $u = u^*$. The fact that

$$(\pi^*\omega + i\partial\bar{\partial}u)^{n+1} = 0$$

follows now from Bedford and Taylor theory adapted to this setting. Uniqueness follows from the maximum principle [7, Theorem 6.4], as we assumed in (2) that the boundary limits are uniform. What is more, it follows from (3) that $t \to u_t$ is uniformly Lipschitz continuous in the t variable.

Although we will not use it here, let us mention that for $u_0, u_1 \in \mathrm{PSH}(X,\omega) \cap C^\infty(X)$ it was proved in [10] that u has bounded Laplacian, and this regularity is optimal, as later observed in [14].

2.2. The Aubin–Mabuchi energy

The Aubin–Mabuchi energy is a concave functional $AM : \mathrm{PSH}(X,\omega) \cap L^\infty(X) \to \mathbb{R}$, given by the formula

$$AM(v) = \frac{1}{(n+1)} \sum_{j=0}^{n} \int_X v\omega^j \wedge (\omega + i\partial\bar{\partial}v)^{n-j}.$$

One can easily compute that for $u, v \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ we have

$$AM(u) - AM(v) = \frac{1}{(n+1)} \sum_{j=0}^{n} \int_X (u - v)(\omega + i\partial\bar{\partial}u)^j \wedge (\omega + i\partial\bar{\partial}v)^{n-j}; \tag{4}$$

in particular,

$$AM(u + c) = AM(u) + c, \quad c \in \mathbb{R}. \tag{5}$$

The Aubin–Mabuchi functional is important for geodesics in the space of Kähler potentials because of the following result.

Theorem 2.1 [3, Proposition 6.2]. *Given a weak subgeodesic $(\alpha, \beta) \ni t \to u_t \in \mathrm{PSH}(X,\omega)$, the correspondence $t \to AM(u_t)$, $t \in (\alpha, \beta)$ is convex. Moreover, the subgeodesic $t \to u_t$ is a geodesic if and only if $t \to AM(u_t)$ is linear.*

The following well-known estimate will be essential in our later investigations about the Aubin–Mabuchi energy. For completeness we include a proof here.

Proposition 2.2 [8, Proposition 2.8]. *For $u \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ with $u \leqslant 0$ the following holds:*

$$\int_X u(\omega + i\partial\bar{\partial}u)^n \leqslant AM(u) \leqslant \frac{1}{n+1} \int_X u(\omega + i\partial\bar{\partial}u)^n.$$

Proof. As $u \leqslant 0$, the second estimate is trivial. To prove the first estimate, we observe that it is enough to argue that

$$\int_X u\omega^k \wedge (\omega + i\partial\bar{\partial}u)^{n-k} \leqslant \int_X u\omega^{k+1} \wedge (\omega + i\partial\bar{\partial}u)^{n-k-1}, \quad k \in \{0, \dots, n-1\}.$$

However, this follows easily, as we have

$$\int_X u\omega^k \wedge (\omega + i\partial\bar\partial u)^{n-k} = \int_X u\omega^{k+1} \wedge (\omega + i\partial\bar\partial u)^{n-k-1} - \int_X i\partial u \wedge \bar\partial u \wedge \omega^k \wedge (\omega + i\partial\bar\partial u)^{n-k-1}. \qquad \square$$

The last result in this section is a kind of domination principle for the Aubin–Mabuchi energy. Although we could not find a reference for it, its proof is implicit in many standard arguments throughout the literature. See, for example, [6, Theorem 1.1].

Proposition 2.3. *Suppose that $u, v \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ with $u \geqslant v$. If $AM(u) = AM(v)$, then $u = v$.*

Proof. Since $u \geqslant v$, it follows from (4) that

$$\int_X (u-v)(\omega + i\partial\bar\partial u)^j \wedge (\omega + i\partial\bar\partial v)^{n-j} = 0, \quad j = 0, \ldots, n. \tag{6}$$

We are finished if we can prove that

$$\int_X i\partial(u-v) \wedge \bar\partial(u-v) \wedge \omega^{n-1} = 0. \tag{7}$$

The first step is to prove that

$$\int_X i\partial(u-v) \wedge \bar\partial(u-v) \wedge \omega \wedge (\omega + i\partial\bar\partial u)^j \wedge (\omega + i\partial\bar\partial v)^{n-j-2} = 0, \quad j = 0, \ldots, n-2. \tag{8}$$

It follows from (6) that

$$\begin{aligned} 0 &= \int_X (u-v) i\partial\bar\partial(u-v) \wedge (\omega + i\partial\bar\partial u)^j \wedge (\omega + i\partial\bar\partial v)^{n-j-1} \\ &= -\int_X i\partial(u-v) \wedge \bar\partial(u-v) \wedge (\omega + i\partial\bar\partial u)^j \wedge (\omega + i\partial\bar\partial v)^{n-j-1}, \quad j = 0, \ldots, n-1. \end{aligned} \tag{9}$$

Using this we now write

$$\begin{aligned} &\int_X i\partial(u-v) \wedge \bar\partial(u-v) \wedge \omega \wedge (\omega + i\partial\bar\partial u)^j \wedge (\omega + i\partial\bar\partial v)^{n-j-2} \\ &= -\int_X i\partial(u-v) \wedge \bar\partial(u-v) \wedge i\partial\bar\partial v \wedge (\omega + i\partial\bar\partial u)^j \wedge (\omega + i\partial\bar\partial v)^{n-j-2} \\ &= \int_X i\partial(u-v) \wedge i\partial\bar\partial(u-v) \wedge \bar\partial v \wedge (\omega + i\partial\bar\partial u)^j \wedge (\omega + i\partial\bar\partial v)^{n-j-2} \\ &= \int_X i\partial(u-v) \wedge \bar\partial v \wedge (\omega + i\partial\bar\partial u) \wedge (\omega + i\partial\bar\partial u)^j \wedge (\omega + i\partial\bar\partial v)^{n-j-2} \\ &\quad - \int_X i\partial(u-v) \wedge \bar\partial v \wedge (\omega + i\partial\bar\partial v) \wedge (\omega + i\partial\bar\partial u)^j \wedge (\omega + i\partial\bar\partial v)^{n-j-2}, \\ &\qquad j = 0, \ldots, n-2. \end{aligned}$$

Using the Cauchy–Schwarz inequality for the Monge–Ampère operator, it follows from (9) that both of the terms in the last sum are zero, proving (8). Continuing this inductive process, we arrive at (7). $\square$

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2.3. The class $\mathcal{E}(X,\omega)$

We recall here a few facts about the class $\mathcal{E}(X,\omega) \subset \mathrm{PSH}(X,\omega)$. For the sake of brevity, we take a very minimalistic approach in our presentation. For a more complete treatment, we refer the reader to [17]. For $\gamma \in \mathrm{PSH}(X,\omega)$, one can define the canonical cutoffs $\gamma_l \in \mathrm{PSH}(X,\omega), l \in \mathbb{R}$ by the formula

$$\gamma_l = \max\{-l, \gamma\}.$$

By an application of the comparison principle, it follows that the Borel measures

$$\chi_{\{\gamma>-l\}}(\omega + i\partial\bar{\partial}\gamma_l)^n$$

are increasing in l. Following [8], despite the fact that γ might be unbounded, one can still make sense of $(\omega+i\partial\bar{\partial}\gamma)^n$ as the limit of these increasing measures:

$$(\omega+i\partial\bar{\partial}\gamma)^n = \lim_{l\to+\infty} \chi_{\{\gamma>-l\}}(\omega+i\partial\bar{\partial}\gamma_l)^n. \tag{10}$$

