

Counting the Number of Integral Points in General n -Dimensional Tetrahedra and Bernoulli Polynomials

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Abstract. Recently there has been tremendous interest in counting the number of integral points in n -dimensional tetrahedra with non-integral vertices due to its applications in primality testing and factoring in number theory and in singularities theory. The purpose of this note is to formulate a conjecture on sharp upper estimate of the number of integral points in n -dimensional tetrahedra with non-integral vertices. We show that this conjecture is true for low dimensional cases as well as in the case of homogeneous n -dimensional tetrahedra. We also show that the Bernoulli polynomials play a role in this counting.

1 Introduction

Let $\Delta(a_1, \dots, a_n)$ be an n -dimensional tetrahedron with nonintegral vertices described by

$$(1.1) \quad \frac{x_1}{a_1} + \frac{x_2}{a_2} + \dots + \frac{x_n}{a_n} \leq 1, \quad x_1 \geq 0, \dots, x_n \geq 0$$

where $a_1 \geq a_2 \geq \dots \geq a_n$ are positive real numbers. Let

$$(1.2) \quad Q_{(a_1, \dots, a_n)} = \#\left\{ (x_1, \dots, x_n) \in (\mathbb{Z}_+ \cup \{0\})^n : \sum_{i=1}^n \frac{x_i}{a_i} \leq 1 \right\}$$

$$(1.3) \quad P_{(a_1, \dots, a_n)} = \#\left\{ (x_1, \dots, x_n) \in \mathbb{Z}_+^n : \sum_{i=1}^n \frac{x_i}{a_i} \leq 1 \right\}.$$

Let $b_i = a_i(1 - \sum_{j=1}^n \frac{1}{a_j})$, $1 \leq i \leq n$. Then

$$(1.4) \quad P_{(a_1, \dots, a_n)} = Q_{(b_1, \dots, b_n)}.$$

In number theory, people are interested in sharp estimates of $Q_{(a_1, \dots, a_n)}$ for application in primality testing and factoring. Given a set \mathcal{P} of primes $p_1 < p_2 < \dots < p_n \leq y$, number theorists want to count the number of integers $m \leq x$ where

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$m = p_1^{\ell_1} p_2^{\ell_2} \cdots p_n^{\ell_n}$, $x = y^u$ for all $u \geq 2$. This is equivalent to counting the number of $(\ell_1, \dots, \ell_n) \in (\mathbb{Z}_+ \cup \{0\})^n$ such that $\ell_1 \log p_1 + \cdots + \ell_n \log p_n \leq \log x$, which is also equivalent to counting the number of $(\ell_1, \dots, \ell_n) \in (\mathbb{Z}_+ \cup \{0\})^n$ such that

$$\frac{\ell_1}{a_1} + \frac{\ell_2}{a_2} + \cdots + \frac{\ell_n}{a_n} \leq 1 \quad \text{where } a_i = \frac{\log x}{\log p_i} = \frac{u \log y}{\log p_i}.$$

This is of course the problem of computing $Q_{(a_1, \dots, a_n)}$.

Let $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a complex analytic function with isolated critical point at the origin. Let $V = \{z \in \mathbb{C}^n : f(z) = 0\}$. The geometric genus p_g of the singularity $(V, 0)$ is defined to be $\dim \Gamma(V - \{0\}, \Omega^{n-1})/L^2(V - \{0\}, \Omega^{n-1})$, where Ω^{n-1} is the sheaf of germs of holomorphic $(n - 1)$ -forms on $V - \{0\}$. It is well known that geometric genus is an important numerical invariant which measures the complexity of the singularity $(V, 0)$.

Let $f(z_1, \dots, z_n) = \sum a_\lambda z^\lambda$, where $z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}$, be the power series expansion of f . The Newton boundary $\Gamma(f)$ is the union of the compact faces of $\Gamma_+(f)$ where $\Gamma_+(f)$ is the convex hull of the union of the subsets $\{\lambda + (\mathbb{R}_+)^n\}$ for λ such that $a_\lambda \neq 0$. The Newton polyhedron $\Gamma_-(f)$ of f is the cone over $\Gamma(f)$ with cone point at 0. For each closed face $\Delta \subseteq \Gamma(f)$, we define $f_\Delta(z) = \sum a_\lambda z^\lambda$, $\lambda \in \Delta$. f is nondegenerate if f_Δ has no critical point in $(\mathbb{C}^*)^n$ for any $\Delta \in \Gamma(f)$ where $\mathbb{C}^* = \mathbb{C} - \{0\}$.

