THE HAMILTON-JACOBI EQUATIONS FOR A RELATIVISTIC CHARGED PARTICLE

J. R. Vanstone

(received June 16, 1962)

Introduction. In the problem of finding the motion of a classical particle one has the choice of dealing with a system of second order ordinary differential equations (Lagrange's equations) or a single first order partial differential equation (the Hamilton-Jacobi equation, henceforth referred to as the H-J equation). In practice the latter method is less frequently used because of the difficulty in finding complete integrals. When these are obtainable, however, the method leads rapidly to the equations of the trajectories. Furthermore it is of fundamental theoretical importance and it provides a basis for quantum mechanical analogues.

The treatment of motion of a relativistic particle is also usually based on Lagrange-type equations. When the H-J viewpoint is adopted the time is customarily singled out as a special parameter. Thus the coordinate-symmetry of the four-dimensional formulation is lost. On the other hand the approach is more easily assimilated into quantum theory.

It is the object of this paper to exhibit an H-J theory for relativistic particles which retains the symmetry of formulation and which yields certain known results by a comparatively easy calculation.

The first section deals with the geometrical background of the approach which is due to H. Rund. Since this is fully described in his paper [1], we shall here content ourselves with a brief outline of the principal results. Section 2 is devoted to the application of the theory to Lagrangians of the type encountered in relativistic electrodynamics. In the last section we examine a specific example.

Canad. Math. Bull. vol. 6, no. 3, September 1963.

1. We consider a dynamical system with $n-1$ degrees of freedom whose Lagrangian has the form $L^*(q^i, t, \dot{q}^i)$. Here t represents the time; $\dot{q}^i$ $(i = 1, 2, \ldots, n-1)$ is the time derivative of the generalized coordinate q^i. The trajectories are given by the variational principle

$$\delta \int L^* \, dt = 0 , \qquad (1)$$

which is equivalent to

$$\delta \int L \, ds = 0 , \qquad (1)'$$

where s is an arbitrary parameter and we have put $q^n = t$, $\dot{q}^i = q'^i / q'^n$, $q'^i = \frac{dq^i}{ds}$ and

$$L = L(q^\alpha, q'^\alpha) = L^*(q^i, q^n, q'^i/q'^n) q'^n$$

$$(\alpha = 1, 2, \ldots, n) . \qquad (2)$$

Clearly L is homogeneous of degree one in the q'^α and we may regard it formally as the metric function of a Finsler space (although it may not satisfy the usual positivity and convexity assumptions).

The canonical momenta p_α are then defined by

$$p_\alpha = \frac{1}{2} \frac{\partial L^2}{\partial q'^\alpha} = L \frac{\partial L}{\partial q'^\alpha} . \qquad (3)$$

Assuming that these equations are solvable for the q'^α as functions of the q^α and p_α we then define the Hamiltonian by

$$H(q^\alpha, p_\alpha) \equiv L[q^\alpha, q'^\alpha(q^\beta, p_\beta)] . \qquad (4)$$

It follows from equations (3) and (4) that

$$\text{(a)} \quad \frac{\partial H}{\partial p_\alpha} = \frac{1}{L} \frac{dq^\alpha}{ds} \qquad \text{(b)} \quad \frac{\partial H}{\partial q^\alpha} = - \frac{\partial L}{\partial q^\alpha} . \qquad (5)$$

Also we note that the Euler-Lagrange equations for the variational principle (1)' read

$$\frac{d}{ds}\frac{\partial L}{\partial q'^{\alpha}} - \frac{\partial L}{\partial q^{\alpha}} = 0 . \tag{6}$$

To this point the parameter s has been arbitrary. If it is now chosen so that

$$ds \equiv L(q^{\alpha}, dq^{\alpha}) , \tag{7}$$

then

$$L(q^{\alpha}, q'^{\alpha}) \equiv 1 , \tag{7)'}$$

and equations (5)(a) and (6) reduce to the canonical equations of motion

$$\frac{\partial H}{\partial p_{\alpha}} = \frac{dq^{\alpha}}{ds} , \qquad \frac{\partial H}{\partial q^{\alpha}} = - \frac{dp_{\alpha}}{ds} , \tag{8}$$

the latter because of (3) and (5)(b).