Using this definition, $(\omega+i\partial\bar{\partial}\gamma)^n$ is called the non-pluripolar Monge–Ampère measure of γ. It is clear from (10) that

$$\int_X (\omega+i\partial\bar{\partial}\gamma)^n \leqslant \int_X \omega^n = \mathrm{Vol}(X).$$

This brings us to the class $\mathcal{E}(X,\omega)$. By definition, $\gamma \in \mathcal{E}(X,\omega)$ if

$$\int_X (\omega+i\partial\bar{\partial}\gamma)^n = \lim_{l\to\infty}\int_X \chi_{\{\gamma>-l\}}(\omega+i\partial\bar{\partial}\gamma_l)^n = \mathrm{Vol}(X). \tag{11}$$

As detailed in [17], most of the classical theorems of Bedford and Taylor theory are valid for the class $\mathcal{E}(X,\omega)$ as well. For brevity we only mention here a version of the domination principle that we will need later.

Proposition 2.4 [9]. *Suppose that $\psi \in \mathcal{E}(X,\omega)$ and $\phi \in \mathrm{PSH}(X,\omega)\cap L^\infty(X)$. If $\psi \geqslant \phi$ a.e. (almost everywhere) with respect to $(\omega+i\partial\bar{\partial}\psi)^n$, then $\psi(x) \geqslant \phi(x), x \in X$.*

Proof. We follow the proof in [9], which in turn is based on an idea of Zeriahi. By translation, we can assume that both ϕ and ψ are negative on X. Then, for all $s > 0$ and small enough $\varepsilon > 0$, we have

$$\{\psi-\phi < -s-\varepsilon\phi\} \subseteq \{\psi-\phi<0\}.$$

Now we can use Lemma 2.3 in [16], which is easily seen to be valid for elements of $\mathcal{E}(X,\omega)$ (one only needs the comparison principle, which is true for elements of $\mathcal{E}(X,\omega)$, as is proved in [17]). According to this result, we have

$$\varepsilon^n \mathrm{Cap}_\omega(\{\psi-\phi<-s-\varepsilon\}) \leqslant \int_{\{\psi-\phi<-s-\varepsilon\phi\}} (\omega+i\partial\bar{\partial}\psi)^n \leqslant \int_{\{\psi-\phi<0\}} (\omega+i\partial\bar{\partial}\psi)^n = 0.$$

Since the Monge–Ampère capacity dominates the Lebesgue measure, it results that $\psi-\phi \geqslant -s-\varepsilon$ a.e. with respect to ω^n. By the sub-mean-value property of psh functions, this estimate extends to X. Now, letting $s,\varepsilon \to 0$, we obtain the desired result. □

Next, we observe that $l \to \gamma_l = \max\{-l, \gamma\}, l \geqslant 0$ is a decreasing weak subgeodesic ray, and hence the map $l \to AM(\gamma_l), l \in (0, +\infty)$ is convex and decreasing, by Theorem 2.1 and (4). From this, it follows that the following quantity is well defined and finite:

$$c_\gamma = \lim_{l\to+\infty} \frac{AM(\gamma_l)}{l} = \lim_{l\to+\infty} \frac{AM(\gamma_l) - AM(\gamma_0)}{l}.$$

Our next result gives a precise formula for c_γ, one that does not use subgeodesic rays. We also obtain a characterization of $\mathcal{E}(X,\omega)$ in terms of this constant.

Theorem 2.5. *Given $\gamma \in \mathrm{PSH}(X,\omega)$, we have that*

$$c_\gamma = \frac{-1}{n+1} \sum_{j=0}^{n} \lim_{l\to+\infty} \int_{\{\gamma\leqslant -l\}} \omega^j \wedge (\omega + i\partial\bar\partial\gamma_l)^{n-j}.$$

In particular, $\gamma \in \mathcal{E}(X,\omega)$ if and only if $c_\gamma = 0$.

Proof. It follows from (5) that for any $d \in \mathbb{R}$ we have $c_{\gamma+d} = c_\gamma$. Hence, we can assume that $\gamma < 0$. First, we prove that

$$\lim_{l\to+\infty} \int_X \frac{\gamma_l}{l} (\omega + i\partial\bar\partial\gamma_l)^n = -\lim_{l\to+\infty} \int_{\{\gamma\leqslant -l\}} (\omega + i\partial\bar\partial\gamma_l)^n. \tag{12}$$

To see this, since

$$\int_X \frac{\gamma_l}{l} (\omega + i\partial\bar\partial\gamma_l)^n = -\int_{\{\gamma\leqslant -l\}} (\omega + i\partial\bar\partial\gamma_l)^n + \int_{\{\gamma>-l\}} \frac{\gamma}{l} (\omega + i\partial\bar\partial\gamma_l)^n,$$

it is enough to show that

$$\lim_{l\to+\infty} \int_{\{\gamma>-l\}} \frac{\gamma}{l} (\omega + i\partial\bar\partial\gamma_l)^n = 0. \tag{13}$$

Since $\gamma < 0$, it is enough to prove that

$$\lim_{l\to+\infty} \int_{\{\gamma>-l\}} \frac{\gamma}{l} (\omega + i\partial\bar\partial\gamma_l)^n \geqslant 0. \tag{14}$$

For any $\varepsilon > 0$, we have

$$\int_{\{\gamma>-l\}} \frac{\gamma}{l} (\omega + i\partial\bar\partial\gamma_l)^n \geqslant -\int_{\{-\varepsilon l\geqslant\gamma>-l\}} (\omega + i\partial\bar\partial\gamma_l)^n - \varepsilon \int_{\{\gamma>-\varepsilon l\}} (\omega + i\partial\bar\partial\gamma_l)^n.$$

The second term in the sum is bounded below by $-\varepsilon\mathrm{Vol}(X)$, whereas the first term can be written as

$$\begin{aligned}\int_{\{-\varepsilon l\geqslant\gamma>-l\}} (\omega + i\partial\bar\partial\gamma_l)^n &= -\int_{\{\gamma>-\varepsilon l\}} (\omega + i\partial\bar\partial\gamma_l)^n + \int_{\{\gamma>-l\}} (\omega + i\partial\bar\partial\gamma_l)^n \\ &= -\int_{\{\gamma>-\varepsilon l\}} (\omega + i\partial\bar\partial\gamma_{\varepsilon l})^n + \int_{\{\gamma>-l\}} (\omega + i\partial\bar\partial\gamma_l)^n,\end{aligned}$$

where the last line follows from locality of the Monge–Ampère measure in the plurifine topology (see [17, formula (2)]). Since the measures $\chi_{\{\gamma>-l\}}(\omega+i\partial\bar{\partial}\gamma_l)^n$ increase with l, taking the limit in the above identity we obtain

$$\lim_{l\to\infty}\int_{\{-\varepsilon l\geqslant\gamma>-l\}}(\omega+i\partial\bar{\partial}\gamma_l)^n=0,$$

proving (14), which in turn implies (12). Following exactly the same line of thought, one can prove that

$$\begin{aligned}&\lim_{l\to+\infty}\int_X\frac{\gamma_l}{l}\omega^j\wedge(\omega+i\partial\bar{\partial}\gamma_l)^{n-j}\\&\quad=-\lim_{l\to+\infty}\int_{\{\gamma\leqslant-l\}}\omega^j\wedge(\omega+i\partial\bar{\partial}\gamma_l)^{n-j},\quad j\in 0,\ldots,n.\end{aligned}\tag{15}$$

After we sum over j in the above equation, we obtain the formula for c_γ in the statement of the theorem.