Theorem 1.1 (Merle-Teissier) *Let $(V, 0)$ be an isolated hypersurface singularity defined by a nondegenerate holomorphic function $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$. Then the geometric genus $p_g = \#\{P \in \mathbb{Z}_+^n \cap \Gamma_-(f)\}$.*

We say that $f(z_1, \dots, z_n)$ is weighted homogeneous of type (w_1, \dots, w_n) , where w_1, \dots, w_n are fixed positive rational numbers, if f can be expressed as a linear combination of monomials $z_1^{i_1} \cdots z_n^{i_n}$ for which $i_1/w_1 + \cdots + i_n/w_n = 1$. Therefore for an isolated singularity defined by a weighted homogeneous polynomial of type (w_1, \dots, w_n) , computing the geometric genus is equivalent to computing the number $P_{(w_1, \dots, w_n)}$, i.e., the number of positive integral points in n -dimensional tetrahedra $\Delta_{(w_1, \dots, w_n)}$.

Computation of $Q_{(a_1, \dots, a_n)}$ has received attention by a lot of distinguished mathematicians. Hardy and Littlewood wrote a series of papers for $n = 2$ [Ha-Li1] [Ha-Li2] [Ha-Li3] [Ha-Li4]. D. Spencer [Sp1] [Sp2] followed up the efforts of Hardy and Littlewood and wrote two papers on this subject. In recent years, there are tremendous activities in finding the exact formula for $Q_{(a_1, \dots, a_n)}$ or $P_{(a_1, \dots, a_n)}$ for a_1, \dots, a_n integers, see [Mo1], [Mor] [Po], [Ca-Sh], [Br-Ve2], [Di-Ro], [Ka-Kh]. The exact formula is complicated. It involves the generalized Dedekind sum. It is difficult to tell how large $P_{(a_1, \dots, a_n)}$ is from the exact formula. Therefore one would like to get a sharp upper estimate of $P_{(a_1, \dots, a_n)}$ in terms of a polynomial in a_1, \dots, a_n . Such a sharp upper polynomial estimate of $P_{(a_1, \dots, a_n)}$ is important because it would have application in the following Durfee Conjecture [Du].

Durfee Conjecture (1978) Let $(V, 0)$ be an isolated hypersurface singularity defined by a holomorphic function $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$. Let

$$\mu = \dim \mathbb{C}\{z_1, z_2, \dots, z_n\} / (f_{z_1}, f_{z_2}, \dots, f_{z_n})$$

be the Milnor number of the singularity. Then $n! p_g \leq \mu$ where p_g is the geometric genus of $(V, 0)$.

If $f(z)$ is a weighted homogeneous polynomial of type (w_1, \dots, w_n) , then the Milnor number μ is given by $\mu = (w_1 - 1)(w_2 - 1) \cdots (w_n - 1)$. Therefore Durfee Conjecture is a special case of the following conjecture.

Conjecture 1.1 Let a_1, \dots, a_n be positive real numbers greater than or equal to two. Then

$$(1.5) \quad n! P_{(a_1, \dots, a_n)} \leq (a_1 - 1)(a_2 - 1) \cdots (a_n - 1).$$

The estimate in the above conjecture is sharper than the following polynomial estimate (1.7) provided by number theorist. Attach a unit cube to the right and above each lattice point of $\Delta(a_1, \dots, a_n)$. Then

$$(1.6) \quad \begin{aligned} Q_{(a_1, \dots, a_n)} &= \Sigma \text{ volume of the unit cube attached to each lattice point} \\ &\leq \text{volume of } \left\{ (x_1, \dots, x_n) \in \mathbf{R}_+^n : \sum_{i=1}^n \frac{x_i - 1}{a_i} \leq 1 \right\} \\ &= \frac{1}{n!} \left(\prod_{i=1}^n a_i \right) \left(1 + \sum_{j=1}^n \frac{1}{a_j} \right)^n. \end{aligned}$$

Hence by (1.4) and (1.6), we have

$$(1.7) \quad \begin{aligned} P_{(a_1, \dots, a_n)} &= Q_{(b_1, \dots, b_n)} \\ &\leq \frac{1}{n!} \left(\prod_{i=1}^n b_i \right) \left(1 + \sum_{j=1}^n \frac{1}{b_j} \right)^n \\ &= \frac{1}{n!} \prod_{i=1}^n a_i. \end{aligned}$$

The estimate of $P_{(a_1, \dots, a_n)}$ given by (1.5) is nice. However it is not sharp enough to provide a solution of the following thirty year old problem in singularities theory.