Consider now a point $P_o(q_o^{\alpha})$ and the family of all geodesics (extremals of (1)') passing through P_o. The points $P(q^{\alpha})$ which lie at a distance s from P_o, as measured along these geodesics, form a hypersurface S which may be represented by

$$S(q^{\alpha}) = s = \text{constant} . \tag{9}$$

Suppose that q'^{α} is the tangent vector to the geodesic through P on this surface and that p_{α} corresponds to q'^{α} as in (3). It can then be shown that

$$p_{\alpha} = \frac{\partial S}{\partial q^{\alpha}} . \tag{10}$$

Hence, in view of the normalizing condition (7)' and (4), the

https://doi.org/10.4153/CMB-1963-028-4 Published online by Cambridge University Press

hypersurface S satisfies the partial differential equation

$$H\left(q^{\alpha}, \frac{\partial S}{\partial q^{\alpha}}\right) = 1 . \tag{11}$$

This condition corresponds to the classical H-J equation for the Hamilton one-point function.

2. We now deal with the case when

$$L(q^{\alpha}, q'^{\alpha}) = (\underset{(1)}{\omega} + \underset{(2)}{\omega})/ds , \tag{12}$$

where (a) $\underset{(1)}{\omega}(q, dq) = a_{\alpha}(q)dq^{\alpha}$,

$$\text{(b)} \quad \underset{(2)}{\omega}(q, dq) = \left(b_{\alpha\beta}(q)dq^{\alpha}dq^{\beta}\right)^{1/2} . \tag{13}$$

As above we assume that α and β range from 1 to n and that the determinant of the $b_{\alpha\beta} = b_{\beta\alpha}$ is not zero, so that the tensor $b^{\alpha\beta}$ inverse to $b_{\alpha\beta}$ exists and satisfies

$$b^{\alpha\beta} b_{\beta\gamma} = \delta^{\alpha}_{\gamma} = \begin{cases} 1 , & \alpha = \gamma \\ 0 , & \alpha \neq \gamma \end{cases} . \tag{14}$$

From (3) we immediately deduce that

$$p_{\alpha} = L \cdot [a_{\alpha} + (b_{\alpha\beta}q'^{\beta})(\underset{(2)}{\omega}/ds)^{-1}] . \tag{15}$$

In order to find the Hamiltonian we should now solve these equations for q'^{α} and substitute in (12). The following method is, however, more direct. First we note that the vector

$$\tilde{p}_{\alpha} \equiv (b_{\alpha\beta}q'^{\beta})(\underset{(2)}{\omega}/ds)^{-1}$$

is a unit vector with respect to the metric tensor $b_{\alpha\beta}$ i.e.

https://doi.org/10.4153/CMB-1963-028-4 Published online by Cambridge University Press

$$b^{\alpha\beta}\tilde{p}_\alpha\tilde{p}_\beta = b^{\alpha\beta}(b_{\alpha\lambda}q'^{\lambda})(b_{\beta\mu}q'^{\mu})(\underset{(2)}{\omega}/ds)^{-2} \equiv 1, \qquad (16)$$

by (14) and (13)(b). But (15) shows that $\tilde{p}_\alpha = p_\alpha L^{-1} - a_\alpha$ and hence (16) yields

$$b^{\alpha\beta}(p_\alpha L^{-1} - a_\alpha)(p_\beta L^{-1} - a_\beta) \equiv 1. \qquad (16)'$$

This equation is (at most) quadratic in L^{-1} and, when solved, expresses L as a function of q^α and p_α. Comparison with (4) then shows that the solution is, in fact, the required Hamiltonian. We will not need the explicit form of H or the q'^α as functions of q^α and p_β, though the former is easily found from (16)' while the latter is given by

$$q'^\alpha = b^{\alpha\beta}(p_\beta - Ha_\beta)\psi^{-1},$$

where

$$\psi = 1 + b^{\alpha\beta}(p_\alpha H^{-1} - a_\alpha)a_\beta,$$

as a straight-forward calculation will verify.