To prove the last claim, observe that from Proposition 2.2 it follows that

$$\int_X\frac{\gamma_l}{l}(\omega+i\partial\bar{\partial}\gamma_l)^n\leqslant\frac{AM(\gamma_l)}{l}\leqslant\frac{1}{n+1}\int_X\frac{\gamma_l}{l}(\omega+i\partial\bar{\partial}\gamma_l)^n,\quad l>0.\tag{16}$$

Putting (16) and (12) together we obtain

$$-\lim_{l\to+\infty}\int_{\{\gamma\leqslant-l\}}(\omega+i\partial\bar{\partial}\gamma_l)^n\leqslant\lim_{l\to+\infty}\frac{AM(\gamma_l)}{l}=c_\gamma\leqslant\frac{-1}{n+1}\lim_{l\to+\infty}\int_{\{\gamma\leqslant-l\}}(\omega+i\partial\bar{\partial}\gamma_l)^n.$$

By the definition of the class $\mathcal{E}(X,\omega)$, $\lim_{l\to+\infty}\int_{\{\gamma\leqslant-l\}}(\omega+i\partial\bar{\partial}\gamma_l)^n=0$ if and only if $\gamma\in\mathcal{E}(X,\omega)$, proving the result. □

Remark 2.6. In the definition of c_γ we could have started with the more general decreasing weak subgeodesic ray

$$l\to\tilde{\gamma}_l=\max\{\beta-l,\gamma\}$$

for any $\beta\in L^\infty(X)\cap \mathrm{PSH}(X,\omega)$. As is easily verified, the resulting constant $\tilde{c}_\gamma=\lim_{l\to+\infty}AM(\tilde{\gamma}_l)/l$ is the same as our original c_γ.

2.4. The weak geodesic rays of Ross and Witt-Nyström

In this section, we review the construction of Ross and Witt-Nyström [20] of weak geodesic rays. Although [20] is written in the setting when the Kähler structure is integral ($[\omega]\in H^2(X,\mathbb{Z})$), the whole construction carries over without changes to our more general situation.

Definition 2.7. A map $\mathbb{R}\ni\tau\to\psi_\tau\in\mathrm{PSH}(X,\omega)$ is called a test curve if the following hold.

(i) $\tau\to\psi_\tau(x)$ is concave in τ for any $x\in X$.

(ii) There exists $C_\psi>0$ such that ψ_τ is equal to some bounded potential $\psi_{-\infty}\in\mathrm{PSH}(X,\omega)\cap L^\infty(X)$ for $\tau<-C_\psi$, and $\psi_\tau=-\infty$ if $\tau>C_\psi$.

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We remark that in the definition of test curves from [20], it is also assumed that each ψ_τ has small unbounded locus. As will be clear in Theorem 2.10 below, this condition can be omitted.

Given a test curve $\tau \to \psi_\tau$, the curve of singularity types $\tau \to [\psi_\tau]$ is called an *analytic test configuration*. As mentioned in the introduction, given b_0, b_1 usc functions on X, one can define the envelopes

$$P(b_0) = \sup\{\psi \leqslant b_0 : \psi \in \mathrm{PSH}(\omega)\},$$
$$P(b_0, b_1) = P(\min\{b_0, b_1\}),$$

which are elements of $\mathrm{PSH}(X,\omega)$. Given $\phi, \psi \in \mathrm{PSH}(X,\omega)$, $\phi \in L^\infty(X)$, one can introduce the envelope of ϕ with respect to the singularity type of ψ. This is given by the formula

$$P_{[\psi]}(\phi) = \mathrm{usc}\Big(\lim_{D\to+\infty} P(\psi + D, \phi)\Big).$$

In the process of obtaining a weak geodesic ray, one starts out with a test curve $\tau \to \psi_\tau$, as defined above, and a potential $\phi \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$. The first step is to 'maximize' the test curve $\tau \to \psi_\tau$ with respect to ϕ by introducing the new test curve

$$\tilde{\psi}_\tau = P_{[\psi_\tau]}(\phi).$$

The second step is to take the 'inverse' Legendre transform of $\tilde{\psi}_\tau$:

$$\phi_t = \mathrm{usc}\Big(\sup_{\tau\in\mathbb{R}}(\tilde{\psi}_\tau + t\tau)\Big) \in \mathrm{PSH}(X,\omega) \cap L^\infty(X), \quad t \in [0,+\infty). \tag{17}$$

Before we state the main result of [20], we remark that in (17) it is possible to omit the upper semi-continuous regularization. In fact, this is more generally true for any test curve.

Proposition 2.8. *Suppose that $\tau \to \psi_\tau$ is a test curve as given in Definition 2.7. Then, for any $t \geqslant 0$, we have*

$$\mathrm{usc}\Big[\sup_\tau(\psi_\tau + t\tau)\Big] = \sup_\tau(\psi_\tau + t\tau).$$

Proof. We denote $u(s,z) = \sup_\tau(\psi_\tau(z) + \mathrm{Re}\, s\tau)$ for $(s,z) \in S \times X$, where $S = \{\mathrm{Re}\, s \geqslant 0\} \subset \mathbb{C}$. Clearly, $\mathrm{usc}\, u \in \mathrm{PSH}(S \times X, \pi^*\omega)$. It will be enough to show that $\mathrm{usc}\, u = u$.

We introduce $E = \{u < \mathrm{usc}\, u\} \subset S \times X$. As both u and $\mathrm{usc}\, u$ are $\mathbb{R}$-invariant, it follows that E is also $\mathbb{R}$-invariant; i.e., there exists $B \subset [0,\infty) \times X$ such that

$$E = B + i\mathbb{R}.$$

As E has capacity 0, it follows that E has Lebesgue measure zero. For $z \in X$, we introduce the slices:

$$B_z = B \cap [0,\infty) \times \{z\}.$$

We have that B_z has Lebesgue measure zero for all $z \in X \setminus F$, where $F \subset X$ is some set of Lebesgue measure zero.

Let $z \in X \backslash F$. We argue that B_z is in fact empty. Both maps $t \to u(t,z)$ and $t \to (\text{usc } u)(t,z)$ are convex on $[0,\infty)$. As they agree on the dense set $[0,\infty)\backslash B_z$, it follows that they have to be the same; hence $B_z = 0$.

For fixed $\tau \in \mathbb{R}$ we clearly have

$$\psi_\tau = \inf_{t \geqslant 0}[u_t - \tau t] \leqslant \chi_\tau := \inf_{t \geqslant 0}[(\text{usc } u)_t - \tau t].$$

Because each B_z is empty for $z \in X \backslash F$, it follows that $\psi_\tau = \chi_\tau$ outside the set of measure-zero F. Since both ψ_τ and χ_τ are ω-psh (the former by definition, and the latter by Kiselman's minimum principle) it follows that $\psi_\tau = \chi_\tau$. Applying the Legendre transform to the curves $\tau \to \psi_\tau$ and $\tau \to \chi_\tau$, we obtain that $u = \text{usc } u$. □

Finally, we state one of the main results of [20].