Problem Let $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a complex analytic function with isolated critical point at the origin. Find an intrinsic characterization for f to be a homogeneous polynomial.

In order to solve the above problem, we need a sharp upper polynomial estimate of $P_{(a_1, \dots, a_n)}$ such that equality holds if and only if $a_1 = a_2 = \dots = a_n = \text{integer}$. The intrinsic characterization of homogeneous polynomial problem was solved for $n = 3$ by Xu and Yau [Xu-Ya1] and for $n = 4$ by Lin and Yau [Li-Ya1]. The purpose of this paper is to formulate a conjectural sharp polynomial upper estimate for $P_{(a_1, \dots, a_n)}$ such that equality holds if and only if $a_1 = a_2 = \dots = a_n = \text{integer}$. In Section 2, we show that our conjecture is true for $n = 3, 4$ and 5. In Section 3, we give an exact formula for $P_{(a_1, \dots, a_n)}$ for homogeneous n -dimensional tetrahedra (i.e., $a_1 = \dots = a_n$). We show that our conjecture is also true in this case. In Section 4 we discuss the possible role of Bernoulli polynomials in this problem.

2 Sharp Polynomial Upper Estimate

Before we formulate our sharp polynomial upper estimate conjecture, it is convenient to introduce some notations. Let a, a_1, \dots, a_n be positive real numbers greater than or equal to $n - 1$. We shall denote

$$(2.1) \quad S_k^{n-1} = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n-1} i_1 i_2 \dots i_k, \quad S_0^{n-1} = 1, \quad S_{n-1}^{n-1} = 1 \cdot 2 \cdot \dots \cdot (n-1)$$

where i_1, i_2, \dots, i_k are integers. Then

$$\begin{aligned} & a(a-1)(a-2) \dots (a-(n-1)) \\ &= a^n - \left(\sum_{i_1=1}^{n-1} i_1 \right) a^{n-1} + (-1)^2 \sum_{1 \leq i_1 < i_2 \leq n-1} i_1 i_2 a^{n-2} \\ (2.2) \quad & + \dots + (-1)^k \sum_{1 \leq i_1 < \dots < i_k \leq n-1} i_1 i_2 \dots i_k a^{n-k} + \dots + (-1)^{n-1} \left(\prod_{i=1}^{n-1} i \right) a \\ &= a^n + (-1) S_1^{n-1} a^{n-1} + (-1)^2 S_2^{n-1} a^{n-2} + \dots + (-1)^k S_k^{n-1} a^{n-k} \\ & + \dots + (-1)^{n-1} S_{n-1}^{n-1} a. \end{aligned}$$

We shall denote

$$(2.3) \quad A_{n-k}^n = \left(\prod_{i=1}^n a_i \right) \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \frac{1}{a_{i_1} a_{i_2} \dots a_{i_k}}$$

$$(2.4) \quad A_n^n = \prod_{i=1}^n a_i, \quad A_0^n = 1.$$

Observe that A_{n-k}^n is a polynomial in a_1, \dots, a_n of degree $n - k$.

The following is our conjecture on sharp polynomial estimate for $P_{(a_1, \dots, a_n)}$.