It follows from (11) and the above remarks that the H-J equation for any dynamical system whose Lagrangian has the form (12) may be written

$$b^{\alpha\beta}\left(\frac{\partial S}{\partial q^\alpha} - a_\alpha\right)\left(\frac{\partial S}{\partial q^\beta} - a_\beta\right) = 1. \qquad (17)$$

The reader who wishes to compare these results with those of the usual theory may check that if

$$q^n = t, \quad L^* \equiv (\underset{(1)}{\omega} + \underset{(2)}{\omega})/dt, \quad p^*_k \equiv \frac{\partial L}{\partial \dot{q}^k}, \quad H^* \equiv p^*_k \dot{q}^k - L^*,$$

then

https://doi.org/10.4153/CMB-1963-028-4 Published online by Cambridge University Press

$$p_k = Lp_k^*, \qquad p_n = -LH^* \qquad (k = 1, 2, \ldots, n-1).$$

The last of these yields the classical H-J equation when L is put equal to 1 and p_α is replaced by $\dfrac{\partial S}{\partial q^\alpha}$.

3. The assumption that

$$n = 4, \qquad a_\alpha = e\varphi_\alpha, \qquad b_{\alpha\beta} = m_o^2 c^2 g_{\alpha\beta}, \tag{18}$$

permits us to interpret L as the Lagrangian of a particle of charge e and rest mass m_o, moving in the presence of a gravitational potential $g_{\alpha\beta}$ and an electromagnetic potential φ_α (cf. Lichnerowicz [2]). From (17), then, the H-J equation is

$$g^{\alpha\beta}\left(\frac{\partial S}{\partial q^\alpha} - e\varphi_\alpha\right)\left(\frac{\partial S}{\partial q^\beta} - e\varphi_\beta\right) = m_o^2 c^2. \tag{19}$$

We remark that if $\varphi_\alpha = 0$ this equation reduces to one introduced by Fock [3], primarily to deal with light rays.

The problem of motion for charged particles (or photons) will be solved by obtaining a complete integral of (19) for pre-assigned $g^{\alpha\beta}$ and φ_α. We examine this process in a special case.

Assume that the line-element $d\sigma \equiv (g_{\alpha\beta}dx^\alpha dx^\beta)^{1/2}$ has the form

$$d\sigma^2 = -\gamma^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \; d\varphi^2 + c^2\gamma \, dt^2. \tag{20}$$

Equation (19) then becomes

$$-\gamma\left(\frac{\partial S}{\partial r} - e\varphi_1\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta} - e\varphi_2\right)^2 - \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \varphi} - e\varphi_3\right)^2$$

https://doi.org/10.4153/CMB-1963-028-4 Published online by Cambridge University Press

$$+\frac{1}{c^2\gamma}\left(\frac{\partial S}{\partial t} - e\varphi_4\right)^2 = m_o^2c^2 . \qquad (21)$$

When γ, φ_α depend only on r and $\varphi_2 = \varphi_4 = 0$ (static spherical symmetry) Nordstrom [4] and Jeffery [5] showed that

$$\gamma = 1 - \frac{2km'}{c^2 r} + \frac{ke'^2}{4\pi c^4 r^2}, \qquad \varphi_4 = \frac{e'}{4\pi r}, \qquad (22)$$

while $\varphi_1(r)$ is arbitrary (it contributes nothing to the electromagnetic field). These results follow from the demand that Einstein's and Maxwell's field equations be satisfied. The constants k and c have their usual meanings, while m' and e' may be considered as constants of integration. It should be noted that the Schwarzschild line-element is a special case of (20) (when $e' = 0$ in (22)). Thus the results based on this line-element will be special cases of the Nordström-Jeffery theory.

Having made the above assumptions, we may further assume, without loss of generality, that $\varphi_1 = 0$ for, if necessary, we may replace S in (21) by $\bar{S} \equiv S - e\int \varphi_1 dr$. Equation (21) then reduces to

$$-\gamma\left(\frac{\partial S}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 - \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2$$

$$+\frac{1}{c^2\gamma}\left(\frac{\partial S}{\partial t} - e\varphi_4\right)^2 = m_o^2c^2. \qquad (23)$$