Theorem 2.9 [20, Theorem 1.1]. *The curve* $[0,+\infty) \ni t \to \phi_t \in \text{PSH}(X,\omega)$ *introduced in (17) is a weak geodesic ray emanating from* ϕ.

As mentioned in the beginning of the section, the original definition of an analytic test configuration $\tau \to [\psi_\tau]$ also assumed that each ψ_τ has small unbounded locus in X. This condition is superfluous, and the above theorem holds in this greater generality. At the recommendation of the referee, we give here a proof provided by Ross and Witt-Nyström.

Proof of Theorem 2.9. Let $\tau \to \psi_\tau$ be a test curve as given in Definition 2.7. Let h_t be the Legendre transform of ψ_τ:

$$h_t := \sup_\tau(\psi_\tau + t\tau), \quad t \geqslant 0;$$

i.e., $t \to h_t$ is a bounded subgeodesic ray with $h_0 = \psi_{-\infty}$. For $D > 0$, let ϕ_t^D denote the supremum of all weak subgeodesic rays bounded from above by $\min(\phi + C_\psi t, h_t + D)$. Clearly, $t \to \phi_t^D$ is also a bounded subgeodesic ray, and we introduce the Legendre transform of ϕ_t^D:

$$\hat{\phi}_\tau^D = \inf_{t \geqslant 0}(\phi_t^D - t\tau).$$

As $\phi_t^D \leqslant \min(\phi + C_\psi t, h_t + D)$, we obtain that in fact $\hat{\phi}_\tau^D \leqslant \min(\psi_\tau + D, \phi)$ for any $\tau \leqslant C_\psi$, and $\hat{\phi}_\tau^D = -\infty$ for $\tau > C_\psi$. Also, by the Kiselman minimum principle, $\inf_t(\phi_t^D - t\tau)$ is ω-psh, and hence

$$\hat{\phi}_\tau^D = \inf_t(\phi_t^D - t\tau) \leqslant P(\psi_\tau + D, \phi).$$

Applying the inverse Legendre transform to the above inequality, we obtain that

$$\phi_t^D = \sup_\tau(P(\psi_\tau + D, \phi) + t\lambda), \tag{18}$$

since $\tau \to P(\psi_\tau + D, \phi)$ is a test curve and the right-hand side is a subgeodesic ray that is a candidate in the definition of ϕ^D. Let ϕ_t denote the limit of ϕ_t^D:

$$\phi_t = \text{usc}\left(\lim_{D \to +\infty} \phi_t^D\right) \in \text{PSH}(X,\omega).$$

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As we will see by the end of the proof, $t \to \phi_t$ introduced this way is the same curve as the one in (17). Using the comparison principle for geodesics, one can show that $t \to \phi_t$ is a weak geodesic ray. Taking the limit in (18), we find

$$\lim_{D\to+\infty} \phi_t^D = \sup_\tau \left(\lim_{D\to+\infty} P(\psi_\tau + D, \phi) + t\tau \right).$$

As $P(\psi_\lambda + C, \phi) \leqslant P_{[\psi_\lambda]}(\phi)$, we have $\phi_t^D \leqslant \sup_\tau (P_{[\psi_\lambda]}(\phi) + t\lambda)$. Since ϕ_t^D increases a.e. to ϕ_t, we get that

$$\phi_t \leqslant \sup_\tau (P_{[\psi_\lambda]}(\phi) + t\lambda). \tag{19}$$

To argue the other direction, note that, since $\phi_t^D \leqslant \phi_t$ and (18) holds, we have

$$\inf_{t\geqslant 0} (\phi_t - t\tau) \geqslant P(\psi_\tau + D, \phi).$$

The left-hand side is ω-psh by Kiselman's minimum principle, and, since $P(\psi_\tau + D, \phi)$ increases a.e. to $P_{[\psi_\tau]}(\phi)$, we obtain that $\inf_t(\phi_t - t\tau) \geqslant P_{[\psi_\tau]}(\phi)$. Taking the Legendre transform of this, we obtain that

$$\phi_t \geqslant \sup_\tau (P_{[\psi_\tau]}(\phi) + t\tau). \tag{20}$$

The inequalities (19) and (20) together imply that $t \to \sup_\tau (P_{[\psi_\tau]}(\phi) + t\tau)$ is indeed a geodesic ray, finishing the proof. $\square$

Lastly, we recall another proposition from [20], one which will be very useful for us later.

Proposition 2.10 [20, Theorem 4.10]. *Suppose that $\phi, \psi \in \mathrm{PSH}(X,\omega)$ with ϕ continuous and ψ possibly unbounded. Then $P_{[\psi]}(\phi)$ is maximal with respect to ϕ; i.e., $P_{[\psi]}(\phi) = \phi$ a.e. with respect to the measure $(\omega + i\partial\bar\partial P_{[\psi]}(\phi))^n$, where $(\omega + i\partial\bar\partial P_{[\psi]}(\phi))^n$ is defined as in (10).*

3. Normalization of weak geodesics

Given a weak subgeodesic segment $(\alpha,\beta) \ni t \to u_t \in \mathrm{PSH}(X,\omega)$, since the correspondence $t \to u_t(x)$ is convex for any $x \in X$, the left and right t-derivatives $\dot u_t^+(x)$ and $\dot u_t^-(x)$ exist at each point $t \in (\alpha,\beta)$. One can also define $\dot u_\alpha^+ = \lim_{t\searrow\alpha} u_t^+$ and $\dot u_\beta^- = \lim_{t\nearrow\beta} u_t^-$, with values possibly equal to $\pm\infty$. Our first lemma is a precise statement about these functions. Given $u_1, u_2 \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$, we denote by $u(u_0,u_1)$ the weak geodesic segment joining u_0 with u_1 constructed by the method of § 2.1.

Lemma 3.1. *Suppose that $u_0, u_1 \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$. For $v = u(u_0,u_1)$, we have the following.*

$$\inf_X \dot v_0^+ = \inf_X (u_1 - u_0).$$

$$\sup_X \dot v_1^- = \sup_X (u_1 - u_0).$$

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Proof. We only prove the first identity, as the second follows trivially from $u(u_0, u_1)_t = u(u_1, u_0)_{1-t}$.

It follows from the construction in § 2.1 that $v_t \geqslant u_0 + \inf_{x\in X}(u_1 - u_0)t$, $t \in [0, 1]$. This implies that $\inf_{x\in X} \dot{v}_0^+ \geqslant \inf_{x\in X}(u_1 - u_0)$. The other direction is easily seen by using the uniform Lipschitz continuity and convexity in the t variable:

$$u_1(x) - u_0(x) = \int_0^1 \dot{v}_t(x)\, dt \geqslant \dot{v}_0^+(x), \quad x \in X. \qquad \square$$

For a weak geodesic segment $(\alpha, \beta) \ni t \to u_t \in \mathrm{PSH}(X, \omega)$ and $a, b \in (\alpha, \beta)$, we denote by $(0, 1) \ni t \to u_t^{ab} \in \mathrm{PSH}(X, \omega)$ the rescaled weak geodesic segment $u_t^{ab} = u_{a+(b-a)t}$.