Main Conjecture Let $P_{(a_1, \dots, a_n)} = \#\{(x_1, \dots, x_n) \in \mathbb{Z}_+^n : \frac{x_1}{a_1} + \dots + \frac{x_n}{a_n} \leq 1\}$, where $a_1 \geq a_2 \geq \dots \geq a_n \geq n - 1, n \geq 3$. Then

$$\begin{aligned}
 (2.5) \quad n! P_{(a_1, \dots, a_n)} &\leq \prod_{i=1}^n a_i + (-1) \frac{S_1^{n-1}}{n} \left(\prod_{i=1}^n a_i \right) \left(\sum_{k=1}^n \frac{1}{a_k} \right) \\
 &\quad + (-1)^2 \frac{S_2^{n-1}}{\binom{n-1}{1}} \left(\prod_{i=1}^{n-1} a_i \right) \left(\sum_{k=1}^{n-1} \frac{1}{a_k} \right) \\
 &\quad + (-1)^3 \frac{S_3^{n-1}}{\binom{n-1}{2}} \left(\prod_{i=1}^{n-1} a_i \right) \left(\sum_{1 \leq i_1 < i_2 \leq n-1} \frac{1}{a_{i_1} a_{i_2}} \right) \\
 &\quad + (-1)^4 \frac{S_4^{n-1}}{\binom{n-1}{3}} \left(\prod_{i=1}^{n-1} a_i \right) \left(\sum_{1 \leq i_1 < i_2 < i_3 \leq n-1} \frac{1}{a_{i_1} a_{i_2} a_{i_3}} \right) \\
 &\quad + \dots + (-1)^{k+1} \frac{S_{k+1}^{n-1}}{\binom{n-1}{k}} \left(\prod_{i=1}^{n-1} a_i \right) \left(\sum_{1 \leq i_1 < \dots < i_k \leq n-1} \frac{1}{a_{i_1} a_{i_2} \dots a_{i_k}} \right) \\
 &\quad + \dots + (-1)^{n-1} \frac{S_{n-1}^{n-1}}{\binom{n-1}{n-2}} (a_1 + a_2 + \dots + a_{n-1}) \\
 &= A_n^n + (-1) \frac{S_1^{n-1}}{n} A_{n-1}^n + (-1)^2 \frac{S_2^{n-1}}{\binom{n-1}{1}} A_{n-2}^{n-1} + (-1)^3 \frac{S_3^{n-1}}{\binom{n-1}{2}} A_{n-3}^{n-1} \\
 &\quad + (-1)^4 \frac{S_4^{n-1}}{\binom{n-1}{3}} A_{n-4}^{n-1} + \dots + (-1)^{k+1} \frac{S_{k+1}^{n-1}}{\binom{n-1}{k}} A_{n-k-1}^{n-1} \\
 &\quad + \dots + (-1)^{n-1} \frac{S_{n-1}^{n-1}}{\binom{n-1}{n-2}} A_1^{n-1},
 \end{aligned}$$

and the equality holds if and only if $a_1 = a_2 = \dots = a_n = \text{integer}$.

For $n = 3$, the Main Conjecture asserts that for $a_1 \geq a_2 \geq a_3 \geq 2$,

$$\begin{aligned}
 3! P_{(a_1, a_2, a_3)} &\leq A_3^3 - \frac{S_1^2}{3} A_2^3 + \frac{S_2^2}{\binom{2}{1}} A_1^2 \\
 &= a_1 a_2 a_3 - \frac{1+2}{3} a_1 a_2 a_3 \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right) + \frac{2}{2} a_1 a_2 \left(\frac{1}{a_1} + \frac{1}{a_2} \right) \\
 &= a_1 a_2 a_3 - (a_1 a_2 + a_1 a_3 + a_2 a_3) + (a_1 + a_2)
 \end{aligned}$$

with equality if and only if $a_1 = a_2 = a_3 = \text{integer}$. This is the main result proved by Xu and Yau in their paper [Xu-Ya1].

For $n = 4$, the Main Conjecture asserts that for $a_1 \geq a_2 \geq a_3 \geq a_4 \geq 3$,

$$\begin{aligned}
 4! P_{(a_1, a_2, a_3, a_4)} &\leq A_4^4 + (-1) \frac{S_1^3}{4} A_3^4 + (-1)^2 \frac{S_2^3}{\binom{3}{1}} A_2^3 + (-1)^3 \frac{S_3^3}{\binom{3}{2}} A_1^3 \\
 &= a_1 a_2 a_3 a_4 - \frac{1+2+3}{4} a_1 a_2 a_3 a_4 \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} \right) \\
 &\quad + \frac{1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3}{3} a_1 a_2 a_3 \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right) \\
 &\quad - \frac{1 \cdot 2 \cdot 3}{3} a_1 a_2 a_3 \left(\frac{1}{a_1 a_2} + \frac{1}{a_1 a_3} + \frac{1}{a_2 a_3} \right) \\
 &= a_1 a_2 a_3 a_4 - \frac{3}{2} (a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4) \\
 &\quad + \frac{11}{3} (a_1 a_2 + a_1 a_3 + a_2 a_3) - 2(a_1 + a_2 + a_3)
 \end{aligned}$$