The variables φ and t are cyclic and, putting $S = S_1(r) + S_2(\theta) + S_3(\varphi) + S_4(t)$, we readily obtain

$$\frac{dS_3}{d\varphi} = k_3, \quad \frac{dS_4}{dt} = k_4, \quad \left(\frac{dS'_2}{d\theta}\right)^2 + \frac{k_3^2}{\sin^2\theta} = k_2^2, \qquad (24)$$

https://doi.org/10.4153/CMB-1963-028-4 Published online by Cambridge University Press

where k_2, k_3, k_4 are constants. Substitution from (24) into (23) then yields

$$\left(\frac{dS_1}{dr}\right)^2 = (k_4 - e\varphi_4)^2(c\gamma)^{-2} - [(m_o c)^2 + (k_2 r^{-1})^2]\gamma^{-1}. \qquad (25)$$

Thus the required complete integral is

$$S = \int [(k_4 - e\varphi_4)^2(c\gamma)^{-2} - (m_o^2 c^2 + k_2^2 r^{-2})\gamma^{-1}]^{1/2}\, dr$$

$$+ \int [k_2^2 - k_3^2 \sin^{-2}\theta]^{1/2} d\theta + k_3\varphi + k_4 t + (\text{constant}),$$

(26)

and the trajectories are found by solving

$$\frac{\partial S}{\partial k_p} = h^p = (\text{constant}) \qquad (p = 2, 3, 4) \qquad (27)$$

for r, θ and φ.

The set of solutions admits a sub-family which lies in the equatorial "plane" $\theta = \pi/2$. To see this we solve the equations

$$\frac{\partial^2 S}{\partial k_p \partial x^i}\frac{dx^i}{dt} + \frac{\partial^2 S}{\partial k_p \partial t} = 0 \qquad (p = 2, 3, 4;\ x' = r,\ x^2 = \theta,\ x^3 = \varphi)$$

for $\frac{d\theta}{dt}$. The solution is

$$\frac{d\theta}{dt} = -c^2\gamma r^{-2}(k_4 - e\,\varphi_4)^{-1}[k_2^2 - k_3^2 \sin^{-2}\theta]^{1/2}.$$

Thus, if $\theta = \pi/2$ initially and we put $k_2 = k_3$, then $d\theta/dt$ remains identically zero. In this case the function S is independent of θ and k_2 is replaced by k_3. Hence the relation

https://doi.org/10.4153/CMB-1963-028-4 Published online by Cambridge University Press

$$\frac{\partial S}{\partial k_3} = \int [(c\gamma)^{-2}(k_4 - e\varphi_4)^2 - (m_o^2c^2 + k_2^2r^{-2})\gamma^{-1}]^{-1/2}(\gamma r^2)^{-1}k_3 dr$$

$$+ \varphi = h^3 \tag{28}$$

gives r as a function of φ. The corresponding differential equation is found by differentiating (28) with respect to r. When we solve the resulting equation for $\frac{dr}{d\varphi}$, square, and set $r = u^{-1}$, we obtain

$$k_3^2\left(\frac{du}{d\varphi}\right)^2 = \frac{1}{c^2}\left(k_4 - \frac{ee'}{4\pi}u\right)^2 - \left(m_o^2c^2 + k_3^2u^2\right)\left(1 - \frac{2km'u}{c^2} + \frac{ke'^2}{4\pi c^4}u^2\right). \tag{29}$$

Allowing for interpretation of the constants k_3, k_4 and the units of charge, this is identical to the equation derived by Jeffery [5].

The author would like to acknowledge his indebtedness to Professor D. K. Sen for several helpful discussions in connection with the above work.

REFERENCES

1. H. Rund, Die Hamiltonsche Funktion bei allgemeinen dynamischen Systemen, Arch. Math. 3, 207-215 (1952).

2. A. Lichnerowicz, Théories Relativistes de la Gravitation et de L'Électromagnétisme, Masson (1955), p. 174.

3. V. Fock, Theory of Space, Time and Gravitation, Pergamon (1959), pp. 117-123.

4. G. Nordström, On the Energy of the Gravitational Field in Einstein's Theory, Proc. Ac. Amsterdam, vol. 20, p. 1238, (1918).

https://doi.org/10.4153/CMB-1963-028-4 Published online by Cambridge University Press

5. G. Jeffery, The Field of an Electron on Einstein's Theory of Gravitation, Proc. Roy. Soc. London, A, vol. 99, p. 123, (1921).

University of Toronto
and
The Summer Research Institute of the
Canadian Mathematical Congress