Lemma 3.2. *Given a weak geodesic segment $(\alpha, \beta) \ni t \to u_t \in \mathrm{PSH}(X, \omega)$, if $\alpha < a < b < \beta$, we have that $u^{ab} = u(u_a, u_b)$.*

Proof. Choose c, d such that $\alpha < c < a < b < d < \beta$. By convexity, for any $t_1, t_2 \in (a, b)$, we have

$$\frac{u_c^{ab} - u_a^{ab}}{c - a} \leqslant \frac{u_{t_1}^{ab} - u_{t_2}^{ab}}{t_1 - t_2} \leqslant \frac{u_b^{ab} - u_d^{ab}}{b - d},$$

and hence $t \to u_t^{ab}$ is uniformly Lipschitz continuous in the t variable. As follows from the construction in § 2.1, this is also true for $u(u_a, u_b)$ as well. Hence, the classical maximum principle can be used to conclude the lemma. $\square$

Lemma 3.3. *Given a weak geodesic segment $(\alpha, \beta) \ni t \to u_t \in \mathrm{PSH}(X, \omega)$, for any $\alpha < a < b < c < \beta$, one has the following.*

(i) $\frac{b-a}{c-a} \inf_X \frac{u_b-u_a}{b-a} + \frac{c-b}{c-a} \inf_X \frac{u_c-u_b}{c-b} \leqslant \inf_X \frac{u_c-u_a}{c-a}$.

(ii) $\frac{b-a}{c-a} \sup_X \frac{u_b-u_a}{b-a} + \frac{c-b}{c-a} \sup_X \frac{u_c-u_b}{c-b} \geqslant \sup_X \frac{u_c-u_a}{c-a}$.

(iii) $\inf_X \frac{u_c-u_b}{c-b} \geqslant \inf_X \frac{u_c-u_a}{c-a}$.

(iv) $\sup_X \frac{u_b-u_a}{b-a} \leqslant \sup_X \frac{u_c-u_a}{c-a}$.

Proof. The first two estimates are trivial, whereas the last two follow from the convexity of the maps $t \to u_t(x)$. $\square$

Theorem 3.4. *Given a weak geodesic $(\alpha, \beta) \ni t \to u_t \in \mathrm{PSH}(X, \omega)$, $(\alpha, \beta \in \mathbb{R} \cup \{-\infty, +\infty\})$, for any $a, b, c, d \in (\alpha, \beta)$, one has the following.*

(i) $\inf_X \frac{u_a-u_b}{a-b} = \inf_X \frac{u_c-u_d}{c-d} = m_u$.

(ii) $\sup_X \frac{u_a-u_b}{a-b} = \sup_X \frac{u_c-u_d}{c-d} = M_u$.

Hence, $t \to u_t$ is Lipschitz continuous in t, with Lipschitz constant $\max\{|M_u|, |m_u|\}$.

Proof. We only prove (i), as the proof of (ii) is similar. We can suppose that $a < b < c < d$. It follows from Lemmas 3.1 and 3.2 that

$$\inf_X \dot{u}_a^+ = \inf_X \frac{u_b - u_a}{b - a} = \inf_X \frac{u_c - u_a}{c - a}. \tag{21}$$

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Putting this into Lemma 3.3(i), we obtain $\inf_{x\in X}\frac{u_c-u_b}{c-b}\leqslant \inf_{x\in X}\frac{u_c-u_a}{c-a}$. Now, Lemma 3.3(iii) implies that

$$\inf_X \frac{u_c-u_b}{c-b}=\inf_X \frac{u_c-u_a}{c-a}. \tag{22}$$

By (21) and (22), we obtain that

$$\inf_X \frac{u_b-u_a}{b-a}=\inf_X \frac{u_c-u_b}{c-b}.$$

Changing the letters (a,b,c) to (b,c,d) in the above identity, we obtain $\inf_X \frac{u_c-u_b}{c-b}=\inf_X \frac{u_d-u_c}{d-c}$, proving part (i) of the theorem. □

We make now a slight digression. For a weak geodesic segment, the correspondence $t\to u_t(x), x\in X$ is convex, and hence the pointwise boundary limits $u_\alpha(x)=\lim_{t\to\alpha}u_t(x)$ and $u_\beta(x)=\lim_{t\to\beta}u_t(x)$ exist even if they might not be bounded or ω-psh. As a corollary to our previous theorem, we obtain that, if α or β is finite, the boundary potentials u_α and u_β are bounded ω-psh functions, and the corresponding limits $\lim_{t\searrow\alpha}u_t=u_\alpha$ and $\lim_{t\nearrow\beta}u_t=u_\beta$ are uniform in X. In essence, this proves the following existence and uniqueness theorem, which is only a slight generalization of a result of Berndtsson [5].

Corollary 3.5. *Given bounded $u_0,u_1\in \mathrm{PSH}(X,\omega)$, there exists a unique solution to the following Dirichlet problem for locally bounded ω-psh functions $u: S_{01}\times X\to\mathbb{R}$:*

$$\begin{gathered}(\pi^*\omega+i\partial\bar{\partial}u)^{n+1}=0\\ u(t+ir,x)=u(t,x)x\in X, t\in(0,1), r\in\mathbb{R} \qquad (23)\\ \lim_{t\to 0}u(t,x)=u_0(x) \text{ and } \lim_{t\to 1}u(t,x)=u_1(x),\quad x\in X,\end{gathered}$$

where the boundary limits are assumed to be only pointwise in X.

Proof. By the method of § 2.1, there exists a solution u that is bounded and assumes the boundary values uniformly, not just pointwise. By our discussion above, any other locally bounded solution v is in fact globally bounded and assumes the boundary values uniformly as well. Hence, by an application of the standard maximum principle adapted to this setting [7, Theorem 6.4], the solution provided by Berndtsson's method is unique. □

We note here that the uniqueness part of the above theorem does not seem to follow from an application of the classical maximum principle alone (without Theorem 3.4), as this result does not work with pointwise boundary limits. Our observation that, in this problem, pointwise boundary limits are in fact uniform seems to be essential.

4. A construction of weak geodesic rays

Before we prove our main theorem about constructing weak geodesic rays, let us recall some notation introduced at the beginning of the paper. For $\phi,\psi\in\mathrm{PSH}(X,\omega)$, $\psi\leqslant\phi$ with ϕ bounded and ψ possibly unbounded, we define the following set of weak geodesic rays:

$$\mathcal{R}(\phi,\psi)=\Big\{v \text{ is a normalized weak geodesic ray with } \lim_{t\to 0}v_t=\phi \text{ and } \lim_{t\to\infty}v_t\geqslant\psi\Big\}.$$

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The set $\mathcal{R}(\phi, \psi)$ is always non-empty as it contains the constant ray $u = \phi$. By $(0, l) \ni t \to u^l_t \in \mathrm{PSH}(X, \omega)$ we denote the unique weak geodesic segments joining ϕ with $\max\{\phi - l, \psi\}, l > 0$.

Theorem 4.1. *For any $\phi, \psi \in \mathrm{PSH}(\omega)$ with ϕ bounded and $\psi \leqslant \phi$, the weak geodesic segments u^l form an increasing family. The upper semicontinuous regularization of their limit $v(\phi, \psi) = \mathrm{usc}(\lim_{l\to\infty} u^l)$ is a weak geodesic ray for which the following hold.*

(i) *$v(\phi, \psi)_t = \mathrm{usc}(\lim_{l\to\infty} u^l_t)$ for any $t \in (0, +\infty)$.*

(ii) *$v(\phi, \psi) \in \mathcal{R}(\phi, \psi)$; more precisely, $v(\phi, \psi) = \inf_{v\in\mathcal{R}(\phi,\psi)} v$. In particular, $t \to v(\phi, \psi)_t$ is constant if and only if $\mathcal{R}(\phi, \psi)$ contains only the constant ray ϕ.*

(iii) *$AM(v(\phi, \psi)_t) = AM(\phi) + c_\psi t$; in particular, $t \to v(\phi, \psi)_t$ is constant if and only if $\psi \in \mathcal{E}(X, \omega)$.*

The proof will be done in a sequence of lemmas.