with equality if and only if $a_1 = a_2 = a_3 = a_4 = \text{integer}$. Xu and Yau [Xu-Ya3] proved that if $a_1 \geq a_2 \geq a_3 \geq a_4 \geq 2$ and $P_{(a_1, a_2, a_3, a_4)} > 0$, then the above result is true. Lin and Yau [Li-Ya1] proved that the condition $P_{(a_1, a_2, a_3, a_4)} > 0$ can be removed if we assume $a_1 \geq a_2 \geq a_3 \geq a_4 \geq 3$. Therefore our Main Conjecture is true for $n = 4$.

For $n = 5$, the Main Conjecture asserts that for $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq 4$.

$$\begin{aligned}
 5! P_{(a_1, a_2, a_3, a_4, a_5)} &\leq A_5^5 + (-1) \frac{S_1^4}{5} A_4^5 + \frac{S_2^4}{\binom{4}{1}} A_3^4 - \frac{S_3^4}{\binom{4}{2}} A_2^4 + \frac{S_4^4}{\binom{4}{3}} A_1^4 \\
 &= a_1 a_2 a_3 a_4 a_5 - \frac{1+2+3+4}{5} a_1 a_2 a_3 a_4 a_5 \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \right) \\
 &\quad + \frac{(1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4)}{4} a_1 a_2 a_3 a_4 \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} \right) \\
 &\quad - \frac{(1 \cdot 2 \cdot 3 + 1 \cdot 2 \cdot 4 + 1 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4)}{\binom{4}{2}} a_1 a_2 a_3 a_4 \\
 &\quad \quad \cdot \left(\frac{1}{a_1 a_2} + \frac{1}{a_1 a_3} + \frac{1}{a_1 a_4} + \frac{1}{a_2 a_3} + \frac{1}{a_2 a_4} + \frac{1}{a_3 a_4} \right) \\
 &\quad + \frac{1 \cdot 2 \cdot 3 \cdot 4}{4} a_1 a_2 a_3 a_4 \left(\frac{1}{a_1 a_2 a_3} + \frac{1}{a_1 a_2 a_4} + \frac{1}{a_2 a_3 a_4} \right) \\
 &= a_1 a_2 a_3 a_4 a_5 - 2(a_1 a_2 a_3 a_4 + a_1 a_2 a_3 a_5 + a_1 a_2 a_4 a_5 + a_1 a_3 a_4 a_5 + a_2 a_3 a_4 a_5) \\
 &\quad + \frac{35}{4} (a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4) \\
 &\quad - \frac{25}{3} (a_1 a_2 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4) + 6(a_1 + a_2 + a_3 + a_4)
 \end{aligned}$$

with equality if and only if $a_1 = a_2 = a_3 = a_4 = a_5 = \text{integer}$. This is the main result proved by Lin and Yau in their paper [Li-Ya3].

3 Number of Positive Integral Points in a Homogeneous n -Dimensional Tetrahedron

In this section we shall prove that our Main Conjecture in Section 2 holds for homogeneous n -dimensional tetrahedra $\Delta(a, a, \dots, a)$. We first start with an elementary summation by parts lemma.

Lemma 3.1 *Let $\{u_k\}, \{v_k\}, k = 1, \dots, n$ be two sequences of numbers. Then*

$$(3.1) \quad \sum_{k=1}^n u_k v_k = u_n \sum_{k=1}^n v_k - \sum_{r=2}^n \left[(u_r - u_{r-1}) \sum_{k=1}^{r-1} v_k \right].$$

Proof We shall prove (3.1) by induction. (3.1) is obviously true for $n = 2$. Assume that (3.1) holds for $n - 1$. Then

$$\begin{aligned} \sum_{k=1}^n u_k v_k &= \sum_{k=1}^{n-1} u_k v_k + u_n v_n \\ &= u_{n-1} \sum_{k=1}^{n-1} v_k - \sum_{r=2}^{n-1} \left[(u_r - u_{r-1}) \sum_{k=1}^{r-1} v_k \right] + u_n v_n \\ &= u_n \sum_{k=1}^n v_k - u_n \sum_{k=1}^{n-1} v_k + u_{n-1} \sum_{k=1}^{n-1} v_k - \sum_{r=2}^{n-1} \left[(u_r - u_{r-1}) \sum_{k=1}^{r-1} v_k \right] \\ &= u_n \sum_{k=1}^n v_k - (u_n - u_{n-1}) \sum_{k=1}^{n-1} v_k - \sum_{r=2}^{n-1} \left[(u_r - u_{r-1}) \sum_{k=1}^{r-1} v_k \right] \\ &= u_n \sum_{k=1}^n v_k - \sum_{r=2}^n \left[(u_r - u_{r-1}) \sum_{k=1}^{r-1} v_k \right] \quad \blacksquare \end{aligned}$$

Theorem 3.1 *Let $n \geq 2$ be a positive integer and $a \geq n$. Let P_n be the number of positive integral solutions of $\frac{x_1}{a} + \frac{x_2}{a} + \dots + \frac{x_n}{a} \leq 1$, i.e., $P_n = \#\{(x_1, x_2, \dots, x_n) \in \mathbb{Z}_+^n : \frac{x_1}{a} + \frac{x_2}{a} + \dots + \frac{x_n}{a} \leq 1\}$. Let $[a]$ be the greatest integer less than or equal to a . Then*

$$(3.2) \quad \begin{aligned} n! P_n &= [a]([a] - 1)([a] - 2) \dots ([a] - n + 1) \\ &\leq a(a - 1)(a - 2) \dots (a - n + 1) \end{aligned}$$

with equality if and only if a is an integer.

Proof For $n = 2$, we have $\frac{x_1}{a} + \frac{x_2}{a} \leq 1$. From the level $x_2 = 1$ to the level $x_2 = [a] - 1$, we have the following positive integral points:

$$\begin{aligned} x_2 = 1 & : (1, 1), (2, 1), \dots, ([a] - 1, 1) \\ x_2 = 2 & : (1, 2), (2, 2), \dots, ([a] - 2, 2) \\ & \vdots \\ x_2 = [a] - 1 & : (1, [a] - 1). \end{aligned}$$

Hence $P_2 = 1 + 2 + \dots + ([a] - 1) = \frac{[a]([a]-1)}{2}$, i.e., $2! P_2 = [a]([a] - 1)$.

Now assume that (3.2) holds for any integer n . Consider a homogeneous $(n + 1)$ -dimensional tetrahedron $\frac{x_1}{a} + \frac{x_2}{a} + \dots + \frac{x_{n+1}}{a} \leq 1$.

From the level $x_{n+1} = 1$ to the level $x_{n+1} = [a] - n$, we have the following homogeneous n -dimensional tetrahedron:

$$\begin{aligned} x_{n+1} = 1 & : \frac{x_1}{a-1} + \frac{x_2}{a-1} + \dots + \frac{x_n}{a-1} \leq 1 \\ x_{n+1} = 2 & : \frac{x_1}{a-2} + \frac{x_2}{a-2} + \dots + \frac{x_n}{a-2} \leq 1 \\ & \vdots \\ x_{n+1} = [a] - n & : \frac{x_1}{a-[a]+n} + \frac{x_2}{a-[a]+n} + \dots + \frac{x_n}{a-[a]+n} \leq 1. \end{aligned}$$

Hence by induction hypothesis we have

$$\begin{aligned} n! P_{n+1} & = \sum_{k=n}^{[a]-1} k(k-1) \dots (k-(n-1)) \\ & = \sum_{k=1}^{[a]-1} k(k-1) \dots (k-(n-1)). \end{aligned}$$