Lemma 4.2. *The weak geodesic segments $\{u^l\}_{l>0}$ form an increasing family. The upper semicontinuous regularization of their limit $v(\phi, \psi) = \mathrm{usc}(\lim_{l\to\infty} u^l)$ is a weak geodesic ray for which*

$$v(\phi, \psi)_t = \mathrm{usc}\left(\lim_{l\to\infty} u^l_t\right), \quad t \in (0, +\infty). \tag{24}$$

Proof. It is clear that $t \to \gamma_t = \max\{\phi - t, \psi\}, t > 0$ is a weak subgeodesic ray. We define the following family of weak subgeodesic rays $\{t \to \gamma^l_t\}_{l\geqslant 0}$:

$$\gamma^l_t = \begin{cases} u^l_t & \text{if } 0 < t < l, \\ \gamma_t & \text{if } t \geqslant l. \end{cases} \tag{25}$$

By the sub-mean-value property of psh functions, it is clear that each γ^l_t is a weak subgeodesic ray. From Berndtsson's construction it also follows that this family is increasing in l. In particular, the family $\{u^l\}_{l>0}$ is also increasing in l. We denote $v(\phi, \psi) = \mathrm{usc}(\lim_{l\to\infty} u^l)$. It follows now from Bedford and Taylor theory that the Monge–Ampère measures $(\omega + i\partial\bar\partial u^l)^{n+1}$ converge weakly to $(\omega + i\partial\bar\partial v(\phi, \psi))^{n+1}$. This implies that

$$(\omega + i\partial\bar\partial v(\phi, \psi))^{n+1}|_{S_{0h}\times X} = 0$$

for any $h > 0$. Hence, $t \to v(\phi, \psi)_t$ is a weak geodesic ray.

We now prove (24). By Theorem 3.4, the limits $u^l_0 = \lim_{t\to 0} u^l_t = \phi$ and $u^l_l = \lim_{t\to l} u^l_t = \gamma_l$ are uniform in X, and hence it follows that

$$M_{u^l} = \sup_X \frac{u^l_l - u^l_0}{l} = \sup_X \frac{\max\{-l, \psi - \phi\}}{l} \leqslant 0,$$

$$m_{u^l} = \inf_X \frac{u^l_l - u^l_0}{l} = \inf_X \frac{\max\{-l, \psi - \phi\}}{l} \geqslant -1.$$

This implies that the u^l are uniformly Lipschitz in the t-variable, and hence so is their limit $\lim_{l\to\infty} u^l$. This in turn implies that $(\mathrm{usc}(\lim_{l\to\infty} u^l))_t = \mathrm{usc}(\lim_{l\to\infty} u^l_t)$, $t \in (0, +\infty)$, proving the desired result. □

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Lemma 4.3. *$v(\phi, \psi) \in \mathcal{R}(\phi, \psi)$; more precisely, $v(\phi, \psi) = \inf_{v \in \mathcal{R}(\phi, \psi)} v$. In particular, $t \to v(\phi, \psi)_t$ is constant if and only if $\mathcal{R}(\phi, \psi)$ contains only the constant ray ϕ.*

Proof. We start out by observing that $\max\{\phi - t, \psi\} = \gamma_t \leqslant u^l_t \leqslant \phi, t \in (0, l), l > 0$. After taking the limit $l \to +\infty$, then regularizing, it follows that

$$\gamma_t \leqslant v(\phi, \psi)_t \leqslant \phi, \quad t > 0. \tag{26}$$

This implies that $\lim_{t\to+\infty} v(\phi, \psi)_t \geqslant \psi$ and $v(\phi, \psi)_0 = \lim_{t\to 0} v(\phi, \psi)_t = \phi$. By Theorem 3.4, this last limit is uniform. Hence, using (26) again and Theorem 3.4, we find that

$$M_{v(\phi,\psi)} = \sup_X \frac{v(\phi, \psi)_t - \phi}{t} \geqslant \sup_X \frac{\gamma_t - \phi}{t}, \tag{27}$$

$$m_{v(\phi,\psi)} = \inf_X \frac{v(\phi, \psi)_t - \phi}{t} \geqslant \sup_X \frac{\gamma_t - \phi}{t}, \tag{28}$$

for any $t > 0$. Since $t \to v(\phi, \psi)_t$ is decreasing, it follows that $M_{v(\phi,\psi)} \leqslant 0$. By taking the limit $t \to +\infty$ in (27), we obtain that

$$M_{v(\phi,\psi)} = 0.$$

Turning to (28), we conclude that

$$m_{v(\phi,\psi)} \geqslant -1.$$

To see that $v(\phi, \psi) \in \mathcal{R}(\phi, \psi)$, it is enough to prove that either $t \to v(\phi, \psi)_t$ is constant or $m_{v(\phi,\psi)} = -1$. To conclude this, first we prove that $v(\phi, \psi) \leqslant h$ for any $h \in \mathcal{R}(\phi, \psi)$. If $h \in \mathcal{R}(\phi, \psi)$, then the limit $h_0 = \lim_{t\to 0} h_t = \phi$ is uniform, and since $m_h = -1$ we have $h_l \geqslant \max\{\phi - l, \psi\} = \gamma_l, l > 0$. By the maximum principle, this implies that $u^l_t \leqslant h_t$, $t \in [0, l]$. Letting $l \to +\infty$ in this estimate, then regularizing, we arrive at $v(\phi, \psi) \leqslant h$.

If $t \to v(\phi, \psi)_t$ is non-constant, then its normalization $\tilde{v} \in \mathcal{R}(\phi, \psi)$ is non-constant as well. Since $v_0(\phi, \psi) = \tilde{v}_0 = \phi$, $v(\phi, \psi) \leqslant \tilde{v}$ and $m_{\tilde{v}} = -1$, it follows from Theorem 3.4 that

$$m_{v(\phi,\psi)} = \inf_X \frac{v(\phi, \psi)_1 - \phi}{1} \leqslant \inf_X \frac{\tilde{v}_1 - \phi}{1} = -1.$$

This implies that $m_{v(\phi,\psi)} = -1$, finishing the proof. □

Lemma 4.4. *$AM(v(\phi, \psi)_t) = AM(\phi) + c_\psi t$; in particular, $v(\phi, \psi)$ is constant if and only if $\psi \in \mathcal{E}(X, \omega)$.*

Proof. Since the segments u^l are weak geodesics, by Theorems 2.1 and 3.4, we have

$$AM(u^l_t) = AM(\phi) + \frac{t}{l}(AM(\max\{\phi - l, \psi\}) - AM(\phi)), \quad t \in (0, l), l > 0.$$

As $l \to \infty$, it follows from (24) that the sequence u^l_t increases a.e. to $v(\phi, \psi)_t$, $t > 0$. Using this, Bedford and Taylor theory implies that $AM(u^l_t) \to AM(v(\phi, \psi)_t)$.