Now we apply Lemma 3.1 with $u_k = (k - 1)(k - 2) \dots (k - (n - 1))$, $v_k = k$. Then $\sum_{k=1}^{r-1} v_k = \sum_{k=1}^{r-1} k = \frac{r(r-1)}{2}$

$$\begin{aligned} u_r - u_{r-1} & = (r - 1)(r - 2) \dots (r - (n - 1)) - (r - 2)(r - 3) \dots (r - n) \\ & = (r - 2)(r - 3) \dots (r - n + 1)((r - 1) - (r - n)) \\ & = (n - 1)(r - 2)(r - 3) \dots (r - n + 1). \end{aligned}$$

From (3.1), we have

$$\begin{aligned} & \sum_{k=1}^{[a]-1} k(k-1)\cdots(k-n+1) \\ &= u_{[a]-1} \frac{[a]([a]-1)}{2} - \sum_{r=2}^{[a]-1} (n-1)(r-2)(r-3)\cdots(r-n+1) \frac{r(r-1)}{2} \\ &= ([a]-2)([a]-3)\cdots([a]-n) \frac{[a]([a]-1)}{2} \\ &\quad - \frac{n-1}{2} \sum_{k=1}^{[a]-1} k(k-1)(k-2)\cdots(k-n+1). \end{aligned}$$

Hence we have

$$\begin{aligned} \left(1 + \frac{n-1}{2}\right) \sum_{k=1}^{[a]-1} k(k-1)(k-2)\cdots(k-n+1) \\ = \frac{1}{2} [a]([a]-1)([a]-2)\cdots([a]-n), \end{aligned}$$

which implies

$$(n+1) \sum_{k=1}^{[a]-1} k(k-1)(k-2)\cdots(k-n+1) = [a]([a]-1)([a]-2)\cdots([a]-n).$$

Therefore we have

$$\begin{aligned} n! P_{n+1} &= \sum_{k=1}^{[a]-1} k(k-1)(k-2)\cdots(k-n+1) \\ &= \frac{1}{n+1} [a]([a]-1)([a]-2)\cdots([a]-n) \end{aligned}$$

i.e., $(n+1)! P_{n+1} = [a]([a]-1)([a]-2)\cdots([a]-n)$. ■

Corollary 3.1 *The Main Conjecture in Section 2 is true for n -dimensional homogeneous tetrahedra $\Delta(a, a, \dots, a)$.*

Proof Set $a_1 = a_2 = \dots = a_n = a$ in the Main Conjecture. We get

$$\begin{aligned} n! P_{(a,\dots,a)} &\leq a^n + (-1)S_1^{n-1}a^{n-1} + (-1)^2S_2^{n-1}a^{n-2} + (-1)^3S_3^{n-1}a^{n-3} \\ &\quad + \dots + (-1)^{k+1}S_{k+1}^{n-1}a^{n-k-1} + \dots + (-1)^{n-1}S_{n-1}^{n-1}a \\ &= a(a-1)(a-2)\cdots(a-(n-1)) \quad \text{by (2.2)} \end{aligned}$$

This is exactly Theorem 3.1. ■

4 Bernoulli Polynomials and Counting the Number of Integral Points in an n -Dimensional Homogeneous Tetrahedron

Let us define the Bernoulli polynomials, $B_0(x), B_1(x), B_2(x), \dots$, by the following equation

$$(4.1) \quad \frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n.$$

Then (4.1) implies

$$(4.2) \quad \begin{aligned} \sum_{n=0}^{\infty} \frac{B_n(x+1)}{n!} t^n &= \frac{te^{t(x+1)}}{e^t - 1} = \frac{te^{tx}}{e^t - 1} [(e^t - 1) + 1] \\ &\Rightarrow \sum_{n=0}^{\infty} \frac{B_n(x+1) - B_n(x)}{n!} t^n = te^{tx} = \sum_{n=0}^{\infty} \frac{x^n}{n!} t^{n+1} \\ &\Rightarrow \frac{B_n(x+1) - B_n(x)}{n!} = \frac{x^{n-1}}{(n-1)!} \\ &\Rightarrow B_n(x+1) - B_n(x) = nx^{n-1}. \end{aligned}$$

Differentiating (4.1) with respect to x , we get

$$(4.3) \quad \begin{aligned} \frac{t^2 e^{tx}}{e^t - 1} &= \sum_{n=0}^{\infty} \frac{B'_n(x)}{n!} t^n \\ &\Rightarrow t \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{B'_n(x)}{n!} t^n \\ &\Rightarrow \frac{B_{n-1}(x)}{(n-1)!} = \frac{B'_n(x)}{n!} \\ &\Rightarrow B'_n(x) = nB_{n-1}(x). \end{aligned}$$

Therefore if we denote $B_n^{(j)}(x)$ the j -th derivative of $B_n(x)$, then we have

$$(4.4) \quad B_n^{(j)}(x) = nB_{n-1}^{(j-1)}(x).$$

Theorem 4.1 *Let $n \geq 2$ be a positive integer and $a \geq n - 1$. Let P_n be the number of positive integral solution of $\frac{x_1}{a} + \frac{x_2}{a} + \dots + \frac{x_n}{a} \leq 1$, i.e., $P_n = \#\{(x_1, x_2, \dots, x_n) \in \mathbb{Z}_+^n : \frac{x_1}{a} + \frac{x_2}{a} + \dots + \frac{x_n}{a} \leq 1\}$. Let $[a]$ be the greatest integer less than or equal to a . then*

$$(n)! P_n = \sum_{k=1}^{n-1} \frac{(-1)^{n-k-1} S_{n-k-1}^{n-2} n}{k+1} (B_{k+1}([a]) - B_{k+1}(1)) \\ = \sum_{k=1}^{n-1} \frac{(-1)^{n-k-1} S_{n-k-1}^{n-2} k!}{(n-1)!} (B_n^{(n-k-1)}([a]) - B_n^{(n-k-1)}(1)).$$

Proof By (4.2), we have

$$\begin{aligned} \frac{1}{n+1}(B_{n+1}([a]) - B_{n+1}([a]-1)) &= ([a]-1)^n \\ \frac{1}{n+1}(B_{n+1}([a]-1) - B_{n+1}([a]-2)) &= ([a]-2)^n \\ &\vdots \\ \frac{1}{n+1}(B_{n+1}(3) - B_{n+1}(2)) &= 2^n \\ \frac{1}{n+1}(B_{n+1}(2) - B_{n+1}(1)) &= 1^n. \end{aligned}$$

Summing the above equations, we have

$$(4.5) \quad \frac{1}{n+1}(B_{n+1}([a]) - B_{n+1}(1)) = 1^n + 2^n + \dots + ([a]-1)^n = \sum_{x=1}^{[a]-1} x^n.$$

Similarly by (4.2) and (4.4), we have

$$\begin{aligned} \frac{1}{n}(B_n([a]) - B_n(1)) &= \frac{(n-1)!}{(n+1)!}(B'_{n+1}([a]) - B'_{n+1}(1)) \\ \frac{1}{n-1}(B_{n-1}([a]) - B_{n-1}(1)) &= \frac{(n-2)!}{(n+1)!}(B''_{n+1}([a]) - B''_{n+1}(1)) \\ &\vdots \\ (4.6) \quad \frac{1}{k}(B_k([a]) - B_k(1)) &= \frac{(k-1)!}{(n+1)!}(B^{(n-k+1)}_{n+1}([a]) - B^{(n-k+1)}_{n+1}(1)) \\ &\vdots \\ \frac{1}{2}(B_2([a]) - B_2(1)) &= \frac{1}{(n+1)!}(B^{(n-1)}_{n+1}([a]) - B^{(n-1)}_{n+1}(1)). \end{aligned}$$

By the proof of Theorem 3.1, we have

$$\begin{aligned} n! P_{n+1} &= \sum_{x=1}^{[a]-1} x(x-1)(x-2) \dots (x-n+1) \\ &= \sum_{x=1}^{[a]-1} \sum_{k=1}^n (-1)^{n-k} S_{n-k}^{n-1} x^k \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^n \sum_{x=1}^{[a]-1} (-1)^{n-k} S_{n-k}^{n-1} x^k \\
&= \sum_{k=1}^n \frac{(-1)^{n-k} S_{n-k}^{n-1}}{k+1} (B_{k+1}([a]) - B_{k+1}(1)) \quad \text{by (4.5)} \\
&= \sum_{k=1}^n \frac{(-1)^{n-k} S_{n-k}^{n-1} \cdot k!}{(n+1)!} (B_{n+1}^{n-k}([a]) - B_{n+1}^{n-k}(1)) \quad \text{by (4.6)}
\end{aligned}$$

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