By Remark 2.6, the right-hand side of the last identity converges to $AM(\phi) + tc_\psi$. We conclude that

$$AM(v(\phi, \psi)_t) = AM(\phi) + tc_\psi, \quad t \in (0, +\infty).$$

Since $v(\phi, \psi)_t \leqslant \phi$, by Proposition 2.3, it follows that $t \to v(\phi, \psi)_t$ is constant if and only if $t \to AM(v(\phi, \psi)_t)$ is constant. This last condition is equivalent to $c_\psi = 0$. Hence, by Theorem 2.5, it follows that $t \to v(\phi, \psi)_t$ is constant if and only if $\psi \in \mathcal{E}(X, \omega)$. □

Corollary 4.5. *Suppose that $t \to u_t$ is a normalized weak geodesic ray such that $u_0 = \phi$ and $u_\infty = \psi$. Then for the normalized ray $t \to v(\phi, \psi)_t$ it is also true that $v(\phi, \psi)_\infty = \psi$.*

Proof. As $t \to u_t$ is normalized, it follows that $u \in \mathcal{R}(\phi, \psi)$. By Theorem 4.1(ii), it follows that $\psi \leqslant v(\phi, \psi)_t \leqslant u_t$ for all $t \in [0, \infty)$. From this, the conclusion follows. □

5. The inverse Legendre transform of a weak geodesic ray and $\mathcal{E}(X, \omega)$

We start this section by proving a result about the maximality of the Legendre transform of a weak geodesic ray.

Proposition 5.1. *Given a weak geodesic ray $(0, +\infty) \ni t \to \phi_t \in \mathrm{PSH}(X, \omega)$, its Legendre transform $\mathbb{R} \ni \tau \to \phi^*_\tau = \inf_{t \in (0,+\infty)}(\phi_t - t\tau) \in \mathrm{PSH}(X, \omega)$ satisfies*

$$\phi^*_\tau = P(\phi^*_\tau + C, \phi_0), \quad \tau \in \mathbb{R}, C > 0.$$

In particular, $P_{[\phi^*_\tau]}(\phi_0) = \phi^*_\tau$.

Proof. Fix $\tau \in \mathbb{R}$. The fact that $\phi^*_\tau \in \mathrm{PSH}(X, \omega)$ follows from Kiselman's minimum principle. Suppose that $\phi^*_\tau \not\equiv -\infty$, and fix $C > 0$. Since $\phi^*_\tau \leqslant \phi_0$, it results that $P(\phi^*_\tau + C, \phi_0) \geqslant \phi^*_\tau$. Hence we only have to prove that

$$P(\phi^*_\tau + C, \phi_0) \leqslant \phi^*_\tau.$$

Let $[0, 1] \ni t \to g^l_t, h_t \in \mathrm{PSH}(X, \omega), l \geqslant 0$ be the weak geodesic segments defined by the formulas

$$g^l_t = \phi_{tl} - tl\tau,$$
$$h_t = P(\phi^*_\tau + C, \phi_0) - Ct.$$

Then we have $h_0 \leqslant \phi_0 = \lim_{t\to 0} g^l_t = g^l_0$ and $h_1 \leqslant \phi^*_\tau \leqslant g^l_1$ for any $l \geqslant 0$. Hence, by the maximum principle, we have

$$h_t \leqslant g^l_t, \quad t \in [0, 1], l \geqslant 0.$$

Taking the infimum in the above estimate over $l \in [0, +\infty)$ and then taking the supremum over $t \in [0, 1]$, we obtain

$$P(\phi^*_\tau + C, \phi_0) \leqslant \phi^*_\tau.$$

Letting $C \to +\infty$, we obtain the last statement of the proposition. □

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The above proposition combined together with Theorems 1 and 2.9 gives the following result of independent interest.

Corollary 5.2. *The construction of § 2.4 gives rise to all geodesic rays* $[0,\infty) \ni t \to v_t \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$.

Proof. We have to argue that

$$\psi_\tau = \inf_{t \geqslant 0}(v_t - t\tau)$$

is a 'maximal' test curve. First we argue that $\tau \to \psi_\tau$ is a test curve. Concavity of $\tau \to \psi_\tau(z)$ follows from the convexity of the curves $t \to v_t(z)$. Condition (ii) of Definition (2.7) is a consequence of Theorem 1. Finally, as $v_0 = \psi_{-\infty}$, maximality of the test curve $\tau \to \psi_\tau$ follows from the previous proposition:

$$\psi_\tau = P_{[\psi_\tau]}(\psi_{-\infty}). \qquad \square$$

We will be interested in applying the above proposition in the case when $t \to \phi_t$ is a normalized geodesic ray and $\tau = 0$. In this case, we have

$$\phi_0^* = \inf_{t \in (0,+\infty)} \phi_t = \lim_{t \to +\infty} \phi_t =: \phi_\infty.$$

Using our construction of weak geodesic rays, we can now characterize $\mathcal{E}(X,\omega)$ in terms of envelopes.

Theorem 5.3. *Suppose that* $\psi \in \mathrm{PSH}(X,\omega)$ *and* $\phi \in \mathrm{PSH}(X,\omega) \cap C(X)$. *Then* $\psi \in \mathcal{E}(X,\omega)$ *if and only if*

$$P_{[\psi]}(\phi) = \phi.$$

Proof. Suppose that $\psi \in \mathrm{PSH}(X,\omega)$ is such that $P_{[\psi]}(\phi) = \phi$. There exists $D > 0$ such that $\psi - D < \phi$. Let $t \to v_t(\phi, \psi - D)$ be the normalized weak geodesic ray constructed in Theorem 4.1. As mentioned above, in this case we have

$$v_0^* = \inf_{t \in (0,+\infty)} v(\phi, \psi - D)_t = \lim_{t+\infty} v_t(\phi, \psi - D) = v_\infty.$$

We also have the following sequence of inequalities:

$$\phi = P_{[\psi]}(\phi) = P_{[\psi - D]}(\phi) \leqslant P_{[v_0^*]}(\phi) = v_0^* \leqslant \phi,$$

where we have used the fact that $\psi - D \leqslant v_\infty = v_0^*$ and Proposition 5.1. It follows from this that $\phi = v_\infty \leqslant v(\phi, \psi - D)_t \leqslant \phi, t > 0$; hence the normalized weak ray $t \to v(\phi, \psi - D)_t$ is constant, equal to ϕ. From this, using Theorem 4.1(iii), we conclude that $\psi \in \mathcal{E}(X,\omega)$.

To prove the other direction, suppose now that $\psi \in \mathcal{E}(X,\omega)$. Since $\psi - D \leqslant P_{[\psi]}(\phi) \leqslant \phi$, it follows from Theorem 2.5 that $P_{[\psi]}(\phi) \in \mathcal{E}(X,\omega)$. By Proposition 2.10, $P_{[\psi]}(\phi)$ is maximal with respect to ϕ; hence $P_{[\psi]}(\phi) \geqslant \phi$ a.e. with respect to $(\omega + i\partial\bar{\partial} P_{[\psi]}(\phi))^n$. Now, Proposition 2.4 implies that $P_{[\psi]}(\phi) \geqslant \phi$ holds everywhere. $\square$

We remark in passing that, when ψ is assumed to have small unbounded locus, then one can give a proof of the above result, using the maximum principle of [8] instead of Theorem 4.1.

6. Connection with analytic test configurations

Again, we consider the weak subgeodesic ray $t \to \gamma_t = \max\{\phi - t, \psi\}, t \geqslant 0$, and its Legendre transform $\tau \to \gamma^*_\tau$:

$$\gamma^*_\tau = \inf_{t \in [0,+\infty)} (\gamma_t - t\tau), \quad \tau \in \mathbb{R}.$$

We can easily verify that γ^*_τ is a test curve, as defined in § 2.4, which can be specifically given:

$$\gamma^*_\tau = \begin{cases} \phi & \tau \in (-\infty, -1), \\ (1+\tau)\psi - \tau\phi & \tau \in [-1, 0], \\ -\infty & \tau \in (0, \infty). \end{cases} \tag{29}$$

As pointed out by Ross and Witt-Nyström, this test curve can be seen as a generalization of a test curve arising from deformations to the normal cone.

In turns out that the geodesic ray constructed out of this test curve using the method of [20] is the same as the one constructed in Theorem 4.1.

Theorem 6.1. *Suppose that $\phi, \psi \in \mathrm{PSH}(\omega)$ with ϕ bounded and $\psi \leqslant \phi$. Then the weak geodesic ray $v(\phi, \psi)$ is the same as the ray obtained from the special test curve $\tau \to \gamma^*_\tau$ using the method of [20].*

Proof. Since $\tau \to \gamma^*_\tau$ is a test curve, one can apply the method of [20] (see § 2.4) to produce a weak geodesic ray

$$r_t = \mathrm{usc}\bigg(\sup_{\tau \in \mathbb{R}} (P_{[\gamma^*_\tau]}(\phi) + t\tau)\bigg), \quad t > 0.$$

We will prove that $r \in \mathcal{R}(\phi, \psi)$ and $r \leqslant u$ for any $u \in \mathcal{R}(\phi, \psi)$. By Theorem 4.1(ii), this is enough to conclude that $r = v(\phi, \psi)$. Much of the remaining argument is similar to the proof of Lemma 4.3.

If $u \in \mathcal{R}(\phi, \psi)$, then $\max\{\phi - t, \psi\} = \gamma_t \leqslant u_t, t \in [0, +\infty)$, and hence also $\gamma^*_\tau \leqslant u^*_\tau$, $\tau \in \mathbb{R}$. This implies that

$$P_{[\gamma^*_\tau]}(\phi) \leqslant P_{[u^*_\tau]}(\phi) = u^*_\tau,$$

where the last identity follows from Proposition 5.1. Hence by the involution property of Legendre transforms we arrive at

$$r_t = \mathrm{usc}\bigg(\sup_{\tau \in \mathbb{R}} (P_{[\gamma^*_\tau]}(\phi) + t\tau)\bigg) \leqslant \mathrm{usc}\bigg(\sup_{\tau \in \mathbb{R}} (u^*_\tau + t\tau)\bigg) = u_t, \quad t \in (0, +\infty).$$

Now we prove that $r \in \mathcal{R}(\phi, \psi)$. Since $\gamma_t \leqslant r_t \leqslant \phi, t > 0$, it follows that $r_0 = \lim_{t \to 0} r_t = \phi$ and $\lim_{t \to +\infty} r_t \geqslant \psi$. As in the proof of Lemma 4.3, it also follows that $M_r = 0$ and $m_r \geqslant -1$. We have to argue that $t \to r_t$ is either constant or normalized.

If $t \to r_t$ is non-constant, then its normalization $\tilde{r} \in \mathcal{R}(\phi, \psi)$ is non-constant as well. Since $r \leqslant \tilde{r}$, $r_0 = \tilde{r}_0 = \phi$ and $m_{\tilde{r}} = -1$, we obtain that $m_r \leqslant -1$. This concludes the proof. □

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References

1. C. Arezzo and G. Tian, Infnite geodesic rays in the space of Kähler potentials, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **2**(4) (2003), 617–630.
2. E. Bedford and B. A. Tayor, A new capacity for plurisubharmonic functions, *Acta Math.* **149** (1982), 1–40.
3. R. Berman, S. Boucksom, V. Guedj and A. Zeriahi, A variational approach to complex Monge–Ampère equations, *Publ. Math. Inst. Hautes Études Sci.* **117** (2013), 179–245.
4. B. Berndtsson, Probability measures related to geodesics in the space of Kähler metrics, Preprint, 2009, arXiv:0907.1806.
5. B. Berndtsson, A Brunn–Minkowski type inequality for Fano manifolds and some uniqueness theorems in Kähler geometry, *Invent. Math.* **200**(1) (2015), 149–200.
6. Z. Blocki, Uniqueness and stability for the Monge–Ampère equation on compact Kähler manifolds, *Indiana Univ. Math. J.* **52** (2003), 1697–1702.
7. Z. Blocki, *The complex Monge–Ampère equation in Kähler geometry, course given at CIME Summer School in Pluripotential Theory, Cetraro, Italy, July 2011* (ed. F. Bracci and J. E. Fornæss), Lecture Notes in Mathematics, Volume 2075, pp. 95–142 (Springer, 2013).
8. S. Boucksom, P. Eyssidieux, V. Guedj and A. Zeriahi, Monge–Ampère equations in big cohomology classes, *Acta Math.* **205** (2010), 199–262.
9. M. M. Branker and M. Stawiska, Weighted pluripotential theory on complex Kähler manifolds, *Ann. Polon. Math.* **95**(2) (2009), 163–177. arXiv:0801.3015.
10. X. X. Chen, The space of Kähler metrics, *J. Differential Geom.* **56**(2) (2000), 189–234.
11. X. X. Chen, Space of Kähler metrics III: on the lower bound of the Calabi energy and geodesic distance, *Invent. Math.* **175**(3) (2009), 453–503.
12. X. X. Chen and Y. Tang, Test configuration and geodesic rays, Géometrie differentielle, physique mathématique, mathématiques et société. I, *Astérisque* **321** (2008), 139–167.
13. T. Darvas, Envelopes and geodesics in spaces of Kähler potentials, Preprint, 2014, arXiv:1401.7318.
14. T. Darvas and L. Lempert, Weak geodesics in the space of Kähler metrics, *Math. Res. Lett.* **19** (2013), 1127–1135.
15. S. K. Donaldson, Symmetric spaces, in *Kähler geometry and Hamiltonian dynamics*, American Mathematical Society Translations, Series 2, Volume 196, pp. 13–33 (American Mathematical Society, Providence RI, 1999).
16. P. Eyssidieux, V. Guedj and A. Zeriahi, Singular Kähler–Einstein metrics, *J. Amer. Math. Soc.* **22** (2009), 607–639.
17. V. Guedj and A. Zeriahi, The weighted Monge–Ampère energy of quasiplurisubharmonic functions, *J. Funct. Anal.* **250**(2) (2007), 442–482.
18. T. Mabuchi, Some symplectic geometry on compact Kähler manifolds I, *Osaka J. Math.* **24** (1987), 227–252.
19. D. H. Phong and J. Sturm, Test configurations for K-stability and geodesic rays, *J. Symplectic Geom.* **5**(2) (2007), 221–247.

20. J. Ross and D. Witt-Nyström, Analytic test configurations and geodesic rays, *J. Symplectic Geom.* **12**(1) (2014), 125–169.
21. J. Ross and D. Witt-Nyström, Envelopes of positive metrics with prescribed singularities, Preprint, 2012, arXiv:1210.2220.
22. Y. A. Rubinstein and S. Zelditch, The Cauchy problem for the homogeneous Monge–Ampère equation, III. Lifespan, Preprint, 2012, arXiv:1205.4793.
23. S. Semmes, Complex Monge–Ampère and symplectic manifolds, *Amer. J. Math.* **114** (1992), 495–550.

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